Notes for a talk at "Model Theory and Mathematical Logic: Conference in honor of Chris Laskowski's 60th birthday," University of Maryland, College Park

Ratio Then and Now

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1 Preface

So that they might serve as a reference for my talk a week later in College Park. I prepared and printed these notes before leaving Istanbul on June 14, 2019. I assembled them, in part from earlier notes and a published article:

- "Euclid Mathematically and Historically," from a 50-minute colloquium talk in the mathematics department of Bilkent University, Ankara, March 7, 2018;
- "Conic Sections With and Without Algebra," from a 20-minute contributed talk at Antalya Algebra Days, Nesin Mathematics Village, May 15, 2019;
- "Affine Geometry," filename `affine.tex`;
- "Thales and the Nine-point Conic" [20].

While in the US, by hand I wrote out five A5-size pages of notes that I might actually write on the boards. In the process, I detected some mistakes, now corrected, in the present notes. After the talk, I prepared a separate typeset document containing notes of what I said and might have said.

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2 Abstract

Having submitted this as plain text, I typeset it here as such.

“Heraclitus holds that the findings of sense-experience are untrustworthy, and he sets up reason [logos, ratio] as the criterion” (Sextus Empiricus)

“It is necessary to know that war is common and right is strife [eris] and all things happen by strife and necessity” (Heraclitus, according to Origen)

1. Strife has arisen between the historian of mathematics and the mathematician who thinks about the past. One must be both, to understand Euclid’s obscure definition of proportion of numbers. Proportion is sameness of ratio. When this occurs between two pairs of numbers, something should be the same about each pair. In Book VII of the Elements, this can only mean that the Euclidean Algorithm has the same steps when applied to either pair of numbers. From this, despite modern suggestions to the contrary, Euclid has rigorous proofs, not only of what we call Euclid’s Lemma, but also of the commutativity of multiplication.

2. Apollonius of Perga gives three ways to characterize a conic section: (i) an equation, involving a latus rectum, that we can express in Cartesian form; (ii) the proportion whereby the square on the ordinate varies as the abscissa or product of abscissas; (iii) an equation of a triangle with a parallelogram or trapezoid. The latter equation holds in an affine plane. With the advent of Cartesian methods in 1637, the equation seems to have been forgotten, because it is not readily translated into the lengths (symbolized by single minuscule letters) that Descartes has taught us to work with. With the affine equation, Apollonius can give a proof-without-words of what today we consider a coordinate change, performed with more or less laborious computations.

3. By interpreting the field where algebra is done in the plane where geometry is done, Descartes does inspire new results. An example still builds on work of an ancient mathematician, Pappus of Alexandria. The model companion of the theory of Pappian affine spaces of unspecified dimension, considered as sets of points with ternary relation of collinearity and quaternary relation of parallelism, is the theory of Pappian affine planes over algebraically closed fields.

3 Notes for talk

3.1 Affinity

Preservation Theorem. *If V and V^* are the points $(1,0)$ and (a,b) on the ellipse or hyperbola with center K given by*

$$x^2 \pm y^2 = 1, \tag{3.1}$$

as in Fig. 3.1, then the curve is preserved under the linear transformation that interchanges V and V^*.

Proof. The transformation is multiplication by

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

for some (c,d), and then

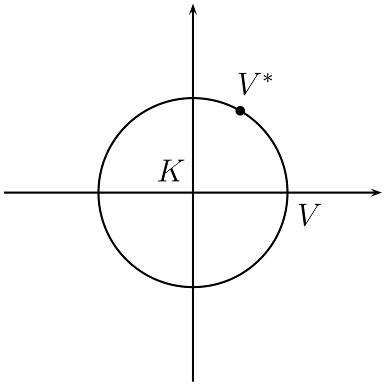
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a^2 + cb \\ ba + db \end{pmatrix},$$

so that

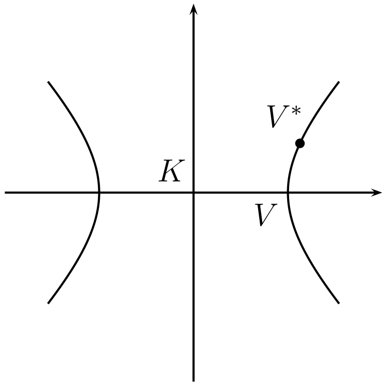
$$c = \pm b, \qquad d = -a,$$

and

$$\begin{pmatrix} a & \pm b \\ b & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \pm by \\ bx - ay \end{pmatrix}.$$



(a) Ellipse



(b) Hyperbola

Figure 3.1: Ellipse and hyperbola

Since
$$a^2 \pm b^2 = 1,$$
we compute
$$(ax \pm by)^2 \pm (bx - ay)^2 = x^2 \pm y^2.$$
Thus the image $(ax \pm by, bx - ay)$ lies on our curve if (x, y) does. $\square$

Corollary. *If $\overrightarrow{KV}$ and $\overrightarrow{KL}$ are independent vectors, as in Fig. 3.2 or Fig. 3.3, and V^* lies on the locus of points P such that*
$$\overrightarrow{KP} = x \cdot \overrightarrow{KV} + y \cdot \overrightarrow{KL},$$
where again (3.1) holds, then the affinity *or affine transformation of the plane that fixes K and interchanges V and V^* fixes the curve.*

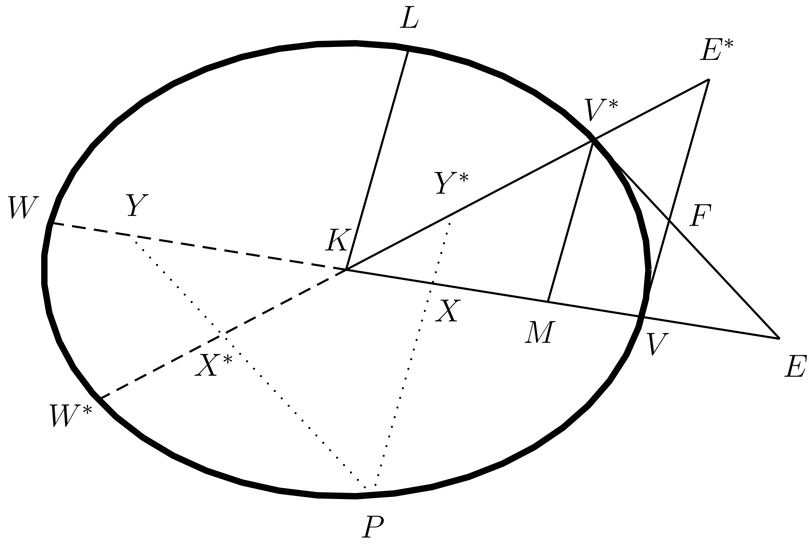


Figure 3.2: Ellipse

After publication of Descartes's *Geometry* [6, 7] in 1637, versions of the preservation theorem, or rather its corollary, were expressed and proved in

1655 by Wallis [24],
1659 by de Witt [11],
1748 by Euler [9], and
1758 by Hugh Hamilton [12, 13].

