Euclid Mathematically and Historically

Notes for a colloquium

David Pierce

Friday, March 9, 2018
Corrected Sunday, June 10, 2018

Mathematics Department
Mimar Sinan Fine Arts University
Istanbul
mat.msgsu.edu.tr/~dpierce/
david.pierce@msgsu.edu.tr

Preface

These notes were prepared for a general colloquium in the mathematics department of Bilkent University, Ankara, on Wednesday, March 7, 2018, 3:40–4:30 PM. I had been invited by Sinan Sertöz, who was engaged in translating all of Euclid's *Elements* into Turkish [12].

As expected, there was not time during the colloquium for all details. Omitted entirely were §§ 1.5, 2.1, and 3.4.

I started the talk by noting the assertion in the abstract that Euclid was in some ways more rigorous than we. An example was his proof of commutativity of multiplication of numbers. Few students today may see this proved, from the Peano Axioms perhaps.[1]

Concerning equality in Euclid, my example was not the parallelograms in the same parallels of Propositions I.35 and 36, as in Figures 5 and 7, but the "complementary" parallelograms in the one large parallelogram of Proposition I.43, as in Figure 6. I added this example to the present text after the talk.

During the talk, I used the example for an excuse to write out Proposition XII.2 of the *Elements* in Greek. This proposition uses the article in three different forms, and Proposition I.43 illustrates the notational usefulness of the gendered article, as

[1] If they do see a proof, it might be as in Landau [28]. I did not go into how Peano himself [32] misunderstood the axioms, in the way that Landau discusses in his "Preface for the Teacher," and I discuss in "Induction and Recursion" [35]. Proof by induction alone does not justify definition by recursion. Dedekind, writing earlier [7], understood this.

discussed in my article "Abscissas and Ordinates" [37].

I ended the talk with the proof of commutativity in §3.3.

I brought some props to the talk:

1. Three chopsticks, marked off into four parts each; and four chopsticks, cut to three-quarter length and marked off into thirds. The sticks are supposed to show that it is not absolutely trivial that three fours are equal to four threes.
2. A triangle, providing both a straightedge and a right angle as allowed by Euclid's postulates.
3. A cord, as allowed by Archimedes's postulate.
4. Bottle caps, for distinguishing measuring from dividing; I had used them to illustrate a blog article about commutativity of multiplication [38].

In the formal question period after the talk, Laurence Barker asked whether Euclid was rigorous in the sense of rejecting arguments that he knew, but considered invalid. On the spot, I could only marvel at how Euclid found it worthwhile to give what for us would be an epsilon–delta proof that circles vary as the squares on their diameters. The Egyptians defined the area of a quadrilateral as the product of the averages of the lengths of opposite sides; Euclid showed how any straight-edged figure was exactly equal to a rectangle on a given base.

I review how I became interested in all of this. In high school I wanted to read Euclid, rather than our textbook by Weeks and Adkins [48]. I read some of the *Elements* [17] on my own, and more of them as a freshman at St John's College [36]. As a sophomore there I read Descartes [10], and I have found his "digitization" of geometry to be directly useful for at least one piece of modern research [34]. In Istanbul, I helped to create, and I often teach, a course in which freshmen read and present Book I of the *Elements* [20]. I began studying Euclid's number-theory to see if it could be incorporated into a later course for our students. I concluded that it could not. Euclid's way of thinking was too different from ours. I discovered that even modern mathematicians, as far as I could tell, were misreading Euclid. My study led to much of what is in the present notes.

When the abstract that I submitted was included in announcements of the colloquium, the definite article "The" had been inexplicably inserted in front of the opening "Mathematics," though the initial capital letter of this had been retained.

A briefer abstract might be that, when we translate a proportion of numbers into an equation of fractions, we may overlook the subtlety with which Euclid works. He even proves, rigorously, something so "obvious" as the commutativity of multiplication of numbers. Few of us today may ever do this, in our Cartesian drive to get new results.

Contents

List of Figures

Abstract

Mathematics can save the world, not through a theorem or application, but as an example of an endeavor where
(1) differences can and must be resolved peacefully;
(2) dissent is encouraged; and
(3) wealth has to be shared to be recognized, and then only gains in value.

Nonetheless, in trying to share in the wealth of mathematical knowledge, we have to think historically as well as mathematically, particularly when the wealth comes down to us from Ancients such as Euclid.

The thirteen books of Euclid's *Elements* have given us a paradigm of mathematical exposition, with axioms and postulates at the beginning, definitions as needed, and propositions stated and proved. We may have improved on the model; but sometimes we misunderstand it. In Greek, the root meaning of *γεωμετρία*, *geometria*, is surveying. Herodotus of Halicarnassus (today's Bodrum) said the Greeks had learned *geometria* from the Egyptians. However, the mathematics that Euclid went on to work out, presumably in Alexandria, did not follow naturally from a need to measure land lost to the annual flooding of the Nile. Neither does Euclid's meaning follow naturally from a superficial reading of his words today. *Anistoresy,* a-historicity, as for example concerning equality and proportion, can lead to misunderstanding and even misdiagnosis of logical error in Euclid. In some ways Euclid's mathematics is more rigorous than ours.

1 Circles

Here are three theorems about circles, or one theorem expressed three ways.

1. In school today we learn a formula for the area of a circle:

$$A = \pi r^2. \tag{1}$$

   The area A of the circle is said to be the number π ("pi") times the "square" of the radius r.[2]

2. Archimedes "squares" the circle as in Figure 1, showing what we might write as

$$A = \frac{1}{2} r\, C,$$

   but he uses words:

[2]I was taught the rule in precisely the form (1) from Weeks and Adkins [48] in 1981. Even in Turkish, where the word for radius is yarıçapı "half diameter," the area of a circle is given as πr^2, at least in the two sources that I consulted [9, 26].

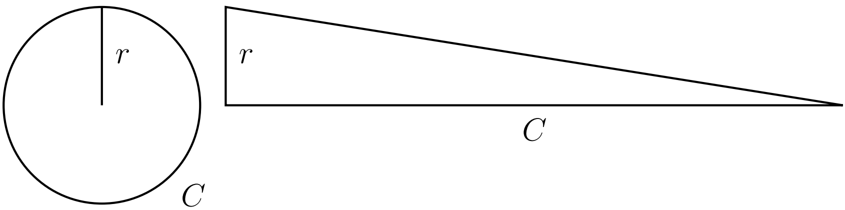

Figure 1: Triangle equal to circle

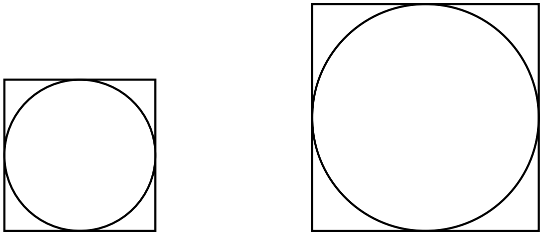

Figure 2: Circles and the squares on their diameters

> Every circle is equal to a right triangle, where the radius is equal to one of the legs, and the circumference to the base. // *Πᾶς κύκλος ἴσος ἐστὶ τριγώνῳ ὀρθογωνίῳ, οὗ ἡ μὲν ἐκ τοῦ κέντρου ἴση μιᾷ τῶν περὶ τὴν ὀρθήν, ἡ δὲ περίμετρος τῇ βάσει.*[3]

3. The rule of Proposition 2 of Book XII of Euclid's *Elements* is,

Οἱ	κύκλοι	πρὸς	ἀλλήλους	εἰσὶν
The	circles	to	one another	are

ὡς	τὰ	ἀπὸ	τῶν	διαμέτρων	τετράγωνα.
as	the	on	the	diameters	squares.

