Conic sections with and without algebra

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Preface

This document concerns a contributed talk, delivered Wednesday, May 15, 2019, 16:10–16:30, the first day of Antalya Algebra Days XXI, Nesin Mathematics Village, Şirince, Selçuk, Izmir, Turkey.

Abstract is as submitted, then edited before publication in the conference booklet (an error remained, as noted).

Notes on abstract spell out the algebra, not covered in the talk itself (except with a reference to the abstract).

Notes of talk are an approximation, from written notes and memory, of what I actually said in the talk, mostly out loud only. On the chalkboards of the Nişanyan Library, I drew the diagrams, but mostly wrote only the symbolic equations and proportions. Since Tuna Altınel had been imprisoned on the previous Saturday, and Ayşe Berkman had discussed the case when opening the meeting, I alluded to the case (and to the blocking of *Wikipedia* in Turkey).

Notes for talk were prepared and printed before travelling from Istanbul to Şirince (via Ankara, that previous Saturday). I knew that I would not have time to write out all of the computations. Only later did I streamline the written talk with the idea denoted by $\propto$.

Additional remarks were originally considered for inclusion in the talk, only there would not be enough time.

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1 Abstract

This is about how history may reveal a forgotten vision of mathematics. Mathematics is universal, but ways of understanding it are not. As has been argued in general terms [3], the Ancients did not secretly use algebra. This should not be understood pejoratively. People without keyboards or even ready supplies of paper may develop ways of understanding that we never feel the need for.

The square on the ordinate varies as the abscissa, in a parabola; in the ellipse and hyperbola, as the rectangle bounded by the two abscissas [1]. In Cartesian terms [2], the proportions are expressed by algebraic equations:

$$y^2 = \ell x, \qquad\qquad y^2 = \frac{\ell x}{2d}(2d - x).$$

With laborious algebraic manipulations, John Wallis [6] reestablished that the curves defined above, if they contain (a, b), and if $c = a + d$,[*] are fixed under the respective affine transformations

$$\left\{\begin{matrix} x' = x - \dfrac{2by}{\ell} + a \\ y' = -y + b \end{matrix}\right\}, \qquad \left\{\begin{matrix} x' = \dfrac{cx}{d} - \dfrac{2by}{\ell} + a \\ y' = -\dfrac{bx}{d} - \dfrac{cy}{d} + b \end{matrix}\right\}.$$

[*] Should be $c = d - a$.

It has been asserted [5] that Apollonius somehow understood this; but all modern proofs that I have found, as by de Witt, Euler, and Hugh Hamilton, lack the clarity of Apollonius's non-algebraic proof, which uses areas in a way that does not reduce to manipulations of lengths in the Cartesian fashion. The key is that the equations for the conics can be written as equations of a parallelogram with a triangle and a trapezoid respectively, all lying on one plane (which need only be an affine plane).

2 Notes on abstract

We confirm some features of the given affine transformations

$$x' = x - \frac{2by}{\ell} + a, \qquad y' = -y + b, \tag{2.1}$$

$$x' = \frac{cx}{d} - \frac{2by}{\ell} + a, \qquad y' = -\frac{bx}{d} - \frac{cy}{d} + b, \tag{2.2}$$

of the parabola and central conic respectively.

2.1 Characterization

That (2.1) interchanges $(0, 0)$ and (a, b) is easy to see. To see that (2.2) does too, we need that, since the central conic is given by

$$y^2 = \frac{\ell x}{2d}(2d - x) \tag{2.3}$$

and contains (a, b), in particular

$$\frac{2b^2}{\ell} = \frac{a}{d}(2d - a) = \frac{a}{d}(c + d), \tag{2.4}$$

where

$$c = d - a \tag{2.5}$$

(this corrects the abstract). Moreover, (2.2) fixes $(d, 0)$ in the central conic, but (2.1) interchanges any $(x, 0)$ and $(a + x, b)$.

2.2 Derivation

Taking $(0,0)$ to (a,b), (2.2) must have the form

$$x' = Ax + By + a, \qquad y' = Cx + Dy + b.$$

Since the transformation fixes $(d,0)$, from (2.5) we have

$$A = \frac{c}{d}, \qquad C = -\frac{b}{d}. \tag{2.6}$$

Since (2.2) will take (a,b) to $(0,0)$, from (2.4) we have

$$0 = \frac{ca}{bd} + B + \frac{a}{b} = B + \frac{a}{bd}(c+d) = B + \frac{2b}{\ell}, \tag{2.7}$$

$$0 = -\frac{a}{d} + D + 1 = D + \frac{c}{d}, \tag{2.8}$$

which yields (2.2). Alternatively, we obtain (2.6) and (2.8), then derive B from $AD - BC = -1$ instead of (2.7).