In each case *lengths* are used, or perhaps *vectors.* Descartes reduced the ancient algebra of areas and volumes to an algebra of lengths alone. Nobody seems to follow Apollonius in proving the preservation theorem using areas, as in Book I of the *Conics* [1, 2]. We may thus have lost something.

Apollonius reasons as follows. Dropping

- VE^* to KV^* and
- V^*M and PX to KV,

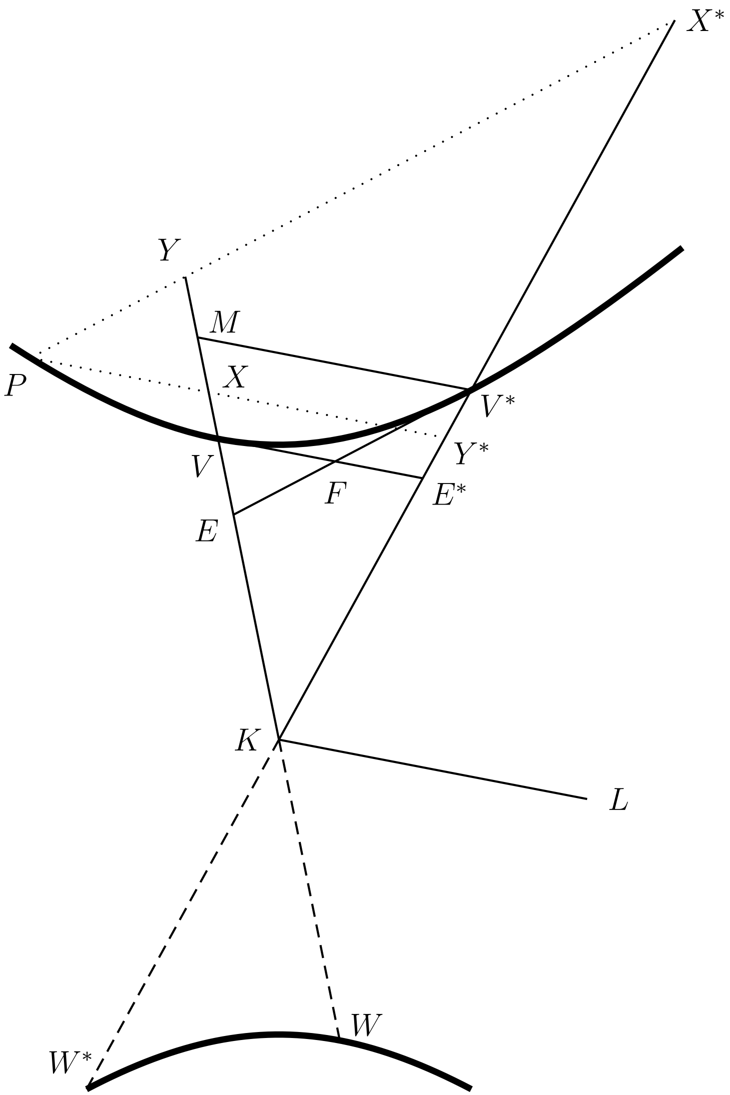


Figure 3.3: Hyperbola

all parallel to KL, writing (3.1) as

$$\pm y^2 = (1-x)(1+x),$$

we have

$$\begin{aligned}\overrightarrow{XP}^2 &\propto \overrightarrow{XV} \cdot \overrightarrow{WX} \\ &\propto \overrightarrow{XV} \cdot (\overrightarrow{KV} + \overrightarrow{KX}) \\ &\propto \overrightarrow{XV} \cdot (\overrightarrow{VE^*} + \overrightarrow{XY}) \\ &\propto VXY^*E^*. \end{aligned} \tag{3.2}$$

Letting E on KV satisfy

$$V^*V \parallel E^*E,$$

equivalently

$$\overrightarrow{KM} : \overrightarrow{KV} :: \overrightarrow{KV} : \overrightarrow{KE},$$

we have

$$MV^*E = VMV^*E^*. \tag{3.3}$$

Dropping PY to KV parallel to V^*E, we have

$$\overrightarrow{XP}^2 \propto XPY$$

and therefore, from (3.2),

$$XPY \propto VXY^*E^*.$$

Since this becomes an equation, namely (3.3), when P is V^*, we conclude

$$XPY = VXY^*E^*. \tag{3.4}$$

This is an alternative defining equation for our curve. The polygons in (3.4) are oriented, and the non-parallel sides of the trapezoid may intersect internally. If either of

$$\overrightarrow{PX} = \overrightarrow{XP'}, \qquad \overrightarrow{PK} = \overrightarrow{KP'}$$

holds, then P' lies on the curve. Thus, because

- VK bisects the chords that are parallel to VE^*, it is a **diameter** of the curve;
- the curve is symmetric about K, this is its **center.**

Consequently,

every line through K is a diameter of the curve.

Moreover, PY cutting V^*K at X^*, if to either side of the defining equation (3.4) we add the quadrilateral

$$YX^*Y^*X,$$

we obtain

$$\begin{aligned} Y^*PX^* &= VYX^*E^* \\ &= EYX^*V^*. \qquad \text{[by (3.3)]} \end{aligned}$$

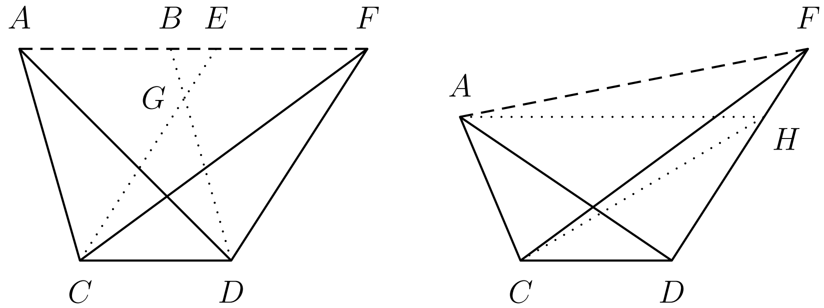
Thus,

with respect to any diameter,
the curve has the same equation.