See Figure 2. More smoothly in English, this is,

[3]The Greek is from Heiberg's text [1, p. 258]; the English is my translation. "The radius" would be more literally "the from-the-center." Also "where" would be more literally "of which," but then, in English, this would seem to apply to "the radius" rather than "the legs." For Archimedes, the legs are "the [sides] about the right [angle]," but I do not know why, having specified one of these, he says then "base" rather than "other leg." Heath's translation is looser, but perhaps clearer to the modern mind: "The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle" [2, p. 91]. The introduction of *area* may be misleading.

Circles are to one another
as the squares on the diameters.

> Other ways to say this include, "Circles have the ratio of the squares on their diameters."[4]

To say these three theorems are one theorem is itself a theorem, perhaps more sophisticated than is generally understood.

1. The modern theorem is about so-called real numbers.
2. Archimedes's theorem is about geometrical figures, except that there is no properly geometrical way to construct the triangle.
3. Euclid's theorem is purely geometrical.[5]

I suggest that Euclid is more rigorous than we are, in part because he does not make all of the unexamined assumptions that lie behind our blithe assertion that the three theorems are the same.

1.1 Equations

What is the *meaning* of (1), namely $A = \pi r^2$? The formula allows us to perform a computation. Given a radius, in decimal notation, we can punch its digits into a pocket calculator, punch a few more keys, and get a result for the area.

The formula (1) assumes that geometrical objects can indeed be assigned numbers. This is like the assumption of some art-museum visitors that every painting on display has a numerical value in dollars.

[4] The Greek again is from Heiberg [15], and the smooth, but now still literal, translation is Heath's [18].

[5] By Euclid here, I mean simply the author or authors of the collection known as Euclid's *Elements.* I make no assertion about what is original with Euclid.

The $\boxed{\pi}$ key on the calculator supplies an exact numerical value, but not that of π. This *has* an exact value, but it is irrational, even transcendental. In principle, we can compute it as finely as we want, as by using the Leibniz formula,

$$\begin{aligned}\frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \\ &= \frac{2}{1 \cdot 3} + \frac{2}{5 \cdot 7} + \frac{2}{9 \cdot 11} + \cdots,\end{aligned}$$

which we can derive from

$$\frac{\pi}{4} = \arctan(1) = \int_0^1 \frac{\mathrm{d}x}{1+x^2} = \lim_{t \to 1^-} \int_0^t \sum_{k=0}^{\infty} (-1)^k x^{2k} \mathrm{d}x.$$

As for the formula (1) itself, we may declare that a circle of radius r is defined by the formula

$$x^2 + y^2 = r^2, \tag{2}$$

so that A is by definition

$$2 \int_{-r}^{r} \sqrt{r^2 - x^2} \; \mathrm{d}\, x.$$

We compute this by letting

$$x = r \sin u, \qquad \mathrm{d}\, x = r \cos u \; \mathrm{d}\, u,$$

so that

$$A = 2r^2 \int_{-\pi/2}^{\pi/2} \cos^2 u \; \mathrm{d}\, u = r^2 \int_{-\pi/2}^{\pi/2} (1 + \cos 2u) \; \mathrm{d}\, u = r^2 \, \pi.$$

Perhaps this assures us more that we have learned calculus properly than that Euclid and Archimedes were correct. This is the idea of Russell and Whitehead, who write in the Preface of the *Principia Mathematica* [50, p. v],

> the chief reason in favour of any theory on the principles of mathematics must always be inductive, *i.e.* it must lie in the fact that the theory in question enables us to deduce ordinary mathematics.

I propose that Euclid can be understood as ordinary mathematics here. The "analytic" style of mathematics introduced by Descartes [11] is inductive in the sense that it is justified by *working*—working to "explain" such mathematics as is already known from the Ancients.

According to Kline in *Mathematical Thought from Ancient to Modern Times* [27, pp. 87–8],

> A critical study of Euclid, with, of course, the advantage of present insights, shows that he uses dozens of assumptions that he never states and undoubtedly did not recognize.

Kline seems to assume that we do mathematics better today. However, who today recognizes all of the assumptions that go into our saying that the formula (2), namely $x^2 + y^2 = r^2$, defines a circle of radius r?[6]

[6] I quote Russell and Whitehead with approval, and I suggest that Euclid is the paradigm for ordinary mathematics. Kline [27, p. 1005] however quotes Russell as saying in 1902 [40], "It has been customary when Euclid, considered as a text-book, is attacked for his verbosity or his obscurity or his pedantry, to defend him on the ground that his logical excellence is transcendent, and affords an invaluable training to the youthful powers of reasoning. This claim, however, vanishes on a close inspection. His definitions do not always define, his axioms are not always indemonstrable, his demonstrations require many axioms of which he is quite unconscious. A valid proof retains its demonstrative force when no figure is drawn, but very many of Euclid's earlier proofs fail before this test . . . the value of his work as a masterpiece of logic has been very grossly exaggerated." The same criticisms apply to modern mathematics, at least as it is taught, though we think we can correct the defects, if we really have to. The dogmatic assertion about proofs and figures is just that. *We* think proofs ought be to

I propose that all of the hidden assumptions of mathematics *today* are (or contribute to) why people are often afraid of it. People equate mathematics with equations. Perhaps what they fear is equations. Equations *ought* to be frightening. They hide a lot of difficulty. They let you do something, without really knowing what it is.

1.2 Decimals

In equation (1), namely $A = \pi r^2$, the letters stand for *positive real numbers.* Such numbers have *decimal expansions.* We blithely performs computations with them; and yet, strictly speaking, there is no proper algorithm for this.

Suppose an "oracle" feeds us the digits of a number, one by one, and the first few digits are 0.33333. Given the job of multiplying by 3, we shall never know whether the answer starts out as 1.0 or as 0.9, unless at some point the oracle actually gives us a digit other than 3.

This bothers almost nobody today.[7] It bothers Euclid. For him, there is no decimal system that he declines to use; but he declines to use anything like what we call π. For him, the ratio of two circles is the *same* as the ratio of the squares on their diameters, and therefore on their radii. We may write this as

$$A : A_1 :: r^2 : {r_1}^2.$$

be "digital" (or digitizable, so that a computer can check them); the Ancients simply do not.

[7] An exception is David Fowler [21], who gives the example (which he attributes to Christopher Zeeman) of computing $1.\overline{2} \times 0.\overline{81}$. The factors being $^{11}/_9$ and $^{81}/_{99}$, the product is 1; but this digit does not arise from any finite computation $1.2\ldots2 \times 0.81\ldots81$.