2.3 Confirmation

Under the change of coordinates given by

$$x = d - t,$$

the transformation (2.2) becomes

$$t' = \frac{ct}{d} + \frac{2by}{\ell}, \qquad y' = \frac{bt}{d} - \frac{cy}{d}, \tag{2.9}$$

and (2.3) becomes

$$y^2 = \frac{\ell}{2d}(d^2 - t^2),$$

$$\frac{t^2}{d^2} + \frac{2y^2}{\ell d} = 1 \tag{2.10}$$

(d is positive in the ellipse, negative in the hyperbola). One easily checks that (t', y') as in (2.9) satisfies (2.10), since (c, b) does in particular, so that

$$\frac{c^2}{d^2} + \frac{2b^2}{\ell d} = 1.$$

3 Notes of talk

This talk is both mathematics and history concerning Apollonius of Perga. Such work has practical application as an example of the principle that being good at one thing (such as mathematics and history) does not necessarily make you good at another thing (such as history or mathematics):

- Being good at getting elected president of a country does not make you good at choosing the rectors of your country's universities.
- Being good at locking people up does not make you good at knowing who should be locked up.

The flier for the Maryam Mirzakhani poster exhibition* says, "Mathematics is universal." I say the same in my abstract, with the qualification that *understanding* is not universal. For example, in the first talk this morning, by Rostislav Grigorchuk, we heard of four papers on the arXiv, two asserting that a certain group was amenable, the other two that it was not. The authors on one side cannot go beat up those on other side with sticks; they cannot denounce them to the police to have them locked up. All authors have to consider the possibility that they are wrong.† In this sense, the world needs more mathematics.

*Hung in the arcade outside the Library.

†Apparently they *were* all wrong, and the status of the group in question, one of the Thompson groups, is yet undecided.

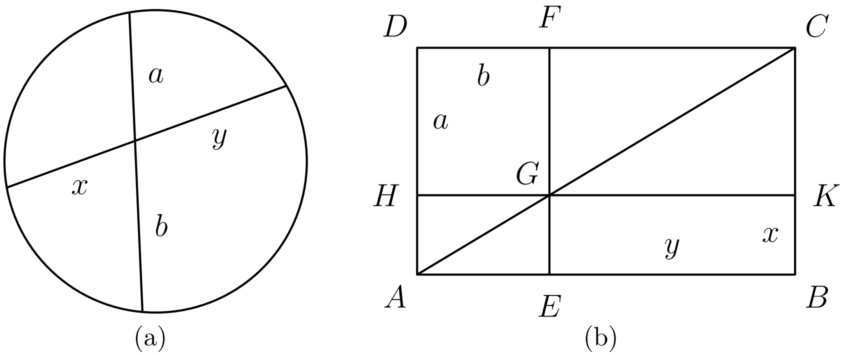


Figure 3.1: Intersecting chords of a circle

By a theorem in Euclid, when two chords of a circle cut one another, the rectangles formed by the parts are equal to one another. Algebraically, from Fig. 3.1a,

$$ab = xy. \tag{3.1}$$

What does this mean? For Euclid, in Fig. 3.1b,[‡]

$$ab = xy \iff AGC \text{ is straight.} \tag{3.2}$$

If we take this as a definition, why is the equality so defined transitive? For Euclid, the equivalence is a theorem: if AGC is indeed straight, then, being congruent, the two large triangles ABC and CDA are equal, as are the small ones (AEG and GHA, and GKC and CFG); equals being subtracted from equals, the remaining rectangles are equal.

In short, for Euclid, transitivity of equality of areas is axiomatic, and (3.2) is a theorem.

[‡]In the talk, I did not label the points, except those along the diagonal.

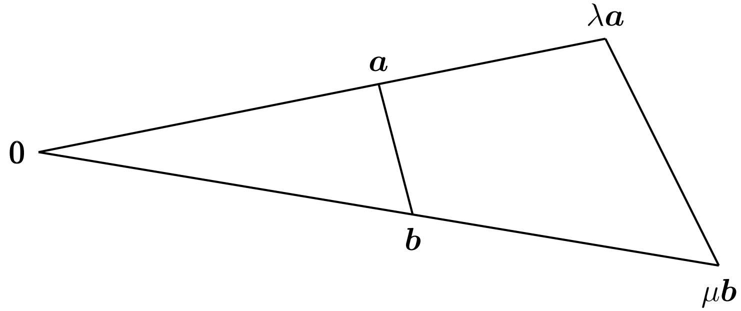


Figure 3.2: Intercept Theorem

Alternatively, one may use axioms like Hilbert's, concerning only equality of lengths; then one has to prove transitivity of equality of areas as defined by (3.2).