This is the Preservation Theorem.

For Rosenfeld in his commentary [22, p. 57] on the theorem,

> Apollonius never mentions parabolic, elliptic, and hyperbolic turns, but no doubt that he used these transformations to generalize the results obtained by his precursors in rectangular coordinates for the cases of oblique coordinates.



(a) Proposition 37 (b) Proposition 39

Figure 3.4: Parallelism

A “turn” is for Coxeter [4, pp. 206–7] a rotation; it is a kind of affinity.

There are all kinds of doubt that Apollonius had any notion of an affinity in our technical sense. This is important because:

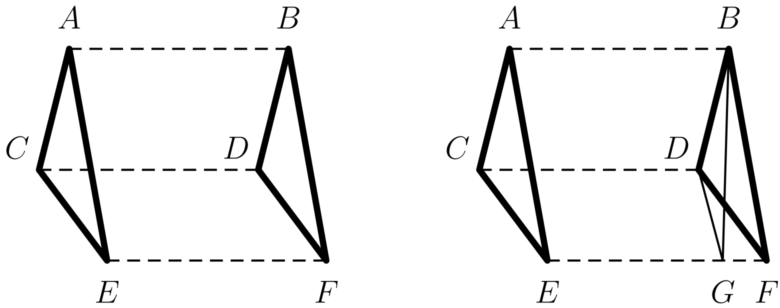
1) Apollonius’s proof is more direct than the best modern proof;
2) misunderstanding of the ancient use of ratios has led some modern mathematicians to accuse Euclid of logical error.

3.2 Affine plane

In Book I of the *Elements,* Propositions 37 and 39 are that, in Fig. 3.4,

$$AF \parallel CD \Leftrightarrow ACD = FCD.$$

We can understand Euclid’s proof as having the following steps, the justifications of which will be sufficient to axiomatize an **affine plane** in the full sense of being acted on tran-



(a) Statement (b) Proof

Figure 3.5: Prism Theorem

sitively and faithfully by a 2-dimensional vector space over a commutative field, albeit of characteristic other than 2.

1. First we assume $AF \parallel CD$. Using **Playfair's Axiom,** that through a point not on a line passes exactly one parallel, we let

$$AC \parallel BD, \qquad\qquad CE \parallel DF.$$

2. Because $ABDC$ and $CDFE$ are parallelograms, as in Fig. 3.5a,

$$ACE = BDF.$$

3. Because equality is a congruence on the abelian group of polygons, by Euclid's Common Notions 2 and 3,

   If equals be added to (or subtracted from) equals,
   the wholes (or remainders) are equal,

we conclude

$$\begin{aligned} ACDB &= ACGB + CDG \\ &= ACE - BGE + CDG \\ &= BDF - BGE + CDG = ECDF \end{aligned}$$

(which is Proposition 35).

4. Because a diagonal bisects a parallelogram (by Proposition 34),
$$2ACD = 2FCD.$$
5. No element of the group of polygons having order 2,
$$ACD = FCD.$$
6. Now suppose $AF \nparallel CD$. Letting $AH \parallel CD$, we conclude
$$ACD = HCD \neq HCD + FCH = FCD,$$
by Euclid's Common Notion 5,

The whole is greater than the part.

Our axioms formally govern a structure with two sorts,

- Σ, of points (τὰ σημεῖα), and
- Π, of polygons (τὰ πολύγωνα).

When $n \geqslant 3$, there is an n-ary map

$$(P_1, \ldots, P_n) \mapsto P_1 \cdots P_n$$

from Σ to Π. In Axiom 3, the group of polygons satisfies the following rules, where Γ and Δ are strings of letters for points:

$$\begin{gathered} A\Gamma = A\Gamma A, \\ A\Gamma = \Gamma A, \\ A\Gamma B + B\Delta A = A\Gamma B\Delta, \\ -A_1 \cdots A_n = A_n \cdots A_1. \end{gathered}$$

We can understand Axiom 6 as a definition of collinearity:

$$\operatorname{coll}(A, B, C) \Leftrightarrow ABC = 0, \tag{3.5}$$

or rather as meaning that if each of two distinct points is collinear, in the sense of (3.5), with two other distinct points, the converse holds as well:

$$A \not\equiv B \;\&\; ACD = 0 \;\&\; BCD = 0 \;\&\; C \not\equiv D \Rightarrow ABC = 0.$$

It is now a theorem that two points determine a line: if the two points are A and B, the line they determine is defined by

$$ABX = 0,$$

and if C and D are two points on this line, then they determine the same line.

To obtain a companionable theory whose model-companion is still a theory of affine planes, we let there be a quaternary relation of **parallelism** on Σ, given by the axiom

$$\begin{aligned} AB \parallel CD \Leftrightarrow \forall X\ \Big(&A \not\equiv B \;\&\; C \not\equiv D \;\&\\ &(ABX = 0 \;\&\; CDX = 0 \Rightarrow ABC = 0 \;\&\; ABD = 0)\Big). \end{aligned}$$

The axiom is $\forall\exists$, by the general rule

$$\big(\sigma \Leftrightarrow \forall x\ \phi(x)\big) \Leftrightarrow \forall x\ \exists y\ \Big(\big(\sigma \Rightarrow \phi(x)\big) \;\&\; \big(\phi(y) \Rightarrow \sigma\big)\Big).$$

Also for companionability, Axiom 5 should be that the abelian group of polygons is either torsion-free or an elementary p-group (thus a vector space) for some odd prime p.

The case of **Desargues's Theorem** that I call the **Prism Theorem,** and that is shown in Fig. 3.5a, is now a theorem.

If AB, CD, and EF are parallel to one another (parallelism being transitive by Playfair's Axiom), then

$$AC \parallel BD \mathbin{\&} CE \parallel DF \Rightarrow AE \parallel BF.$$

We may assume $ACE \neq 0$. By Axiom 2,

$$ACE = BDF.$$

If $AE \nparallel BF$, let

$$AE \parallel BG.$$

By Axiom 2 again,

$$ACE = BDG.$$

By Euclid's Proposition 39 as above,

$$BD \parallel EF,$$

contradicting the (implicit) assumption that AB and CD are distinct lines.