By **alternation,** which is Proposition 16 of Book V of the *Elements,*

$$A : r^2 :: A_1 : {r_1}^2.$$

Thus the ratio of a circle to the square on its radius is independent of the radius. We can call this ratio π; but Euclid does not consider it, presumably because there is no use for it in *proving* anything.

1.3 Postulates

Archimedes does consider the ratio that we call π. He finds[8]

$$\frac{1}{7} < \pi - 3 < \frac{10}{71}.$$

This needs a **postulate:**[9] not only is the chord shorter than the circular arc, but the arc is shorter than the circumscribed

[8] As Archimedes himself puts it, Παντὸς *κύκλου ἡ περίμετρος τῆς διαμέτρου τριπλασίων ἐστί, καὶ ἔτι ὑπερέχει ἐλάσσονι μὲν ἢ ἑβδόμῳ μέρει τῆς διαμέτρου, μείζονι δὲ ἢ δέκα ἑβδομηκοστομόνοις* [1, p. 263], "Of every circle, the circumference is triple the diameter, and yet exceeds by less than the seventh part of the diameter, and by more than ten seventy-first [parts]." The big Liddell–Scott lexicon cites just this passage under *ἑβδομηκοστόμονος*, though other authors are cited for the words for 72nd, 75th, and 73rd. The 1925 Preface of Jones acknowledges the contribution of Heath to the entries for mathematical terminology [29, p. vii]. For Archimedes's result, Heath [2, p. 93] uses the mixed fractions $3\frac{1}{7}$ and $3\frac{10}{71}$. In Smyth's *Greek Grammar* [42, ¶347, p. 103], "seventieth" is *ἑβδομηκοστός*, "twentieth" is *εἰκοστός*, but "twenty-first" is *πρῶτος καὶ εἰκοστός*. A reason to look at these words is that *English* is strange to use the suffix "-first." Our word "first" is cognate with *πρῶτος* and is used as the ordinal form of "one," but is in origin the superlative of what is now the preposition "for," which once meant *before* [25].

[9] Archimedes calls it a *λαμβανόμενον* [1, p. 8–10], a *thing taken*; Netz [3, p. 36] translates this as "postulate."

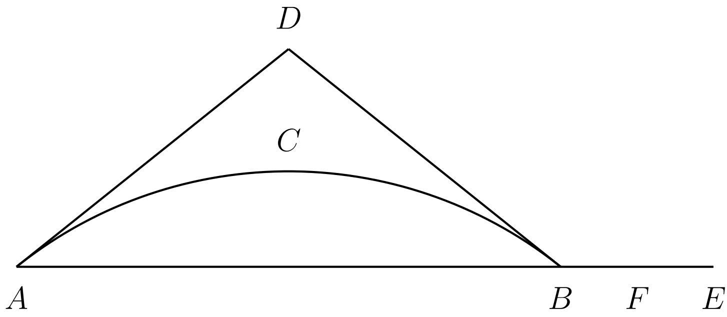

Figure 3: Archimedes's postulate

angle, so that, in Figure 3,

$$AB < ACB < ADB.$$

Four of *Euclid's* five postulates are more basic.[10] They give us a toolkit, comprising ruler, compass, and set square (or triangle), for

(1) drawing straight lines,
(2) extending straight lines,
(3) drawing circles, and
(4) checking whether angles are right.[11]

Practically speaking, in place of ruler and compass, we could use a cord, a string—a *line.*[12]

[10]Even the word for a postulate is different here; now it is *αἴτημα* [14, p. 8], a *demand* or *request.*

[11]Strictly, the fourth postulate is that all right angles are equal to one another; this is symbolized by the set square, which carries a right angle from place to place. There is no need to postulate that we can draw right angles, since this can be proved as in Propositions 11 and 12 of Book I of the *Elements.*

[12]The word "line" comes to English by two or three routes, from Latin words related to LINUM *flax* [25]; "linen" means *made of flax.* In

We could use a line for measuring an arc by wrapping the arc with the line, then straightening it. Archimedes's postulate tells us what would happen. In Figure 3, by purely Euclidean means, we can find the point E so that

$$AE = AD + DB.$$

With our line, when we find the point F so that

$$AF = ACB,$$

Archimedes tells us F will lie between B and E. There is no *construction* for this point, other than by the method that we have described. Practically speaking, the method requires *cutting out* the arc, to give it an edge for holding the line.

Euclid's fifth postulate tells us what will happen when we extend two straight lines; but this is something that we already know how to do.[13] The straight lines will intersect, so as to form the sides of a triangle, *provided that* the base angles will together be less than two right angles, as in Figure 4.[14]

Greek, Turkish, and other languages, a straight line is called for short a "straight"; in English, perversely, a "line." However, we are now considering this in the proper sense of a *flexible* one-dimensional object, not necessarily pulled taut.

[13] This observation tends to justify the assertion of Seidenberg [41] that, unlike the method of Archimedes, that of Euclid is not axiomatic. But then Seidenberg's closing remark is odd: "BOLYAI, writing to his father about his work on the theory of parallels, said: 'From nothing I have created another wholly new world.' EUCLID might very well have taken this proud declaration as his motto." However, earlier in the paper, Seidenberg's point seems to be that, unlike Archimedes, Euclid is *not* making anything up: "It would never have occurred to him that to prove a theorem ('the arc is greater than the chord'), it is all right to generalize it, and then assume the generalization."

[14] The converse, that any two angles of a triangle are less than two right angles, is Proposition 17 of Book I of the *Elements.*

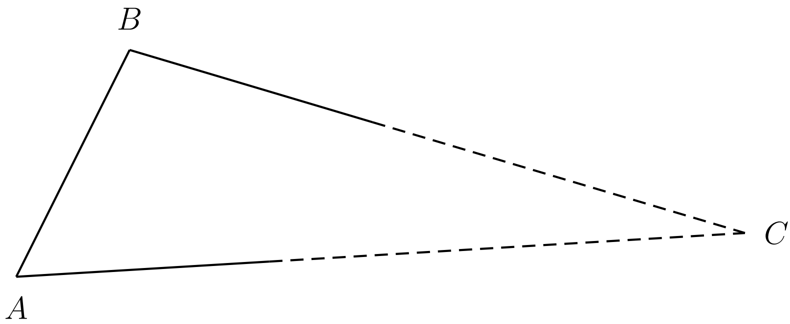

Figure 4: Euclid's 5th postulate

1.4 Proportions

Archimedes knows Euclid, whose rule again (as on page 10) for the area of a circle is,

Οἱ	κύκλοι	πρὸς	ἀλλήλους	εἰσὶν
The	circles	to	one another	are

ὡς	τὰ	ἀπὸ	τῶν	διαμέτρων	τετράγωνα.
as	the	on	the	diameters	squares.

In several translations, this is:

English:

- Circles are to one another as the squares on the diameters (Heath [18]).
- Circles are to one another as the squares on (their) diameters (Fitzpatrick [19]).

Turkish:

- Daireler, çaplarındaki kareler gibi birbirinedir (my attempt at literalism).
- Dairelerin birbirine oranı çapları üzerine çizilen karelerin oranına eşittir (Sertöz [12]).