One may rewrite (3.1) as

$$a : x :: y : b,$$

"the ratio of a to x is the same as the ratio of y to b." Transitivity of sameness of ratio is a consequence of, or *is,* what *Wikipedia* calls the Intercept Theorem.[§] In one expression of this theorem, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are independent vectors as in Fig. 3.2, and λ and μ are nonzero scalars, then

$$\boldsymbol{a} - \boldsymbol{b} \parallel \lambda\boldsymbol{a} - \mu\boldsymbol{b} \iff \lambda = \mu.$$

[§]In Fig. 3.1b, assuming only $HG \parallel DC$, we may define a quaternary relation of proportionality by $AH{:}HD :: AG{:}GC$. If, like the Ancients, we call this a *sameness* of ratios, this sameness should be transitive (like any relation called "sameness"). Thus, since also $GE \parallel CB$ (but without using that $ABCD$ is a parallelogram), it should follow that $HE \parallel DB$. This is (a case of) Desargues's Theorem, which however follows trivially from the Intercept Theorem, if ratios are independently defined (as in the Eudoxan definition given by Euclid, or in a vector space), so that sameness of ratio is automatically transitive.

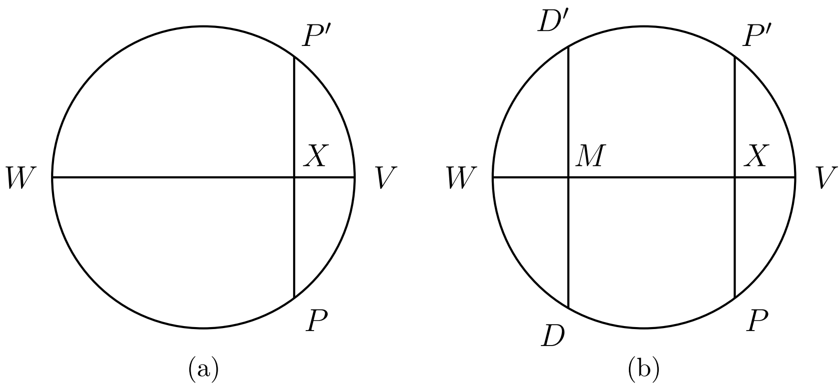


Figure 3.3: Circle with diameter and orthogonal chords

The *Wikipedia* article uses a *normed* vector space, but the norm is needless, unless one wants a ratio of $\boldsymbol{a}$ to $\boldsymbol{b}$. I might edit the article, but the Turkish state blocks our access. We have ways of *reading Wikipedia,* but they do not allow editing; thus Turkey deprives the world of our knowledge and understanding.

Returning to the circle, if, as in Fig. 3.3a, a circle has diameter VW, and chord PP' is orthogonal to this, we have

$$XP^2 = VX \cdot XW.$$

This too assumes a norm, even to define the circle itself. However, without the norm, in an affine plane, we can express

$$XP^2 \propto VX \cdot XW,$$

which means, if we draw chord DD' parallel to PP' as in Fig. 3.3b,

$$(\overrightarrow{XP} : \overrightarrow{MD})^2 :: (\overrightarrow{VX} : \overrightarrow{VM})(\overrightarrow{XW} : \overrightarrow{MW}).$$

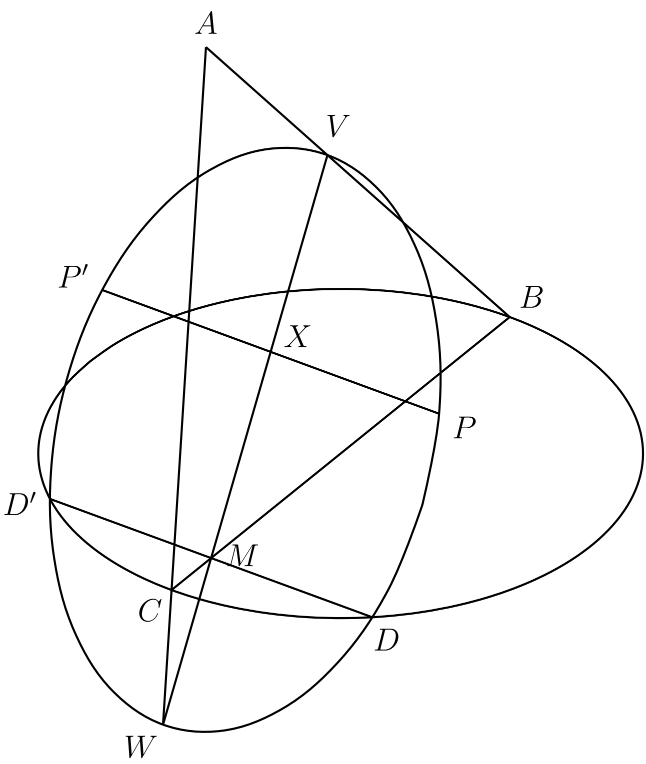


Figure 3.4: Cone, sectioned

Apollonius shows that the same proportion holds for a section of a cone as in Fig. 3.4, where BC is a diameter of the circular base, DD' is orthogonal to this, and the apex A is any point not in the plane of the base. Considering the section in isolation, as in Fig. 3.5, if we draw DE at an arbitrary angle, and let PY be parallel, then