Just as Pappus does, using Euclid's Propositions 37 and 39, we prove **Pappus's Theorem** [18, 17, 16] that if the vertices of $ABCDEF$ lie alternately on two lines, as in Fig. 3.6, then

$$BC \parallel EF \mathbin{\&} CD \parallel FA \Rightarrow AB \parallel DE.$$

For, under the hypothesis,

$$\begin{aligned}
ABE &= ABC + CBE \\
&= ABC + CBF \\
&= ABF + FAC \\
&= ABF + FAD \\
&= ABD,
\end{aligned}$$

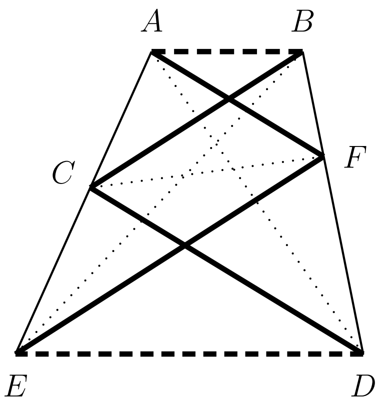


Figure 3.6: Pappus's Theorem

or more briefly,

$$ABE = ABFC = ABD.$$

Selecting now a proper triangle IOI', we define a multiplication on OI as in Fig. 3.7a. The operation is commutative by Pappus's Theorem, as in Fig. 3.7b. For associativity, use Fig. 3.8, where is shown

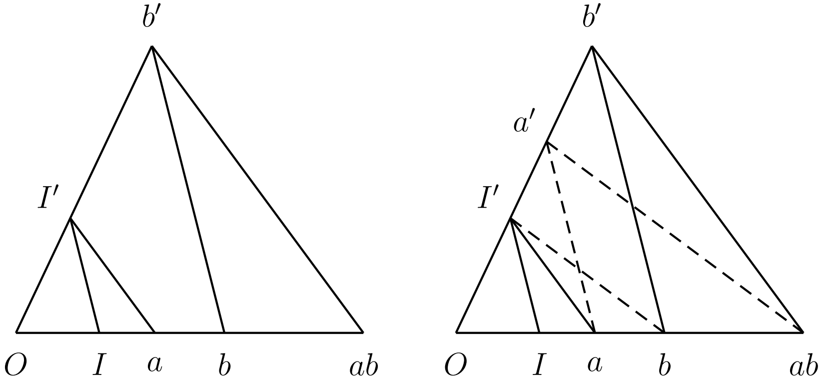
$$(ab)c = (ac)b,$$

from which, by commutativity, associativity follows.

The Prism Theorem lets us define define parallel directed segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ as **equal** if and only if

- the segment are not collinear, and $AC \parallel BD$, or
- they are collinear and equal to a third in the sense just defined.

A **vector** is the class of directed segments equal to a given one. We can add non-parallel vectors by completing the parallelogram; parallel, by placing them end to end, as in Fig. 3.9, where



(a) Definition (b) Commutativity

Figure 3.7: Multiplication

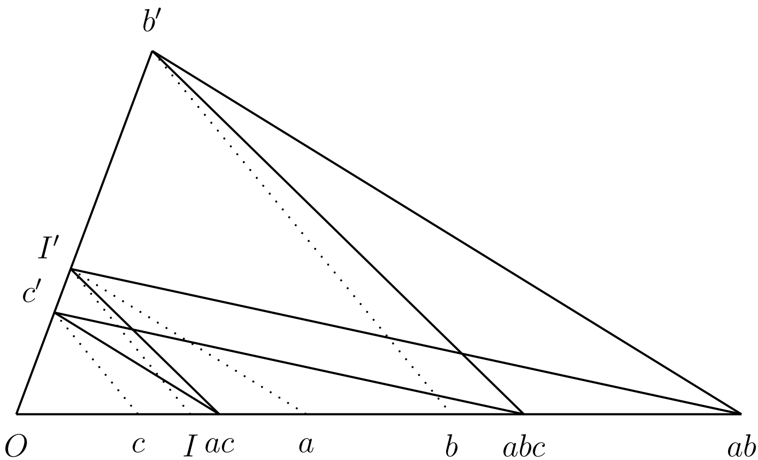


Figure 3.8: Associativity

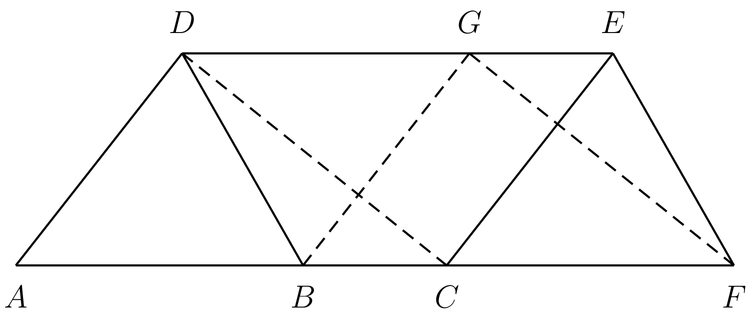


Figure 3.9: Commutativity of addition

$$\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AF} = \overrightarrow{AC} + \overrightarrow{AB},$$

by definition and by Pappus's Theorem.

3.2.1 Associativity

To prove associativity of addition of vectors, we have three cases to consider.

1. When A is not collinear with any two of B, C, and D, as in Fig. 3.10, then

$$\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AE}, \quad \overrightarrow{AE} + \overrightarrow{AD} = \overrightarrow{AF}, \quad \overrightarrow{AC} + \overrightarrow{AD} = \overrightarrow{AG},$$

and then

$$\overrightarrow{AB} + \overrightarrow{AG} = \overrightarrow{AF} \iff AB \parallel GF \;\;\&\;\; AG \parallel BF. \tag{3.6}$$

They *are* parallel, by the Prism Theorem, as follows. We can apply the Theorem first to triangles ACE and DGF, so that, since

$$CG \parallel AD \parallel EF, \qquad AC \parallel DG, \qquad AE \parallel DF,$$