Latin:

- CIRCULI INTER SE SUNT VT DIAMETRORUM QUADRATA (Commandinus [13]).
- CIRCULI EAM INTER SE RATIONEM HABENT QUAM QUADRATA DIAMETRORUM (Heiberg [15]).

This is highly concentrated, like the formula (1), or $A = \pi r^2$. However, the language is ordinary. There are no

(1) algebraic notation,
(2) number π, or
(3) technical terms,

except the words for circle, diameter, and square; but these are what the proposition is about.

There is of course a technical meaning to the ordinary words. Euclid can also say

- circles **are analogously as** the squares on their diameters;[15]
- the **ratio** of two circles is the **same** as the ratio of the squares on the diameters.

There are two key terms in the Greek:

Λόγος

- root meaning of *speech*
- in Latin, RATIO, whence "reason"
- gives ἀνάλογον "analogous(ly)" with prefix ἀνά
  - noun form ἀναλογία "analogy"
  - in Latin, PRO PORTIONE "proportional(ly)"

Αὐτός, -ή, -ό

- means *same* or *self*
- yields "auto-," as in "automorphism"
- first element of αὐθέντης, a person who acts on his or her own behalf: an *authority*

[15] Alternatively, "circles are *analogous to* the squares on their diameters"; but I think this is less literal.

– "authentic" in English
– in Turkish, both efendi and otantik.[16]

Not used here is ἴσος -η, -ον *equal,* appearing in English as in *isomorphism.* Two ratios are never *equal,* but they may be the *same.*[17]

1.5 Language

We can do mathematics in any language. However, if we do borrow our technical terminology from other languages, or even if we just give strange meanings to ordinary words, then our subject becomes **esoteric:** understood only by an inner circle.[18]

Mathematics will always be esoteric, after a certain point. Students will be able to dig to a certain depth, or climb to a certain height, and then they will be overwhelmed.[19]

[16] I use Chantraine [6] for Greek etymology; Nişanyan [31] for Turkish. The word αὐθέντης can also mean a murderer. In this case, the form αὐτοέντης is also used, as in lines 106–7 of *Oedipus Rex*, spoken to Oedipus by his brother-in-law (and unknown uncle) Creon: "The god commanded clearly: let some one / punish with force this dead man's murderers" [43]—the dead man being Laius, killed by Oedipus. The example is pointed out in the LSJ [29].

[17] Turkish has the native oran for ratio; but the root meaning seems to be not speaking or thinking, but cutting or splitting. As I understand from Nişanyan [31], oran is related to orak "sickle" and to the verb yarmak "split in two," whence yarım "half"; also related is ara "interval." I suppose the idea is that things can have a relation, a ratio, only if they have been split apart.

[18] This is a theme of my article "Abscissas and Ordinates" [37].

[19] The point was made by the poet Robert Fitzgerald in his address to the graduating class of St John's College, Annapolis, 1984, at the end of my freshman year. In 1983, Mary Kay Zuravleff, writer-in-residence at St Albans School in Washington, told us students how

However, if we do mathematics only with our native tongue, then the mathematical meaning of our words is may harmonize better with their ordinary meaning.

Conscious reform may not achieve this. It would be pointless to replace all uses of the Greek *logos* with "native" English words. Such reform is why English has the needless "foreword" alongside the perfectly good "preface."

she had studied mathematics as an undergraduate, but encountered the Axiom of Choice as a stumbling-block. In *Mathematics: A Very Short Introduction* [23, p. 132], Timothy Gowers observes that, in the learning of our subject, "Every so often, a new idea is introduced which is very important and markedly more sophisticated than those that have come before, and each one provides an opportunity to fall behind. An obvious example is the use of letters to stand for numbers . . . "

2 Equality and sameness

2.1 Equality before the law

In ordinary language, equality is not sameness. Article 7 of the Universal Declaration of Human Rights reads (emphasis mine),[20]

> All are **equal** before the law and are entitled without any discrimination to **equal** protection of the law. All are entitled to **equal** protection against any discrimination in violation of this Declaration and against any incitement to such discrimination.

In Turkish, this is Madde 7 of the İnsan hakları evrensel beyannamesi:[21]

> Kanun önünde herkes **eşittir** ve farksız olarak kanunun **eşit** korumasından istifade hakkını haizdir. Herkesin işbu Beyannameye aykırı her türlü ayırdedici mualeleye karşı ve böyle bir ayırdedici muamele için yapılacak her türlü kışkırtmaya karşı **eşit** korunma hakkı vardır.

The idea goes back to the Funeral Oration of Pericles, in Athens, as recounted by Thucydides [46, II.37, p. 145]:

> Let me say that our system of government does not copy the institutions of our neighbours. It is more the case of

[20] From `http://www.un.org/en/universal-declaration-human-rights/`, accessed September 2, 2016. Proclaimed by the United Nations General Assembly in Paris on December 10, 1948.

[21] From `http://www.ohchr.org/EN/UDHR/Pages/Language.aspx?LangID=trk`, accessed September 2, 2016.

> our being a model to others, than of our imitating anyone else. Our constitution is called a democracy because power is in the hands not of a minority but of the whole people. **When it is a question of settling private disputes, everyone is equal before the law;** when it is a question of putting one person before another in positions of public responsibility, what counts is not membership of a particular class, but the actual ability which the man possesses.[22]

The language of Thucydides is notoriously difficult, but the English of Rex Warner seems not quite right; Pericles says power is in the hands, not of the *whole* people, but of the majority.[23] The Turkish of Furkan Akderin [47, p. 82] seems more faithful in this regard, though perhaps not in others:

> Siyasi yapımızın komşularımızdan bir farkı yok. Hatta onlardan üstün olduğumuzu bile söyleyebiliriz. Çünkü biz onlara göre değil, onlar bize göre yasalarını yapıyorlar. Bizim devletimiz azınlığın değil çoğunluğun çıkarlarını gözetmektedir. Bu nedenle de ismi demokrasidir. **Herhangi bir anlaşmazlık anında herkes**

[22] Pericles's idea of "what counts in positions of public responsibility" may be compared with the findings of Jenny White: "Hierarchies characterizing Turkish political life are brittle because they are not founded in organizational competence, in rules and procedures, in merit, or even on a relationship of trust between leader and followers. These networks instead constitute what I call a spindle autocracy, grounded in loyalty and obedience to a single, central person instead of the organization itself or to the concept of merit as a marker for leadership and promotion" [49].

[23] Thucydides's Greek is, *χρώμεθα γὰρ πολιτείᾳ οὐ ζηλούσῃ τοὺς τῶν πέλας νόμους, παράδειγμα δὲ μᾶλλον αὐτοὶ ὄντες τισὶν ἢ μιμούμενοι ἑτέρους. καὶ ὄνομα μὲν διὰ τὸ μὴ ἐς ὀλίγους ἀλλ' ἐς πλείονας οἰκεῖν δημοκρατία κέκληται· μέτεστι δὲ κατὰ μὲν τοὺς νόμους πρὸς τὰ ἴδια διάφορα πᾶσι τὸ ἴσον, κατὰ δὲ τὴν ἀξίωσιν, ὡς ἕκαστος ἔν τῳ εὐδοκιμεῖ, οὐκ ἀπὸ μέρους τὸ πλέον ἐς τὰ κοινὰ ἢ ἀπ' ἀρετῆς προτιμᾶται* [45].