$$XP^2 \propto XPY,$$

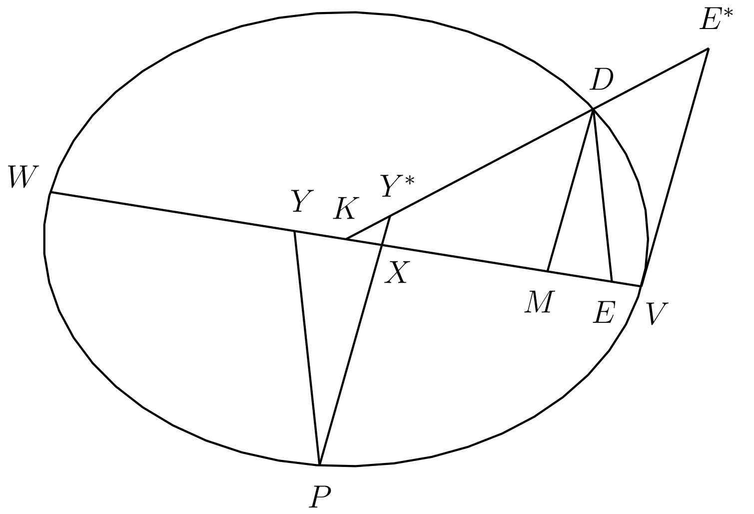


Figure 3.5: Conic section, isolated

while

$$XW = XK + KW = XK + VK \propto XY^* + VE^*,$$

so that

$$VX \cdot XW \propto VXY^*E^*.$$

Thus

$$XPY \propto VXY^*E^*.$$

We turn this into an equation by making

$$MDE = VMDE^*,$$

as in Fig. 3.6, by ensuring

$$\overrightarrow{KM} : \overrightarrow{KV} :: \overrightarrow{KV} : \overrightarrow{KE}.$$

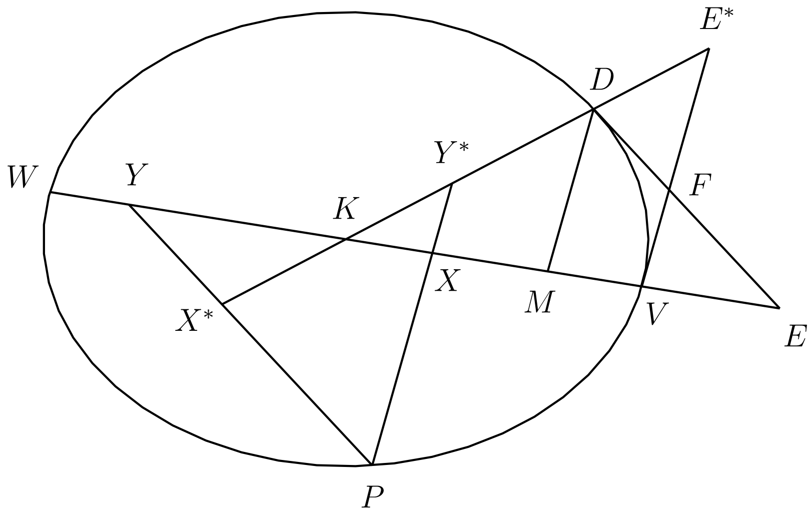


Figure 3.6: Conic section, final analysis

Now

$$XPY = VXY^*E^*.$$

We obtain from this

$$Y^*PX^* = YX^*E^*V$$

by adding the quadrilateral XYX^*Y^*. Thus we show that the conic section is preserved by what, in modern terms, is the affine transformation fixing K and interchanging V and D. The transformation is expressed algebraically in the abstract. One modern commentator says that such a transformation is what Apollonius is using; but he is not using a Cartesian algebra of lengths, he is using areas.

4 Notes for talk

Suppose, as in Fig. 4.1a, diameter BC of a circle bisects parallel chords DD' and JJ' at M and N respectively. From

$$NJ^2 = BN \cdot NC, \qquad MD^2 = BM \cdot MC$$

follows

$$(\overrightarrow{NJ} : \overrightarrow{MD})^2 :: (\overrightarrow{BN} : \overrightarrow{BM})(\overrightarrow{NC} : \overrightarrow{MC}), \tag{4.1}$$

an **affine** proportion (all ratios are of *parallel* directed segments). Now let the circle be the base of a cone with apex A, as in Fig. 4.1b. Let a plane containing MD, but not A, cut AB at V and AC at W (or perhaps not at all). We obtain, as in Fig. 4.2, a **conic section,** with arbitrary point P. Let