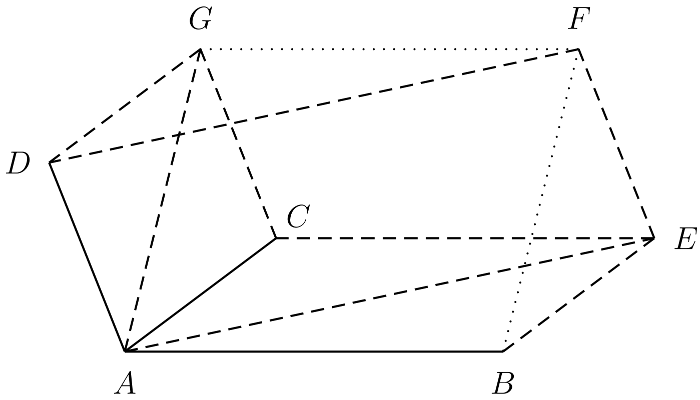


Figure 3.10: Associativity of addition: easy case

we can conclude $CE \parallel GF$. Since also $AB \parallel CE$, we obtain $AB \parallel GF$. Now we can apply the Prism Theorem to triangles ACG and BEF, obtaining $AG \parallel BF$. So we have the right-hand side of (3.6), and therefore the left-hand side, which means

$$\overrightarrow{AB} + (\overrightarrow{AC} + \overrightarrow{AD}) = (\overrightarrow{AB} + \overrightarrow{AC}) + \overrightarrow{AD}. \tag{3.7}$$

2. When AB contains C, but not D, then (3.7) still holds, since in Fig. 3.11,

$$\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AF}, \quad \overrightarrow{AF} + \overrightarrow{AD} = \overrightarrow{AK}, \quad \overrightarrow{AC} + \overrightarrow{AD} = \overrightarrow{AE},$$

so that

$$\overrightarrow{AK} = \overrightarrow{AB} + \overrightarrow{AE} \iff AE \parallel BK.$$

The parallelism follows from the Parallel Theorem, applied to the hexagon $ADBKFE$.

3. Finally, when AB contains both C and D, then, making use of commutativity, in Fig. 3.12 we have

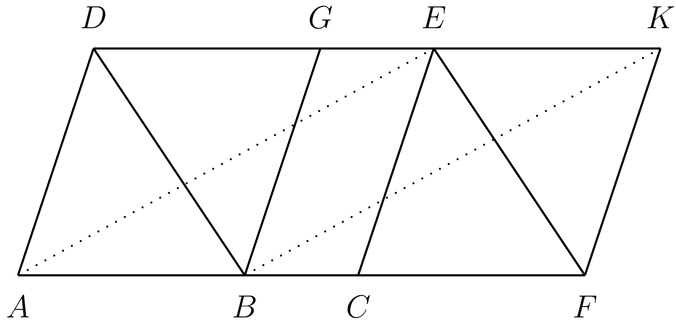


Figure 3.11: Associativity of addition: less easy case

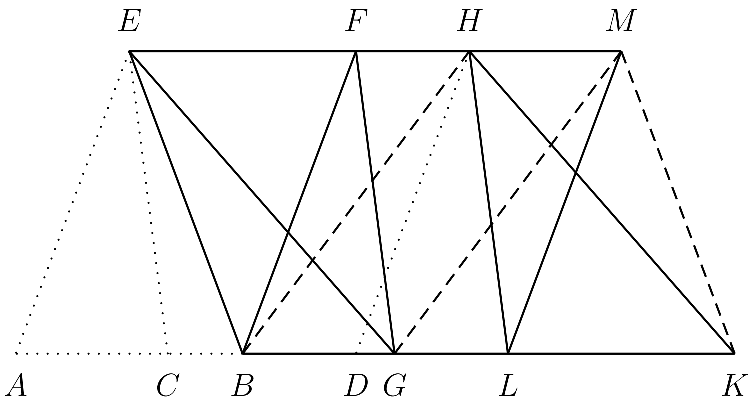


Figure 3.12: Associativity of addition: hardest case

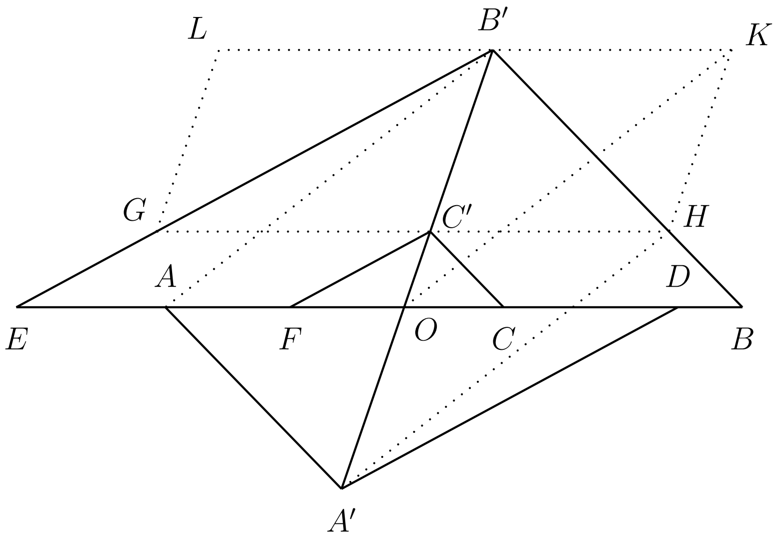


Figure 3.13: Distributivity

$$\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AC} + \overrightarrow{AB} = \overrightarrow{AG},$$
$$\overrightarrow{AG} + \overrightarrow{AD} = \overrightarrow{AK}, \qquad \overrightarrow{AC} + \overrightarrow{AD} = \overrightarrow{AL},$$
$$\overrightarrow{AB} + \overrightarrow{AL} = \overrightarrow{AK} \iff BE \parallel MK.$$

The Parallel Theorem, applied to $BFGMLH$, yields $BH \parallel GM$; then, applied to $KHBEGM$, $BE \parallel MK$.

3.2.2 Distributivity

Addition of vectors makes any line into an abelian group. Multiplication in the line OI makes that line into a field and every line through O into a one-dimensional vector space. Indeed, in Fig. 3.13, where

$$\overrightarrow{OA} = \boldsymbol{a}, \qquad \overrightarrow{OB} = \boldsymbol{b}, \qquad \overrightarrow{OC} = \boldsymbol{a} + \boldsymbol{b},$$

and then

$$\overrightarrow{OD} = c\boldsymbol{a}, \qquad \overrightarrow{OE} = c\boldsymbol{b}, \qquad \overrightarrow{OF} = c(\boldsymbol{a}+\boldsymbol{b}),$$