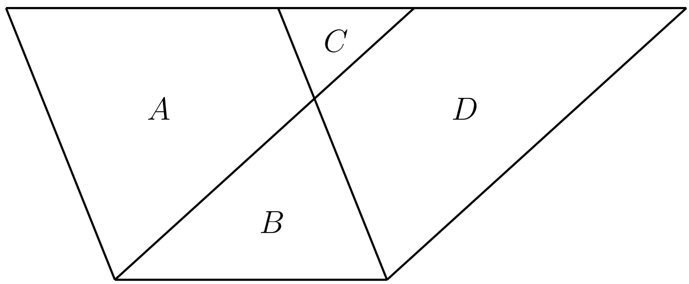

Figure 5: Parallelograms on the same base

> **yasalar karşısında eşittir.** Ancak konu kamu yaşamına katılmak olduğunda kim diğerlerinden daha üstünse yönetimde o bulunur.

As persons, we are equal before the law; but we are not all the same person.

2.2 Equality in Euclid

In mathematics today, we confuse equality with sameness. Equal numbers are the same number. Euclid does not make this confusion. Thus for example in Book I of the *Elements,* Proposition 35 is that parallelograms on the **same** base in the same parallels are **equal** to one another, as $A + B$ and $B + D$ in Figure 5. For $A + C$ and $C + D$ are **congruent** triangles, and therefore

$$A + C = C + D, \qquad A = D, \qquad A + B = B + D.$$

Congruence *implies* equality, but not conversely.

Another example is Proposition 43, as in Figure 6, where by congruence

$$A + B + C = D + E + F, \qquad A = D, \qquad C = F,$$

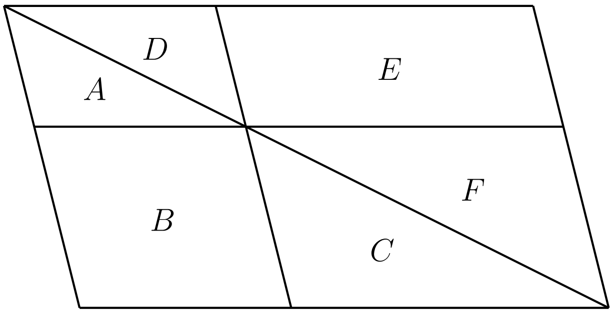

Figure 6: Equal "complementary" parallelograms

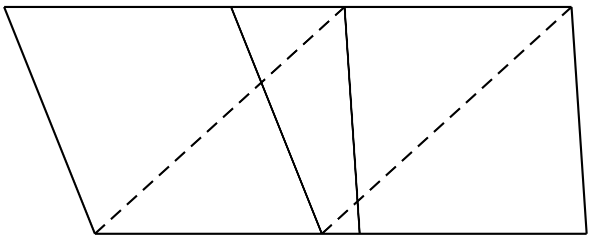

Figure 7: Parallelograms on equal bases

and therefore $B = E$.

Congruent figures **coincide.** This is the word that Heath uses in translating one of the so-called Common Notions at the beginning of the *Elements* [17]:

> Things which coincide with one another
> are equal to one another.

However, the Latin translations of Commandinus and Heiberg use the verb CONGRVO, whose second element means "fall" and gives us "ruin" in English.

Since parallelograms in the same parallels on the *same* base are equal, so are parallelograms on *equal* bases (Figure 7). I'm not sure how clearly students today see the distinction, since they have been trained to assign *numbers* as lengths to line segments, and "equal" numbers are the same number.

Ratios for Euclid are never equal. One ratio may be greater than another. Otherwise, the two ratios are not really two at all, because they are the *same.*

2.3 Sameness of ratio

When are two ratios the same? We can say when, as Euclid does in Book V of the *Elements,* without actually defining ratios. Euclid says only,

> A **ratio** is a sort of relation in respect of size between two magnitudes of the same kind.
>
> Magnitudes are said to **have a ratio** to one another which are capable, when multiplied, of exceeding one another.

Thus, to have a ratio, magnitudes must meet what we call an **Archimedean** condition, though Archimedes came later.

Suppose A and B have a ratio, and C and D have a ratio. The ratios are the **same,** so that the four magnitudes themselves are **proportional,** and we may write

$$A : B :: C : D,$$

provided that, for all multipliers k and m (we call them positive integers),

$$A \cdot k \gtreqless B \cdot m \iff C \cdot k \gtreqless D \cdot m.$$

This is the **Eudoxan** definition of proportion, so called since a scholium attributes the definition to Eudoxus [of Cnidus], teacher of Plato.[24]

[24]Heath [17, Vol. 2, p. 112] quotes the scholium from Heiberg [16, p. 280] and suggests that it may be by Proclus.

If A and B are **incommensurable,** so that $A \cdot k = B \cdot m$ never, then the ordered pairs of multipliers are cut into two disjoint sets:

$$\{(m,k)\colon B \cdot m < A \cdot k\}, \qquad \{(m,k)\colon B \cdot m > A \cdot k\}.$$

One may prefer to write (m,k) as ${}^m/_k$. The two sets compose what we call a **Dedekind cut.**

Dedekind was told [7, pp. 39–40] that his idea could already be found in Bertrand's *Traité d'arithmetique* [4]. Dedekind pointed out that *Bertrand's* idea could be found in Euclid: distinct ratios determine distinct cuts. Dedekind's innovation was to use cuts in place of ratios, thus obtaining all of what we now call the positive real numbers, without need for geometry.

By Euclid's definition of **greater ratio,**

$$3 : 2 > 7 : 5$$

because, for some multipliers k and m,

$$3 \cdot k > 2 \cdot m, \qquad 7 \cdot k = 5 \cdot m.$$

Indeed, just let $k = 5$ and $m = 7$.

Thus we have a test for proportionality of numbers:

$$A : B :: C : D \iff A \times D = B \times C. \tag{3}$$

Let us call this the **Eudoxan** definition of proportion of *numbers.* It assumes that multiplication of numbers is commutative. Euclid *proves* this, *using* proportions. Thus the **Euclidean** definition of proportion of numbers will be different.

3 Numbers

3.1 Proportion

At the head of Book VII of the *Elements,* we are told:

1. "A number is a multitude of units."[25]
2. "Numbers are **proportional** when the first is
   - the same multiple, or
   - the same part, or
   - the same parts,

   of the second that the third is of the fourth."
3. The following are equivalent for numbers.
   a) B is a multiple of A,
   b) A is a part of B,
   c) A **measures** B.
4. When neither multiple nor part of B, A is **parts** of B.[26]

Measuring is dividing in *extension,* but not in *intension*:

(1) We can **measure** 12 apples evenly by 4 apples.
(2) In the process, we **divide** the apples into 3 groups.[27]

[25] John Dee *created* the word "unit" precisely to translate Euclid's μονάς, as he notes his "Mathematicall Preface" [8] to Billingsley's 1570 English translation of the *Elements.* The existing alternative was "unity." See my article "On commensurability and symmetry" [39].