$$PP' \parallel DD'.$$

Then

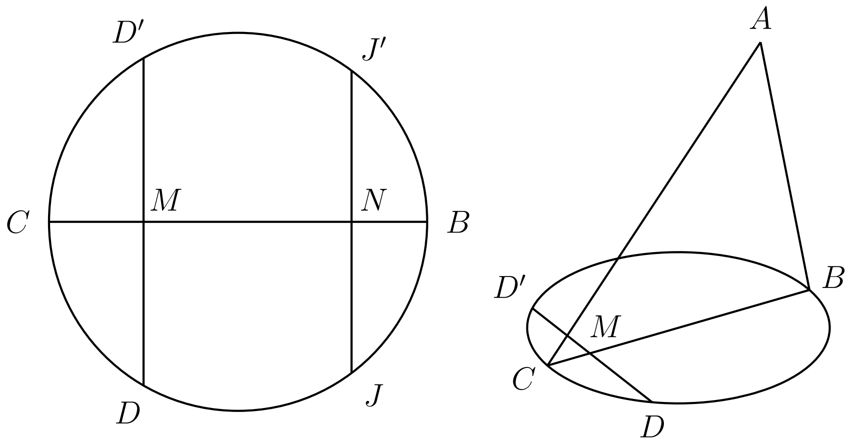
$$PX = XP',$$

so VM is a **diameter** (διάμετρος) of the section. Let also

$$QR \parallel BC.$$

From (4.1) and Thales's Theorem, without using W, we compute

$$(\overrightarrow{XP} : \overrightarrow{MD})^2 :: (\overrightarrow{VX} : \overrightarrow{VM})(\overrightarrow{RX} : \overrightarrow{CM}), \tag{4.2}$$



(a) Circle (b) Cone

Figure 4.1: Circle as base of cone

since, in detail,

$$\begin{aligned}(\overrightarrow{XP} : \overrightarrow{MD})^2 &:: (\overrightarrow{XP} : \overrightarrow{NJ})^2(\overrightarrow{NJ} : \overrightarrow{MD})^2\\ &:: (\overrightarrow{QX} : \overrightarrow{BN})(\overrightarrow{XR} : \overrightarrow{NC})(\overrightarrow{BN} : \overrightarrow{BM})(\overrightarrow{NC} : \overrightarrow{MC})\\ &:: (\overrightarrow{QX} : \overrightarrow{BM})(\overrightarrow{XR} : \overrightarrow{MC})\\ &:: (\overrightarrow{VX} : \overrightarrow{VM})(\overrightarrow{RX} : \overrightarrow{CM}).\end{aligned}$$

When W exists, we have now

$$(\overrightarrow{XP} : \overrightarrow{MD})^2 :: (\overrightarrow{VX} : \overrightarrow{VM})(\overrightarrow{XW} : \overrightarrow{MW}). \tag{4.3}$$

Let the midpoint of VW be K. We show every line, as DK, through K is a new diameter, with respect to which a proportion as in (4.3) is satisfied. In the plane of the section, as in Fig. 4.3, we let point E satisfy