We have

$$\begin{aligned} \overrightarrow{OA} &= \overrightarrow{BC} && \text{[definition of +]} \\ &= \overrightarrow{C'B} && \text{[Prism Theorem]} \\ &= \overrightarrow{B'K}, && \text{[Prism Theorem]} \end{aligned}$$

then

$$AB' \parallel OK,$$

so by the converse of the Prism Theorem,

$$\overrightarrow{A'O} = \overrightarrow{HK} = \overrightarrow{C'B'}.$$

Now the converse of the Prism Theorem yields

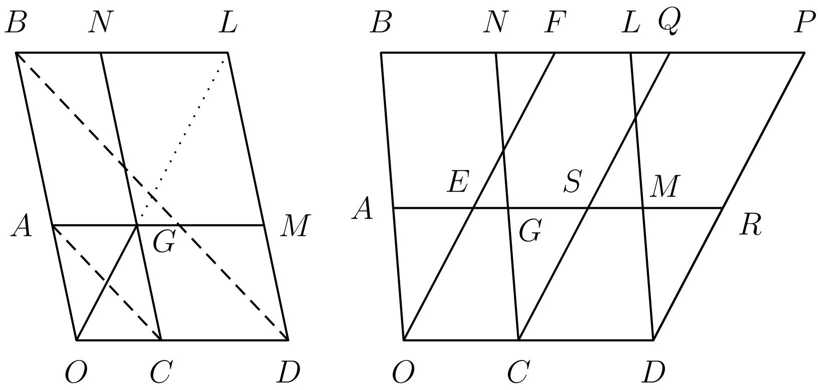
$$\overrightarrow{OD} = \overrightarrow{LB'} = \overrightarrow{GC'} = \overrightarrow{EF},$$

and therefore

$$c\boldsymbol{a} + c\boldsymbol{b} = c(\boldsymbol{a}+\boldsymbol{b}).$$

3.2.3 Vector space

To ensure finally that, with O selected, the plane is a vector space over the field of points along OI, we show that the field is independent of choice of I'. It is independent, because Desargues's Theorem holds in the special case shown in "Thales and the Nine-point Conic" [20]. First we need to establish that, in Fig. 3.14a,



(a) Euclid I.43 (b)

Figure 3.14: Conditions for parallelism

$$AC \parallel BD \iff AGNB = GCDM \iff OGL = 0.$$

Then, in Fig. 3.14b, when we assume

$$AC \parallel BD, \qquad\qquad AE \parallel BF \parallel OD.$$

Then

$$CDRS = CDMG = AGNB = ESQF,$$

so

$$EC \parallel FD.$$

3.3 Proportion

At the head of Book VII of the *Elements,* we are told, among what are labelled as definitions:

1. "A number is a multitude of units."[1]
2. Numbers A, B, C, and D are **proportional** when A is
   - the same multiple, or
   - the same part, or
   - the same parts,

   of B that C is of D.
3. The following are equivalent for numbers.
   a) B is a multiple of A,
   b) A is a part of B,
   c) A **measures** B.
4. If neither multiple nor part of B, A is **parts** of B.[2]

Measuring will be dividing in *extension,* but not in *intension*:

(1) We can **measure** 12 apples evenly by 4 apples.
(2) In the process, we **divide** the apples into 3 groups.[3]

It is clear when A is the **same multiple** or **same part** of B that C is of D; but not **same parts.**

If

$$A : B :: C : D,$$

this should mean at least that for some numbers E and F, for some multipliers k and m (either of which can be unity),

$$\begin{aligned} A &= E \cdot k, & C &= F \cdot k, \\ B &= E \cdot m, & D &= F \cdot m. \end{aligned} \tag{3.8}$$

[1] As he notes his "Mathematicall Preface" [5] to Billingsley's 1570 English translation of the *Elements,* John Dee *created* the word "unit" precisely to translate Euclid's $\mu o \nu \acute{\alpha} \varsigma$. The existing alternative was "unity." See my article "On commensurability and symmetry" [21].

[2] The text says only that the *less* is parts of the greater when not measuring the greater; but the definition of proportion implies that the greater is parts of the less when not a multiple of the less.

[3] Euclid uses dividing, as far as I know, only to say that an even number can be divided in two. Alexandre Borovik discusses measuring and dividing apples [3], though not with the terminology of measuring.

- Some Moderns call this the **Pythagorean** definition of proportion of numbers.[4]
- For the proper **Euclidean** definition, I say, we need also

$$E = \textsc{gcm}(A, B), \qquad F = \textsc{gcm}(C, D), \tag{3.9}$$

  where GCM means **greatest common measure;** equivalently, k and m in (3.8) are coprime.

Without (3.9),

(1) sameness of ratio is not *immediately* transitive;
(2) thus proofs in Book VII are inadequate;
(3) Proposition 4 makes little sense, the enunciation, "Any number is either a part or parts of any number, the less of the greater," only restating a definition, though the proof is nontrivial.

[4]Heath thinks (1) the theory of Book VII is due to the Pythagoreans [8, Vol. 2, p. 294], and (2) its definition of proportion is the one that we are calling Pythagorean [14, p. 190]. In Thomas's first Loeb volume of *Greek Mathematical Works* [23], the chapter "Pythagorean Arithmetic" gives first the definitions that head Book VII of the *Elements,* but nothing ensues that requires a careful interpretation of the definition of proportion. That this definition ought *immediately* to imply transitivity of sameness of ratio: this might seem belied by Proposition 11 in Book V, which proves the transitivity for arbitrary magnitudes under the Eudoxan definition; however, the proof is trivial. Nonetheless, Pengelley and Richman [19, pp. 196, 199] accept Heath's judgment, and Mazur [15, n. 6] accepts *their* judgment; for convenience, I imitate them in using the term *Pythagorean.* I have seen no suggestion that the Pythagoreans proved general theorems like the commutativity of multiplication or Euclid's Lemma; thus perhaps they had no theoretical need for the transitivity of sameness of ratio.

3.4 Anthyphaeresis

In Book VII, **Propositions 1–3** show, for two or more numbers,

(1) *how* to find a GCM, and
(2) *that* it is measured by all common measures.

The proofs use the **Euclidean Algorithm,** namely,

(1) replace the greater of two magnitudes with its remainder, if there is one, after measurement by the less;
(2) repeat.

When the greater is measured exactly by the less, this is the GCM. Thus from

$$\begin{aligned}80 &= 62 \cdot 1 + 18,\\ 62 &= 18 \cdot 3 + 8,\\ 18 &= 8 \cdot 2 + 2,\\ 8 &= 2 \cdot 4,\end{aligned}$$

we have $\text{GCM}(80, 62) = 2$, Also,

$$80 = 2 \cdot 40, \qquad\qquad 62 = 2 \cdot 31, \tag{3.10}$$

and the multipliers 40 and 31 are automatically coprime.