[26] The text says only that the *less* is parts of the greater when not measuring the greater; but the definition of proportion implies that the greater is parts of the less when not a multiple of the less.

[27] Euclid uses dividing, as far as I know, only to say that an even number can be divided in two. Alexandre Borovik discusses measuring and dividing apples [5], though not with the terminology of measuring.

It is clear when A is the **same multiple** or **same part** of B that C is of D; but not **same parts.**

If

$$A : B :: C : D,$$

this should mean at least that for some numbers E and F, for some multipliers k and m,[28]

$$\begin{aligned} A &= E \cdot k, & C &= F \cdot k, \\ B &= E \cdot m, & D &= F \cdot m. \end{aligned} \tag{4}$$

- This much is called the **Pythagorean** definition of proportion of numbers.
- For the **Euclidean** definition, we need

$$E = \text{GCM}(A, B), \qquad F = \text{GCM}(C, D), \tag{5}$$

  where GCM means **greatest common measure;** equivalently, k and m in (4) are coprime.

Without (5),

(1) sameness of ratio is not *immediately* transitive;
(2) thus proofs in Book VII are inadequate;
(3) Proposition 4 as a whole makes little sense.

Therefore I say that the Euclidean definition must be the one that Euclid means.[29]

---

[28] If one of the multipliers k and m is unity, then we have "same part" or "same multiple"; otherwise, "same parts."

[29] Heath thinks (1) the theory of Book VII is due to the Pythagoreans [17, Vol. 2, p. 294], and (2) its definition of proportion is the one that we are calling Pythagorean [24, p. 190]. In Thomas's first Loeb volume of *Greek Mathematical Works* [44], the chapter "Pythagorean Arithmetic" gives first the definitions that head Book VII of the *Elements,* but nothing ensues that requires a careful interpretation of the definition of proportion. That this definition ought *immediately*

3.2 Anthyphaeresis

In Book VII, **Propositions 1–3** show, for two or more numbers,

(1) *how* to find a GCM, and
(2) *that* it is measured by all common measures.

The **Euclidean Algorithm** is used, namely,

(1) of two magnitudes, replace the greater with its remainder, if there is one, after measurement by the less;
(2) repeat.

When the greater is measured exactly by the less, this is the GCM. Thus from

$$\begin{gathered}80 = 62 \cdot 1 + 18,\\ 62 = 18 \cdot 3 + 8,\\ 18 = 8 \cdot 2 + 2,\\ 8 = 2 \cdot 4,\end{gathered}$$

we have $\text{GCM}(80, 62) = 2$, Also,

$$80 = 2 \cdot 40, \qquad 62 = 2 \cdot 31, \tag{6}$$

and the multipliers 40 and 31 are automatically coprime.

The enunciation of **Proposition 4** is, "Any number is either a part or parts of any number, the less of the greater."

---

to imply transitivity of sameness of ratio: this might seem belied by Proposition 11 in Book V, which proves the transitivity for arbitrary magnitudes under the Eudoxan definition; however, the proof is trivial. Nonetheless, Pengelley and Richman [33, pp. 196, 199] accept Heath's judgment, and Mazur [30, n. 6] accepts *their* judgment; for convenience, I imitate them in using the term *Pythagorean.* I have seen no suggestion that the Pythagoreans proved general theorems like the commutativity of multiplication or Euclid's Lemma; thus perhaps they had no theoretical need for the transitivity of sameness of ratio.

Euclid "proves" this by finding GCM's, showing implicitly (in my view) that Euclid intends the Euclidean definition of proportion. From

$$\begin{aligned} 120 &= 93 \cdot 1 + 27, \\ 93 &= 27 \cdot 3 + 12, \\ 27 &= 12 \cdot 2 + 3, \\ 12 &= 3 \cdot 4, \end{aligned}$$

we have $\text{GCM}(120, 93) = 3$, and also

$$120 = 3 \cdot 40, \qquad 93 = 3 \cdot 31. \tag{7}$$

By the repetition in (7) of multipliers from (6),

$$80 : 62 :: 120 : 93.$$

The same follows, just from the repetition of the multipliers $(1, 3, 2, 4)$ in the steps of the Algorithm. Indeed, we can write either of the fractions $^{80}/_{62}$ and $^{120}/_{93}$ as the continued fraction

$$1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{4}}}.$$

In Greek, the Algorithm is **anthyphaeresis** or "alternating subtraction."[30] There is good evidence that before the Eudoxan definition, there was an **anthyphaeretic** definition of proportion for arbitrary magnitudes, whereby the proportion

$$A : B :: C : D$$

[30]The term derives ultimately from ἀνθυφαιρέ-ω (*anthyphaire-ô*) "alternately subtract," the verb that Euclid uses to describe his Algorithm. The analysis is ἀντί + ὑπό + αἱρέ-ω (*anti* + *hypo* + *haire-ô*), the core verb meaning *take.*

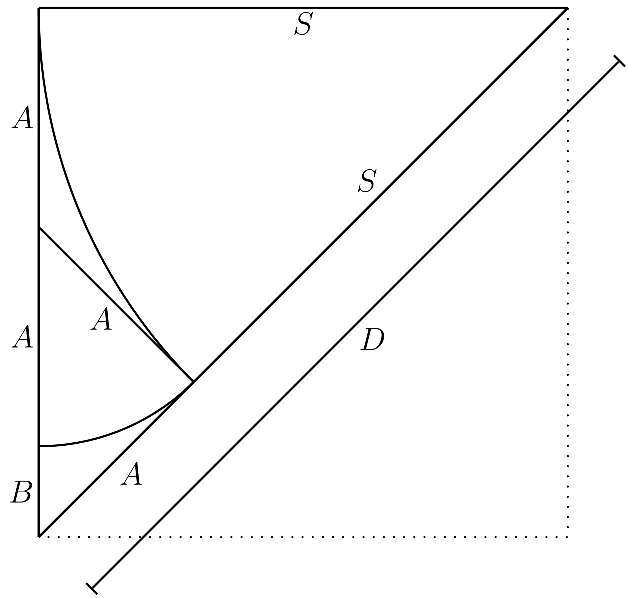

Figure 8: Anthyphaeresis of diagonal and side of square

means the Euclidean Algorithm has the same steps, whether applied to A and B or C and D.[31] The Euclidean definition is a simplification of this for numbers; but the anthyphaeretic definition applies even to incommensurable magnitudes,[32] such as the diagonal and side of a square, as in Figure 8, where

[31] See Thomas [44, pp. 504–9] or Fowler [22].

[32] In particular, there is no reason to think that the Eudoxan theory was "developed to handle incommensurable magnitudes." Pengelley and Richman [33, p. 199] suggest that it was, even though they cite the book [22] of Fowler, who says, "I now disagree with everything in this line of interpretation"—the line whereby the Pythagoreans based mathematics on commensurable magnitudes, until the discovery of incommensurability, whose problems were not resolved until the Eudoxan theory was formulated.

$$D = S + A, \qquad\qquad S = A \cdot 2 + B,$$

and ever after, the less goes twice into the greater, so that

$$S : A :: A : B, \qquad\qquad D : S \text{ is constant.}$$

Understanding proportion is important because Euclid uses it to prove
(1) commutativity of multiplication, and
(2) **Euclid's Lemma,** that a prime measuring a product measures one of the factors.