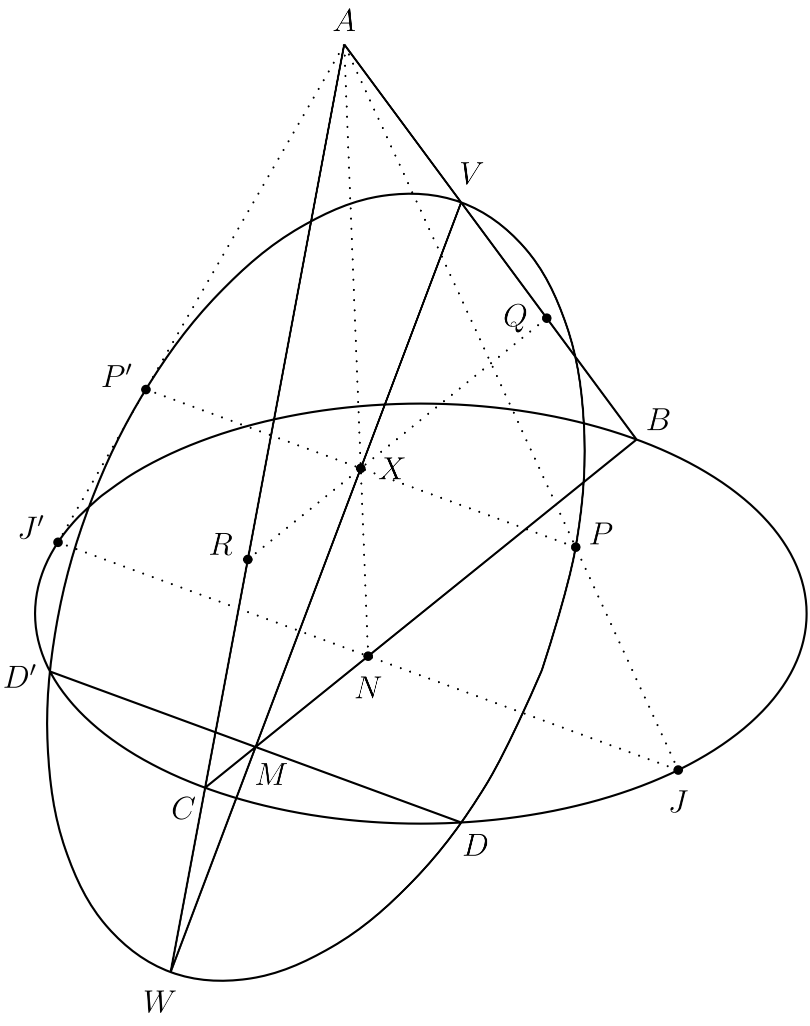


Figure 4.2: Ellipse in cone

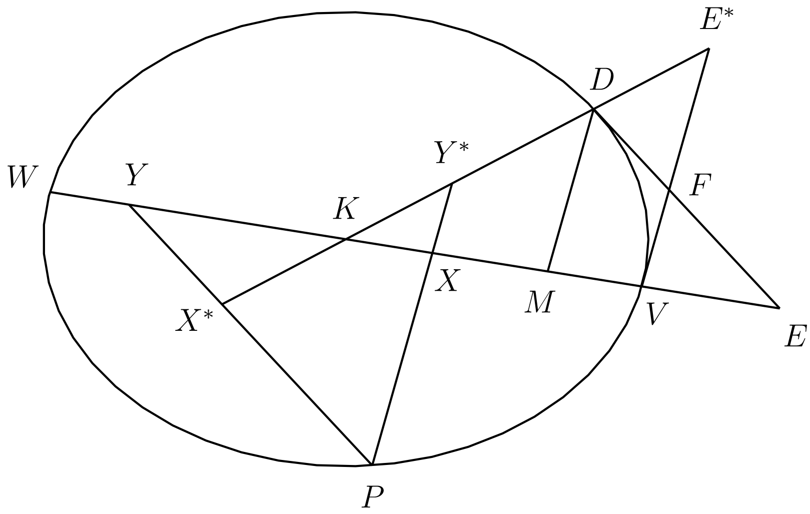


Figure 4.3: Ellipse, isolated

$$\overrightarrow{KM} : \overrightarrow{KV} :: \overrightarrow{KV} : \overrightarrow{KE}, \tag{4.4}$$

and then Y, and E^* satisfy

$$ED \parallel PY, \qquad\qquad MD \parallel VE^*.$$

By similarity of the triangles,

$$(\overrightarrow{XP} : \overrightarrow{MD})^2 :: XPY : MDE. \tag{4.5}$$

Also

$$(\overrightarrow{VX} : \overrightarrow{VM})(\overrightarrow{XW} : \overrightarrow{MW}) :: XY^*E^*V : MDE^*V, \tag{4.6}$$

since

$$\overrightarrow{XW} : \overrightarrow{MW} :: (\overrightarrow{XK} + \overrightarrow{KW}) : (\overrightarrow{MK} + \overrightarrow{KW})$$

$$:: (\overrightarrow{XK} + \overrightarrow{VK}) : (\overrightarrow{MK} + \overrightarrow{VK})$$
$$:: (\overrightarrow{XY^*} + \overrightarrow{VE^*}) : (\overrightarrow{MY^*} + \overrightarrow{VE^*}).$$

Since also, by (4.4),

$$VFE = E^*FD, \tag{4.7}$$

so that

$$MDE = MDE^*V,$$

this with (4.3), (4.5), and (4.6) yield

$$XPY = XY^*E^*V, \tag{4.8}$$

an alternative formulation of (4.3). Adding YX^*Y^*X to either side of (4.8), we obtain

$$Y^*PX^* = YX^*E^*V,$$

since, in detail,

$$XPY + YX^*Y^*X = XPYX^*Y^* = PX^*Y^*,$$
$$YX^*Y^*X + XY^*E^*V = YX^*Y^*E^*V = YX^*E^*V.$$

From (4.7) again,

$$VYX^*E^*V = YX^*DE,$$

and therefore

$$Y^*PX = YX^*DE,$$

which is (4.8) with respect to the new diameter.

5 Additional remarks

In English, as in Greek,

1) a **parable** ($\pi\alpha\rho\alpha\beta o\lambda\acute{\eta}$) illustrates a moral;
2) **hyperbole** (ὑπερβολή) exaggerates;
3) **ellipsis** (ἔλλειψις) leaves out.