Euclid proves **Proposition 4** (again "Any number is either a part or parts of any number, the less of the greater") by finding GCM's, showing implicitly (in my view) that he intends what I am calling the Euclidean definition of proportion. From

$$\begin{aligned}120 &= 93 \cdot 1 + 27,\\ 93 &= 27 \cdot 3 + 12,\\ 27 &= 12 \cdot 2 + 3,\\ 12 &= 3 \cdot 4,\end{aligned}$$

we have $\text{GCM}(120, 93) = 3$, and also

$$120 = 3 \cdot 40, \qquad 93 = 3 \cdot 31. \tag{3.11}$$

By the repetition in (3.11) of multipliers from (3.10),

$$80 : 62 :: 120 : 93.$$

The same follows, just from the repetition of the multipliers $(1, 3, 2, 4)$ in the steps of the Algorithm. Indeed, we can write either of the fractions $^{80}/_{62}$ and $^{120}/_{93}$ as the continued fraction

$$1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{4}}}.$$

In Greek, the Algorithm is **anthyphaeresis** or "alternating subtraction."[5]

- In Book V, for arbitrary magnitudes, Euclid gives the **Eudoxan** definition of proportion, whereby a ratio is effectively a Dedekind cut.
- Before this was known, there was an **anthyphaeretic** definition, whereby the proportion

  $$A : B :: C : D$$

  means the Euclidean Algorithm has the same steps, whether applied to A and B or C and D.[6]

[5] The term derives ultimately from ἀνθυφαιρέ-ω (*anthyphaire-ô*) "alternately subtract," the verb that Euclid uses to describe his Algorithm. The analysis is ἀντί + ὑπό + αἱρέ-ω (*anti* + *hypo* + *haire-ô*), the core verb meaning *take*.

[6] See Thomas [23, pp. 504–9] or Fowler [10].

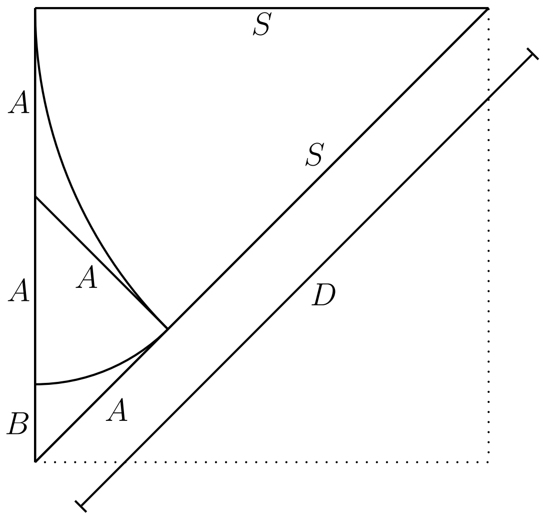


Figure 3.15: Anthyphaeresis of diagonal and side of square

- The Euclidean definition is a simplification of this for numbers.

The anthyphaeretic definition applies even to incommensurable magnitudes,[7] such as the diagonal and side of a square, as in Figure 3.15, where

$$D = S + A, \qquad\qquad S = A \cdot 2 + B,$$

[7]In particular, there is no reason to think that the Eudoxan theory was "developed to handle incommensurable magnitudes." Pengelley and Richman [19, p. 199] suggest that it was, even though they cite the book [10] of Fowler, who says, "I now disagree with everything in this line of interpretation"—the line whereby the Pythagoreans based mathematics on commensurable magnitudes, until the discovery of incommensurability, whose problems were not resolved until the Eudoxan theory was formulated.

and ever after, the less goes twice into the greater, so that

$$S : A :: A : B,$$

and also the ratio $D : S$ is independent of D.

Understanding proportion is important because Euclid uses it to prove

(1) commutativity of multiplication, and
(2) **Euclid's Lemma,** that a prime measuring a product measures one of the factors.

Under the Euclidean definition, the proofs are rigorous.

3.5 Commutativity

In Book VII of the *Elements,* from either the anthyphaeretic or the Euclidean definition of proportion of numbers, we obtain **Propositions 5–8:**

$$A : B :: C : D \implies A : B :: A \pm C : B \pm D.$$

Repeated application gives **Proposition 9:**

$$E : F :: E \cdot m : F \cdot m.$$

This gives, *by transitivity,* **Proposition 10:**

$$E \cdot k : F \cdot k :: E \cdot m : F \cdot m. \tag{3.12}$$

Automatically, if k and m are coprime,

$$E \cdot k : E \cdot m :: F \cdot k : F \cdot m. \tag{3.13}$$

Since every proportion can be written in this form, the implication (3.13) $\Rightarrow$ (3.12) is **Proposition 13, Alternation:**

$$A : B :: C : D \implies A : C :: B : D.$$

Since

$$1 : A :: B : B \times A, \tag{3.14}$$

by Alternation, $1 : B :: A : B \times A$, so by symmetry

$$1 : A :: B : A \times B. \tag{3.15}$$

Comparing (3.14) and (3.15) yields **Proposition 16, Commutativity:**

$$A \times B = B \times A.$$

3.6 Euclid's Lemma

Proposition 17 is like 9:

$$C : D :: C \times A : D \times A. \tag{3.16}$$

Hence **Proposition 18,** $C : D :: A \times C : A \times D$, or with different letters,

$$A : B :: C \times A : C \times B. \tag{3.17}$$

From (3.16), (3.17), and *transitivity,* we get **Proposition 19,**

$$A : B :: C : D \iff D \times A = C \times B,$$

the "Eudoxan" definition of proportion of numbers. **Proposition 20** is that, if A and B are the least X and Y such that

$$X : Y :: C : D,$$

then A measures C,[8] for by *Alternation*

$$A : C :: B : D,$$

[8] And B measures D the same number of times, as Euclid says.

and so A is the same part or parts of C that B is of D; but it cannot be parts, by minimality.[9] Here A and B are also coprime.[10] The converse is **Proposition 21.**

Immediately from the definitions, **Proposition 29:** every prime is coprime with its every non-multiple.

For **Proposition 30, Euclid's Lemma,** suppose a prime P measures $A \times B$, so that for some C,

$$P \times C = A \times B.$$

By 19 (the Eudoxan definition),

$$P : A :: B : C.$$

If P does not measure A, then

- P and A are coprime by 29,
- they are the least numbers having their ratio by 21,
- P measures B by 20.[11]

Euclid uses words alone to describe proportions. This could be because the Ancients were more used to *hearing* mathematics than seeing it. Modern commentators use fractions and the equals sign. I have tried to preserve the distinction between

[9] Mazur says, "Now I don't quite follow Euclid's proof of this pivotal proposition, and I worry that there may be a tinge of circularity in the brief argument given in his text" [15, p. 243]; then he cites Pengelley and Richman [19]. Mazur's own proof uses what he calls Propositions 5 and 6, though 7 and 8 are also needed, to conclude $A : B :: C - A \cdot k : D - B \cdot k$.