Under the Euclidean definition, the proofs are rigorous.

3.3 Commutativity

In Book VII of the *Elements,* from either the anthyphaeretic or the Euclidean definition of proportion of numbers, we obtain **Propositions 5–8:**

$$A : B :: C : D \implies A : B :: A \pm C : B \pm D.$$

Repeated application gives **Proposition 9:**

$$E : F :: E \cdot m : F \cdot m.$$

This gives, *by transitivity,* **Proposition 10:**

$$E \cdot k : F \cdot k :: E \cdot m : F \cdot m. \tag{8}$$

Automatically, if k and m are coprime,

$$E \cdot k : E \cdot m :: F \cdot k : F \cdot m. \tag{9}$$

Since every proportion can be written in this form, the implication (9) ⇒ (8) is **Proposition 13, Alternation:**

$$A : B :: C : D \implies A : C :: B : D.$$

Since

$$1 : A :: B : B \times A, \tag{10}$$

by Alternation, $1 : B :: A : B \times A$, so by symmetry

$$1 : A :: B : A \times B. \tag{11}$$

Comparing (10) and (11) yields **Proposition 16, Commutativity:**

$$A \times B = B \times A.$$

3.4 Euclid's Lemma

Proposition 17 is like 9:

$$C : D :: C \times A : D \times A. \tag{12}$$

Hence **Proposition 18,** $C : D :: A \times C : A \times D$, or with different letters,

$$A : B :: C \times A : C \times B. \tag{13}$$

From (12), (13), and *transitivity,* we get **Proposition 19,**

$$A : B :: C : D \iff D \times A = C \times B,$$

which is (3), the Eudoxan definition of proportion of numbers on page 27. **Proposition 20** is that, if A and B are the least X and Y such that

$$X : Y :: C : D,$$

then A measures C,[33] for by *Alternation*

$$A : C :: B : D,$$

and so A is the same part or parts of C that B is of D; but it cannot be parts, by minimality.[34] Here A and B are also coprime.[35] The converse is **Proposition 21.**

Immediately from the definitions, **Proposition 29:** every prime is coprime with its every non-multiple.

For **Proposition 30, Euclid's Lemma,** suppose a prime P measures $A \times B$, so that for some C,

$$P \times C = A \times B.$$

By 19 (the Eudoxan definition),

$$P : A :: B : C.$$

If P does not measure A, then

- P and A are coprime by 29,
- they are the least numbers having their ratio by 21,
- P measures B by 20.[36]

---

[33] And B measures D the same number of times, as Euclid says.

[34] Mazur says, "Now I don't quite follow Euclid's proof of this pivotal proposition, and I worry that there may be a tinge of circularity in the brief argument given in his text" [30, p. 243]; then he cites Pengelley and Richman [33]. Mazur's own proof uses what he calls Propositions 5 and 6, though 7 and 8 are also needed, to conclude $A : B :: C - A \cdot k : D - B \cdot k$.

[35] This is Proposition 22, but we shall not need it.

[36] For Mazur [30, p. 242], "that if a prime divides a product of two numbers, it divides . . . one of them, is essentially Euclid's Proposition 24 of Book VII." Strictly, this is that the product of numbers prime to a number is also prime to it. Like that of 30, the proof relies on 20, which again for Mazur is problematic.

Euclid uses words alone to describe proportions. This could be because the Ancients were more used to *hearing* mathematics than seeing it. Modern commentators use fractions and the equals sign. I have tried to preserve the distinction between proportions and equations, while making Euclid's rigor *visible* in the way that we Moderns are used to.

Bibliography

[1] Archimedes. *Archimedis Opera Omnia Cum Commentarius Eutochii*, volume I. B. G. Teubner, 1880. Recension of the Florentine Codex, with Latin translation and notes by J. L. Heiberg.

[2] Archimedes. *The Works of Archimedes*. University Press, Cambridge, UK, 1897. Edited in Modern Notation With Introductory Chapters by T. L. Heath.

[3] Archimedes. *The Two Books* On the Sphere and the Cylinder, volume I of *The Works of Archimedes*. Cambridge University Press, Cambridge, 2004. Translated into English, together with Eutocius' commentaries, with commentary, and critical edition of the diagrams, by Reviel Netz.

[4] Joseph Bertrand. *Traité d'Arithmétique*. Hachette, Paris, 1849. Electronic version from Gallica Bibliotèque Numérique (`gallica.bnf.fr`).

[5] Alexandre Borovik. Metamathematics of elementary mathematics. `www.matematikdunyasi.org/yazokulu/borovik_1b.pdf`, July 2008. Lecture at the Nesin Mathematics Village, Şirince, Selçuk, Izmir, Turkey.

[6] Pierre Chantraine. *Dictionnaire étymologique de la langue grecque. Histoire des mots*. Klincksieck, Paris, 1968–1980. In four volumes.

[7] Richard Dedekind. *Essays on the Theory of Numbers. I: Continuity and Irrational Numbers. II: The Nature and Meaning of Numbers*. Dover Publications, New York, 1963. Authorized 1901 translation by Wooster Woodruff Beman of "Stetigkeit und irrationale Zahlen" and "Was sind und was sollen die Zahlen" (second edition, 1893; first, 1887).

[8] John Dee. Mathematicall preface. In *The Elements of Geometrie of the most auncient Philosopher Euclid of Megara*. Iohn Daye, London, 1570. Facsimile in `pdf` format.

[9] Abdurrahman Demirtaş. *Matematik Sözlüğü*. Bilim Teknik Kültür Yayınları, Ankara, 1986.

[10] René Descartes. *The Geometry of René Descartes*. Dover Publications, New York, 1954. Translated from the French and Latin by David Eugene Smith and Marcia L. Latham, with a facsimile of the first edition of 1637.

[11] René Descartes. *La Géométrie*. Jacques Gabay, Sceaux, France, 1991. Reprint of Hermann edition of 1886.

[12] Euclid. Öklid'in Elemanları. Turkish translation and notes by Sinan Sertöz. Version of February 20, 2018. `sertoz.bilkent.edu.tr/oklid/elemanlar.pdf`.

[13] Euclid. *Euclidis Elementorum Libri XV*. Jacobus Chriegher, Pisauri (Pesaro), 1572. Latin version by Federico Commandino. Digitized by Google. `www.wilbourhall.org/`, accessed November 24, 2017.

[14] Euclid. *Euclidis Elementa*, volume I of *Euclidis Opera Omnia*. Teubner, Leipzig, 1883. Edited with Latin interpretation by I. L. Heiberg. Books I–IV.

[15] Euclid. *Elementa*, volume IV. Teubner, 1885. Edited with Latin interpretation by I. L. Heiberg. Books XI–XIII. Digitized by Google. `www.wilbourhall.org/`, accessed February 23, 2018.

[16] Euclid. *Elementa*, volume V. Teubner, 1888. Edidit et Latine interpretatvs est I. L. Heiberg.

