Apollonian Proof

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$L \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

In an **affine plane,** the
locus of $O + x \cdot \overrightarrow{OV} + y \cdot \overrightarrow{OL}$, where

$V^* \begin{pmatrix} a \\ b \end{pmatrix}$

$$x^2 \pm y^2 = 1,$$

is (for any V^*, namely $O + a \cdot \overrightarrow{OV} + b \cdot \overrightarrow{OL}$, on the locus)
fixed by the *affinity* fixing O and interchanging V and V^*.

$O \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$V \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

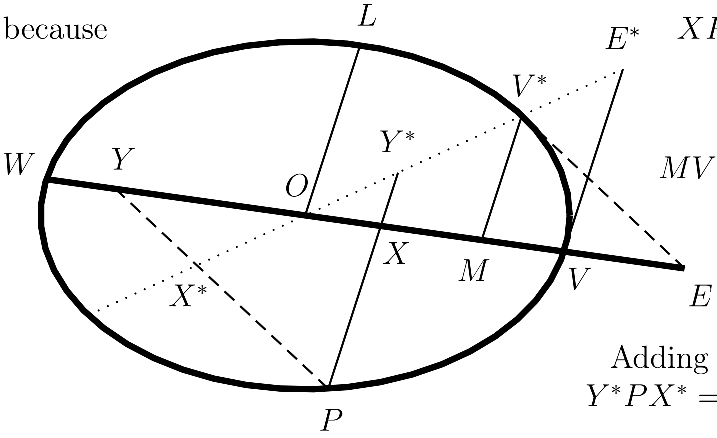
Modern proof. The affinity is $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & \pm b \\ b & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. □

1. Apollonius's proof uses **areas.** 2. So does an **axiomatization** of affine planes in which the *theorems* of **Pappus** and **Desargues** are just that.

Proof of Apollonius. The locus of P is given by

$$XPY = VXY^*E^*,$$

because

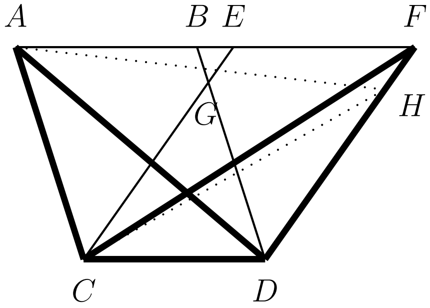


$$\begin{aligned} XPY &\propto XP^2 \\ &\propto XV \cdot XW \\ &\propto VXY^*E^*, \\ MV^*E &= VMV^*E^*. \end{aligned}$$

Adding XYX^*Y^* yields $Y^*PX^* = VYX^*E^*$, then

$$Y^*PX^* = EYX^*V^*.$$

Fundamental to the geometry of areas is **Euclid I.37, 39.**



1. Assuming $AF \parallel CD$, let
$$AC \parallel BD, \quad CE \parallel DF.$$

2. By translation,
$$ACE = BDF.$$

3. By polygon algebra,
$$ACDB = ECDF.$$

4. By bisection,
$$ACDB = 2ACD,$$
$$ECDF = 2FCD.$$

5. By halving,
$$ACD = FCD. \qquad (*)$$

6. If $AF \nparallel CD$, let $AH \parallel CD$.
$$ACD = HCD,$$
$$FCD = HCD + FCH,$$

so $(*)$ fails.

In order 3, 6, 1, 5, 4, 2, the steps are justified by:

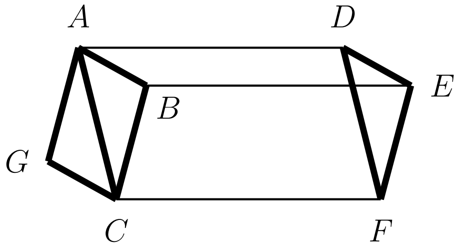
Axiom 1. The polygons compose an abelian group Π where, $*$ and $\dagger$ being strings of vertices,

$$A* \; = A*A = \; *A,$$
$$A*B+B\dagger A = A*B\dagger,$$
$$-ABC\cdots = \cdots CBA.$$

Axiom 2. $ABC = 0$ means that A, B, and C are collinear.