[10] This is Proposition 22, but we shall not need it.

[11] For Mazur [15, p. 242], "that if a prime divides a product of two numbers, it divides . . . one of them, is essentially Euclid's Proposition 24 of Book VII." Strictly, this is that the product of numbers prime to a number is also prime to it. Like that of 30, the proof relies on 20, which again for Mazur is problematic.

proportions and equations, while making Euclid's rigor *visible* in the way that we Moderns are used to.

Bibliography

[1] Apollonius of Perga. *Apollonii Pergaei Qvae Graece Exstant Cvm Commentariis Antiqvis*, volume I. Teubner, 1974. Edited with Latin interpretation by I. L. Heiberg. First published 1891.

[2] Apollonius of Perga. *Conics. Books I–III.* Green Lion Press, Santa Fe, NM, revised edition, 1998. Translated and with a note and an appendix by R. Catesby Taliaferro, with a preface by Dana Densmore and William H. Donahue, an introduction by Harvey Flaumenhaft, and diagrams by Donahue, edited by Densmore.

[3] Alexandre Borovik. Metamathematics of elementary mathematics. `www.matematikdunyasi.org/yazokulu/borovik_1b.pdf`, July 2008. Lecture at the Nesin Mathematics Village, Şirince, Selçuk, Izmir, Turkey.

[4] H. S. M. Coxeter. *Introduction to Geometry.* John Wiley, New York, second edition, 1969. First edition, 1961.

[5] John Dee. Mathematicall preface. In *The Elements of Geometrie of the most auncient Philosopher Euclid of Megara.* Iohn Daye, London, 1570. Facsimile in `pdf` format.

[6] René Descartes. *The Geometry of René Descartes.* Dover Publications, New York, 1954. Translated from the French and Latin by David Eugene Smith and Marcia L. Latham, with a facsimile of the first edition of 1637.

[7] René Descartes. *La Géométrie.* Jacques Gabay, Sceaux, France, 1991. Reprint of Hermann edition of 1886.

[8] Euclid. *The Thirteen Books of Euclid's* Elements. Dover Publications, New York, 1956. Translated from the text of Heiberg with introduction and commentary by Thomas L. Heath. In three volumes. Republication of the second edition of 1925. First edition 1908.

[9] Leonard Euler. Introductio in analysin infinitorum. www.17centurymaths.com/contents/introductiontoanalysisvol1.htm, accessed March 31, 2019. Translated and annotated by Ian Bruce.

[10] David Fowler. *The Mathematics of Plato's Academy: A new reconstruction.* Clarendon Press, Oxford, second edition, 1999.

[11] Albert W. Grootendorst, editor. *Jan de Witt's Elementa Curvarum Linearum, Liber Primus.* Springer, 2000. Text, translation, introduction, and commentary. First edition 1659.

[12] Hugh Hamilton. *De Sectionibus Conicis. Tractatus Geometricus. In quo, ex natura ipsius coni, sectionum affectioens [sic] facillime deducuntur. Methodo nova.* London, 1758. archive.org/details/desectionibusco00hamigoog/page/n7, accessed April 1, 2019.

[13] Hugh Hamilton. *A geometrical treatise of the conic sections, in which the properties of the sections are derived from the nature of the cone, in an easy manner, and by a new method.* J. Nourse, London, 1773. Translated from the Latin original into English. catalog.hathitrust.org/Record/000208065, accessed March 31, 2019.

[14] Thomas Heath. *A History of Greek Mathematics. Vol. I. From Thales to Euclid.* Dover Publications, New York, 1981. Corrected reprint of the 1921 original.

[15] Barry Mazur. How did Theaetetus prove his theorem? In P. Kalkavage and E. Salem, editors, *The Envisioned Life: Essays in honor of Eva Brann.* Paul Dry Books, 2007. www.math.harvard.edu/~mazur/preprints/Eva.pdf, accessed September 20, 2012.

[16] Pappus of Alexandria. *Pappus Alexandrini Collectionis Quae Supersunt*, volume II. Weidmann, Berlin, 1877. E libris manu scriptis edidit, Latina interpretatione et commentariis instruxit Fridericus Hultsch.

[17] Pappus of Alexandria. *Book 7 of the Collection. Part 1. Introduction, Text, and Translation. Part 2. Commentary, Index, and Figures.* Springer Science+Business Media, New York, 1986. Edited With Translation and Commentary by Alexander Jones.

[18] Pappus of Alexandria. Öklid'in Porizmalar'ı için Derleme'nin yedinci kitabı'nın 38 lemmasından ilk 19 lemması. `mat.msgsu.edu.tr/~dpierce/Dersler/Geometriler/`, March 2019. The First 19 of the 38 Lemmas for Euclid's Porisms in the Seventh Book of the Collection. Turkish translation by David Pierce.

[19] David Pengelley and Fred Richman. Did Euclid need the Euclidean algorithm to prove unique factorization? *Amer. Math. Monthly*, 113(3):196–205, 2006.

[20] David Pierce. Thales and the nine-point conic. *The De Morgan Gazette*, 8(4):27–78, 2016. `education.lms.ac.uk/2016/12/thales-and-the-nine-point-conic/`, accessed June 1, 2017.

[21] David Pierce. On commensurability and symmetry. *J. Humanist. Math.*, 7(2):90–148, 2017.

[22] Boris Rosenfeld. The up-to-date complete scientific English translation of the great treatise of Apollonius of Perga with the most comprehensive commentaries. `http://www.personal.psu.edu/sxk37/Apollonius.html` (hosted by Svetlana Katok), accessed April 8, 2019.

[23] Ivor Thomas, editor. *Selections Illustrating the History of Greek Mathematics. Vol. I. From Thales to Euclid.* Number 335 in Loeb Classical Library. Harvard University Press, Cambridge, Mass., 1951. With an English translation by the editor.

[24] John Wallis. *De Sectionibus Conicis, Nova Methodo Expositis, Tractatus.* Oxford, 1655. `archive.org/details/bub_gb_03M_AAAAcAAJ/page/n4` accessed April 1, 2019.