[17] Euclid. *The Thirteen Books of Euclid's* Elements. Dover Publications, New York, 1956. Translated from the text of Heiberg with introduction and commentary by Thomas L. Heath. In three volumes. Republication of the second edition of 1925. First edition 1908.

[18] Euclid. *The Bones: A handy where-to-find-it pocket reference companion to Euclid's* Elements. Green Lion Press, Santa Fe, NM, 2002. Conceived, designed, and edited by Dana Densmore.

[19] Euclid. *Euclid's Elements of Geometry*. Published by the editor, revised and corrected edition, 2008. Edited, and provided with a modern English translation, by Richard Fitzpatrick, `farside.ph.utexas.edu/euclid.html`.

[20] Euclid. *Öklid'in* Öğeler*'inin 13 Kitabından Birinci Kitap*. Mathematics Department, Mimar Sinan Fine Arts University, Istanbul, 4th edition, September 2014. The first of the 13 books of Euclid's Elements. Greek text, with Turkish version by Özer Öztürk & David Pierce.

[21] David Fowler. Dedekind's theorem: $\sqrt{2} \times \sqrt{3} = \sqrt{6}$. *Amer. Math. Monthly*, 99(8):725–733, 1992.

[22] David Fowler. *The Mathematics of Plato's Academy: A new reconstruction*. Clarendon Press, Oxford, second edition, 1999.

[23] Timothy Gowers. *Mathematics: A Very Short Introduction*. Very Short Introductions. Oxford University Press, Oxford, 2002.

[24] Thomas Heath. *A History of Greek Mathematics. Vol. I. From Thales to Euclid*. Dover Publications, New York, 1981. Corrected reprint of the 1921 original.

[25] T. F. Hoad, editor. *The Concise Oxford Dictionary of English Etymology*. Oxford University Press, Oxford, 1986.

[26] Sami Kaya. *YGS-LYS Stratejiler: Geometri*. Delta Kitap, Ankara, 2011. Small cards, roughly A7, joined by a ball chain.

[27] Morris Kline. *Mathematical Thought from Ancient to Modern Times*. Oxford University Press, New York, 1972.

[28] Edmund Landau. *Foundations of Analysis. The Arithmetic of Whole, Rational, Irrational and Complex Numbers*. Chelsea Publishing, New York, third edition, 1966. Translated by F. Steinhardt; first edition 1951; first German publication, 1929.

[29] Henry George Liddell and Robert Scott. *A Greek-English Lexicon*. Clarendon Press, Oxford, 1996. "Revised and augmented throughout by Sir Henry Stuart Jones, with the assistance of Roderick McKenzie and with the cooperation of many scholars. With a revised supplement." First edition 1843; ninth edition 1940.

[30] Barry Mazur. How did Theaetetus prove his theorem? In P. Kalkavage and E. Salem, editors, *The Envisioned Life: Essays in honor of Eva Brann*. Paul Dry Books, 2007. `www.math.harvard.edu/~mazur/preprints/Eva.pdf`, accessed September 20, 2012.

[31] Sevan Nişanyan. *Sözlerin Soyağacı: Çağdaş Türkçenin Etimolojik Sözlüğü.* Adam Yayınları, İstanbul, 3rd edition, 2007. "The Family Tree of Words: An Etymological Dictionary of Contemporary Turkish." Genişletilmiş gözden geçirilmiş ("expanded and revised").

[32] Giuseppe Peano. The principles of arithmetic, presented by a new method. In Jean van Heijenoort, editor, *From Frege to Gödel: A source book in mathematical logic, 1879–1931*, pages 83–97. Harvard University Press, Cambridge, MA, 1889. First published 1889.

[33] David Pengelley and Fred Richman. Did Euclid need the Euclidean algorithm to prove unique factorization? *Amer. Math. Monthly*, 113(3):196–205, 2006.

[34] David Pierce. Model-theory of vector-spaces over unspecified fields. *Arch. Math. Logic*, 48(5):421–436, 2009.

[35] David Pierce. Induction and recursion. *The De Morgan Journal*, 2(1):99–125, 2012. `education.lms.ac.uk/2012/04/david-pierce-induction-and-recursion/`.

[36] David Pierce. St John's College. *The De Morgan Journal*, 2(2):62–72, 2012. `education.lms.ac.uk/wp-content/uploads/2012/02/st-johns-college.pdf`, accessed October 1, 2014.

[37] David Pierce. Abscissas and ordinates. *J. Humanist. Math.*, 5(1):223–264, 2015. `scholarship.claremont.edu/jhm/vol5/iss1/14`.

[38] David Pierce. The geometry of numbers in Euclid. *Polytropy*, January 2017. Blog article, `polytropy.com/2017/01/02/the-geometry-of-numbers-in-euclid/`.

[39] David Pierce. On commensurability and symmetry. *J. Humanist. Math.*, 7(2):90–148, 2017. `scholarship.claremont.edu/jhm/vol7/iss2/6`.

[40] Bertrand Russell. The teaching of Euclid. *The Mathematical Gazette*, 2(33):165–7, 1902. Quoted at `www-groups.dcs.st-and.ac.uk/history/Extras/Russell_Euclid.html`, accessed March 4, 2018.

[41] A. Seidenberg. Did Euclid's Elements, Book I, develop geometry axiomatically? *Arch. History Exact Sci.*, 14(4):263–95, 1975.

[42] Herbert Weir Smyth. *Greek Grammar.* Harvard University Press, Cambridge, Massachusetts, 1980. Revised by Gordon M. Messing, 1956. Eleventh Printing. Original edition, 1920.

[43] Sophocles. *Sophocles I: Oedipus the King, Oedipus at Colonus, Antigone.* University of Chicago, 1954.

[44] Ivor Thomas, editor. *Selections Illustrating the History of Greek Mathematics. Vol. I. From Thales to Euclid.* Number 335 in Loeb Classical Library. Harvard University Press, Cambridge, Mass., 1951. With an English translation by the editor.

[45] Thucydides. *Historiae.* Oxford, 1942. Greek text. Perseus Digital Library `data.perseus.org/texts/urn:cts:greekLit:tlg0003.tlg001`, accessed February 22, 2018.

[46] Thucydides. *The Peloponnesian War.* Penguin, Harmondsworth, Middlesex, England, 1982. Translated by Rex Warner, first published 1954.

[47] Thukydides. *Peloponnessos Savaşları.* Belge, Istanbul, 2010. Translated from Greek to Turkish by Furkan Akderin.

[48] Arthur W. Weeks and Jackson B. Adkins. *A Course in Geometry: Plane and Solid.* Ginn and Company, Lexington MA, 1970.

[49] Jenny White. Spindle autocracy in the New Turkey. *The Brown Journal of World Affairs*, XXIV(1), Fall/Winter 2017. `bjwa.brown.edu/24-1/spindle-autocracy-in-the-new-turkey/`, accessed March 6, 2018.

[50] Alfred North Whitehead and Bertrand Russell. *Principia Mathematica*, volume I. University Press, Cambridge, 1910. Reprinted by Merchant Books, Breinigsville, PA, USA, 04 November 2009, with suppression of the name of the original publisher.