By one account, that of Hilbert and Cohn-Vossen [4], these terms are assigned respectively to certain curves for the following reason. As is proved by Pappus and was apparently known to Euclid, each of the curves is the locus of P, where

$$\frac{|PF|}{|Pd|} = e,$$

where

- F is the **focus,**
- d is the **directrix,** and
- e is the **eccentricity,** which respectively
  (1) equals,
  (2) exceeds, and
  (3) falls short of unity,

as in Fig. 5.1. In the Cartesian plane, the curve where

- F is $(0,0)$ and
- d is $y = -1$

has polar equation

$$\frac{r}{1 + r\sin\theta} = e$$

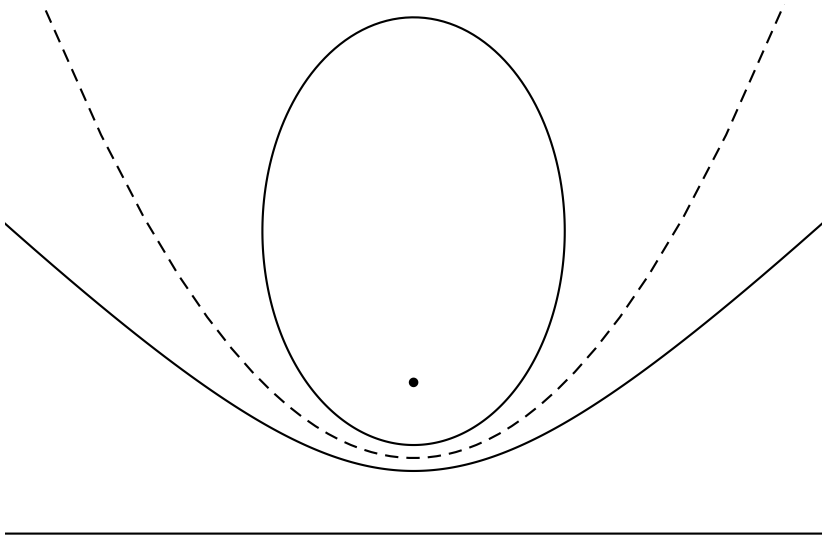


Figure 5.1: Conics with eccentricities $\sqrt{2}$, 1, and $1/\sqrt{2}$

or

$$r = \frac{e}{1 - e\sin\theta}.$$

Hibert and Cohn-Vossen show how any curve so defined is a section of a *right* circular cone.

Apollonius works with an arbitrary circular cone, possibly oblique. One will not get the focus-directrix property this way. Apollonius introduces the terms (1) *parabola,* (2) *hyperbola,* and (3) *ellipse* for another reason, namely that, for each point of the curve, the square on the *ordinate* respectively

(1) equals,
(2) exceeds, and
(3) falls short of

the rectangle bounded by the *abscissa* and the *latus rectum.* In particular, in the **parabola** of Fig. 5.2, we have

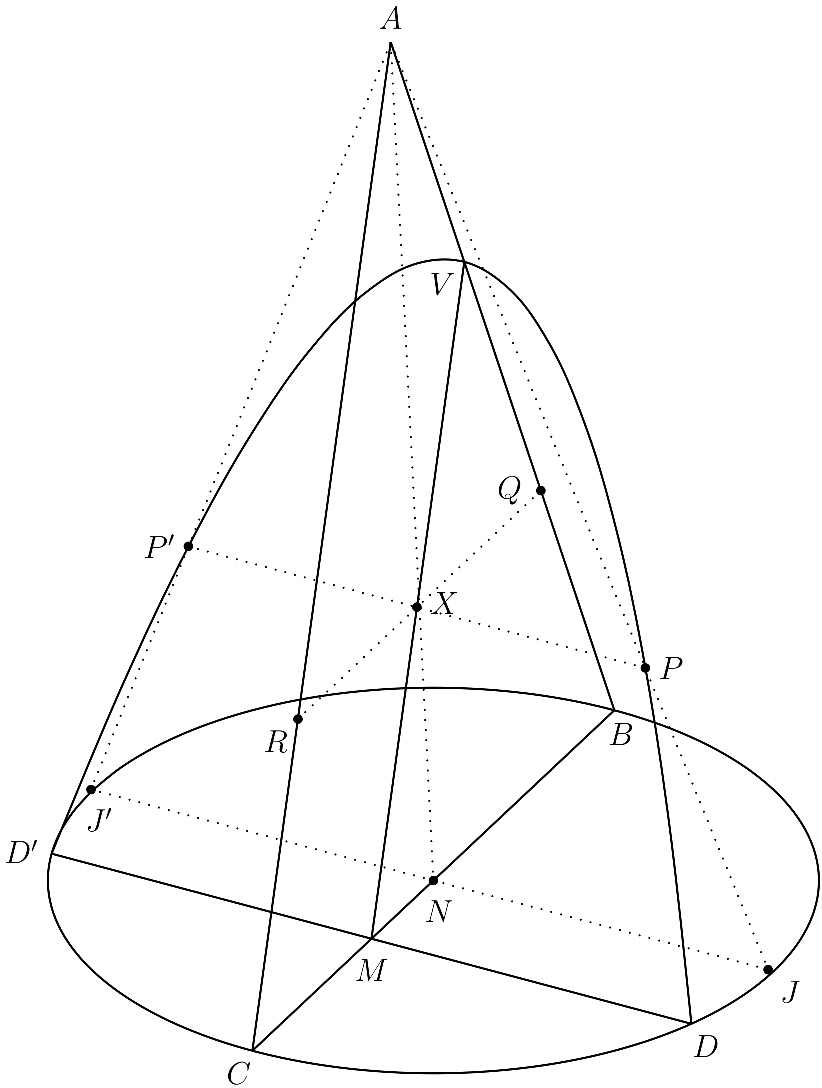


Figure 5.2: Parabola

$$MV \parallel AC,$$

and then (4.2) becomes

$$(\overrightarrow{XP} : \overrightarrow{MD})^2 :: \overrightarrow{VX} : \overrightarrow{VM},$$