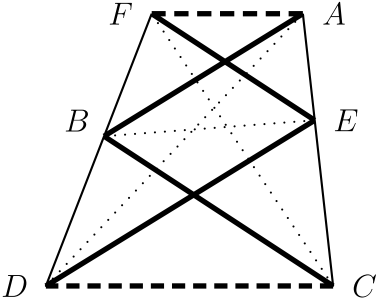
Axiom 3. Playfair's Axiom.

Axiom 4. All nonzero elements of Π have the same order, not 2.



Axiom 5, 6. If $ABCG$, $ABED$, and $BCFE$ are parallelograms, then

$$CGA = ABC = DEF.$$



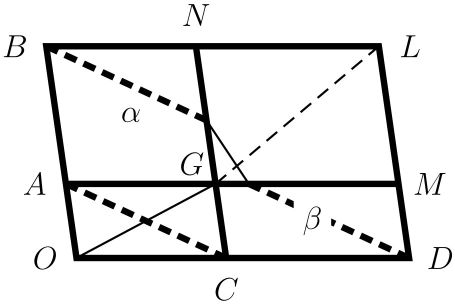
From Euclid I.*37, 39:*

- **Pappus's Hexagon Theorem,** by his proof: In hexagon $ABCDEF$, if

$$AB \parallel DE, \qquad BC \parallel EF,$$

then $FAD = FAEB = FAC$, so

$$CD \parallel FA.$$



- **Euclid I.43 plus.**

$$\begin{aligned} OGL = 0 &\iff \alpha = \beta \\ &\iff BD \parallel AC. \end{aligned}$$

- *Desargues's Theorem.*

Desargues's Theorem. If

$$AB \parallel DE \ \ \& \ \ AC \parallel DF,$$

then $BC \parallel EF$, so $ABC \sim DEF$.

Proof.

- True when $AB \parallel OC$, by I.43+.
- Enough now that, since

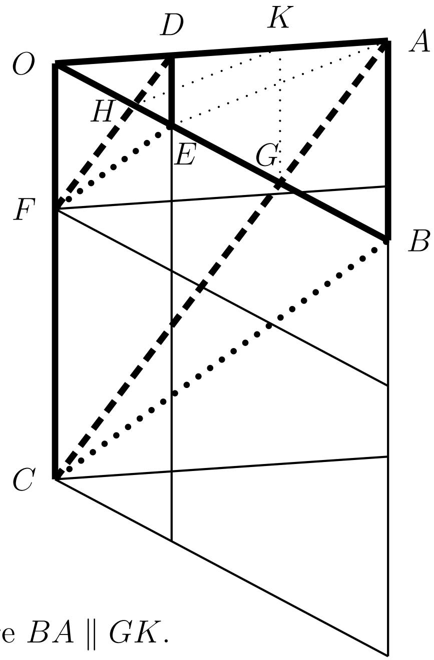
$$BAG \sim EDH,$$

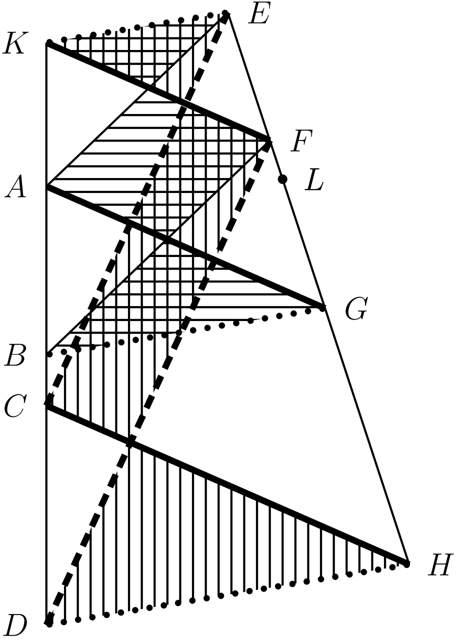
for all X (not shown) on OA,

$$BXG \sim EYH$$

for some Y on OA.

Note $BAE \sim GKH$ by Pappus, where $BA \parallel GK$.





Lemma. Given

$$AEC \sim BFD,$$

we noted

$$AEB \sim CLD$$

for some L on EF. Now let

$$KF \parallel AG \parallel CH.$$

By Pappus twice,

$$BG \parallel KE \parallel DH,$$

whence

$$AGB \sim CHD.$$