meaning the square of the ratio of the ordinates is the ratio of the abscissas. In the Euclidean plane, if

$$\overrightarrow{XP} = y, \qquad\qquad \overrightarrow{VX} = x$$

in Cartesian fashion, for some ℓ,

$$y^2 = \ell x.$$

Here ℓ is the length of the ***latus rectum*** or upright side. Applied to this, the square on the ordinate XP becomes a rectangle whose base is the abscissa VX.

An **hyperbola** is shown in Fig. 5.3; computations are as for the ellipse of Fig. 4.2, so that the square of the ratio of ordinates is the product of the ratios of the abscissas. Again in the Euclidean plane, if now

$$\overrightarrow{VW} = 2d,$$

then for some ℓ,

$$y^2 = \frac{\ell x}{2d}(2d - x) = \ell x - \frac{\ell x^2}{2d},$$

meaning the square on the ordinate

- exceeds (when $d < 0$) and
- falls short of (otherwise)

the rectangle bounded by abscissa and *latus rectum* by the rectangle on the abscissa similar to that bounded by *latus transversum* and *latus rectum.*

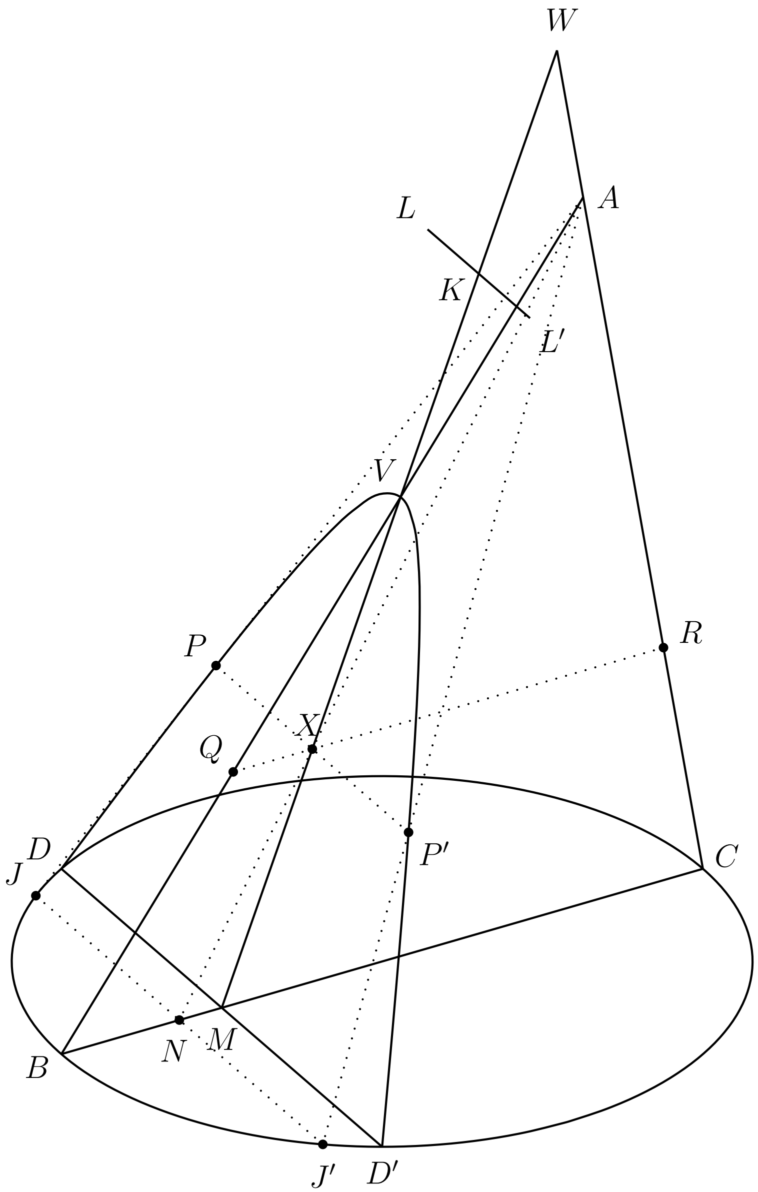


Figure 5.3: Hyperbola

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