A Method for Companionability, Applied to Group Actions and Valuations*

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Abstract

A method for finding model companions is applied to the theory of group actions and to the theory of fields with both an automorphism and a valuation.

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1 The Method

For every system of ordinary differential polynomial equations over a differential field of characteristic 0, the consistency of the system—its solubility in some possibly larger differential field—is a first-order function of the parameters of the system. Abraham Seidenberg [11] showed this, and from it, Abraham Robinson [9, §5.5] derived the theory DCF_0 of **differentially closed fields** of characteristic 0, which is to the theory DF_0 of all differential fields of characteristic 0 as the theory ACF of algebraically closed fields is to the theory of all fields.

Specifically, DCF_0 is the *model completion* of DF_0. To say what this means, we denote by $\operatorname{diag}(\mathfrak{M})$ the **diagram** of $\mathfrak{M}$, namely the theory of structures in which $\mathfrak{M}$ embeds [9, §2.1]; this theory is axiomatized by the atomic and negated atomic sentences, with parameters, that are true in $\mathfrak{M}$.

A theory T^* is the **model completion** of a theory T in the same signature [9, §4.3] if

1) $T \subseteq T^*$,
2) $T^* \cup \operatorname{diag}(\mathfrak{M})$ is consistent whenever $\mathfrak{M} \vDash T$,
3) $T^* \cup \operatorname{diag}(\mathfrak{M})$ axiomatizes a complete theory whenever $\mathfrak{M} \vDash T$.

When T has the model completion T^*, then immediately,

1) every model of T^* embeds in a model of T,
2) every model of T embeds in a model of T^*,
3) T^* is **model complete,** that is, $T^* \cup \operatorname{diag}(\mathfrak{M})$ is complete whenever $\mathfrak{M} \vDash T^*$.

Under these conditions alone, T^* is called the **model companion** of T (the notion was introduced by "Eli Bers" in 1969 [2, p. 609]).

Given a theory T, we define a **system** of T to be a conjunction of atomic and negated atomic formulas in the signature of the theory. T is **inductive** if axiomatized by $\forall\exists$ sentences, equivalently, every union of a chain of models is a model.

Theorem 1. *If it exists, the model-companion T^* of a theory T is axiomatized by $T_\forall$ and sentences*

$$\forall \boldsymbol{x}\, \forall \boldsymbol{y}\, \exists \boldsymbol{z}\, \big(\vartheta(\boldsymbol{x},\boldsymbol{y}) \to \varphi(\boldsymbol{x},\boldsymbol{z})\big),$$

where

- *φ is a system of atomic and negated atomic formulas,*
- *ϑ is from a set Θ_φ of formulas, and*
- *for all models $\mathfrak{M}$ of $T_\forall$ with parameters $\boldsymbol{a}$,*

  $\vartheta(\boldsymbol{a},\boldsymbol{y})$ is soluble in $\mathfrak{M}$ for some ϑ in Θ_φ $\iff$
  $\varphi(\boldsymbol{a},\boldsymbol{z})$ is soluble in a model of $T_\forall \cup \operatorname{diag}\mathfrak{M}$.

One can use Compactness to replace Θ_φ with a single formula. Also, if T has a model-*completion,* this single formula can be required to be quantifier-free (and conversely): this is what Robinson proved. Seidenberg had already shown that DF met the condition on T. It was then observed, first by Blum [1, 8, 6], that not all systems of DF need be considered. Especially, every ordinary differential polynomial equation can be written as

$$f(\dots,\delta^j x_i,\dots)=0,$$

where f is an ordinary polynomial; and this equation is equivalent to the result of replacing each derivative with a new variable, then conjoining the equation of the derivative with the variable:

$$f(\dots,x_i^{(j)},\dots)=0 \land \bigwedge_{i,j} \delta x_i^{(j-1)} = x_i^{(j)}.$$

This approach of isolating the singulary operation δ is useful for other theories involving singulary operations, specifically the theories of

1) group actions (in work with Ayşe Berkman),
2) fields with automorphism and valuation (in work with Özlem Beyarslan, Daniel Max Hoffmann, Gönenç Onay).

The general result that we use is the following.

Porism. *In the hypothesis of Theorem 1, it is enough that* $\varphi(\vec{x}, \vec{y})$ *range over a collection* Φ *of systems in the signature of* T *containing,*

(a) ***for all*** *systems* $\psi(\vec{x}, \vec{u})$ *of* T*,*
(b) ***for all*** *models* $\mathfrak{M}$ *of* T*,*
(c) ***for all*** *choices* $\vec{a}$ *of parameters from* M*,*

a system $\varphi(\vec{x}, \vec{u}, \vec{v})$ *such that,*

(i) *if* $\exists \vec{u}\ \psi(\vec{a}, \vec{u})$ *is consistent with* $T \cup \operatorname{diag} \mathfrak{A}$*, then so is* $\exists \vec{u}\ \exists \vec{v}\ \varphi(\vec{a}, \vec{u}, \vec{v})$*, and*

(ii) $T \cup \operatorname{diag}(\mathfrak{M}) \vdash \forall \vec{u}\ \forall \vec{v}\ \bigl(\varphi(\vec{a}, \vec{u}, \vec{v}) \to \psi(\vec{a}, \vec{u})\bigr)$.

2 Group Actions

We can understand a **group action** as kind of two-sorted structure (G, A), equipped with a function

$$(\xi, y) \mapsto \xi\, y$$

from $G \times A$ to A. The structure should be a model of the theory GA, which has the following axioms:

$$\forall \xi\ \exists \eta\ \forall z\ (\xi\, \eta\, z = z \land \eta\, \xi\, z = z),$$
$$\forall \xi\ \forall \eta\ \exists \zeta\ \forall v\ \xi\, \eta\, v = \zeta\, v,$$
$$\exists \xi\ \forall y\ \xi\, y = y.$$

In words, if the elements of G are called *functions,* GA says

1) every function has an inverse,
2) any two functions have a composite,
3) there is an identity function.

There are no symbolized operations of actually taking inverses, forming composites, or being the identity. The theory FGA of **faithful** group actions is axiomatized by GA along with

$$\forall\xi\ \forall\eta\ \exists z\ (\xi \neq \eta \to \xi\, z \neq \eta\, z).$$

Of a theory T, its **universal part** $T_\forall$ is the theory axiomatized by the universal sentences in T; this is the theory of all substructures of models of T. Then T and $T_\forall$ have the same model companion, if there is one. To obtain a model companion for $\mathsf{GA}_\forall$, it is enough to look at systems of equations $\xi\, y = z$ and inequations $z \neq w$.

Theorem 2 (Berkman, P.)**.**

1. *GA and FGA are not inductive, but they have the same universal part, which is axiomatized by*

$$\forall\xi\ \forall y\ \forall z\ (y \neq z \to \xi\, y \neq \xi\, z),$$

meaning all functions are injective.

2. *Each of GA and FGA has a model companion, GA^*, which is axiomatized by $\mathsf{GA}_\forall$, along with*

$$\forall\xi\ \forall y\ \exists z\ \xi\, z = y,$$

$$\exists x\ \exists y\ x \neq y,$$

$$\forall \boldsymbol{x}\ \exists \boldsymbol{\xi}\ \left(\bigwedge_{i<j<n} x_i \neq x_j \to \varphi_n(\boldsymbol{\xi}, \boldsymbol{x})\right),$$

$$\forall \boldsymbol{\xi}\ \exists \boldsymbol{x}\ \left(\bigwedge_{\substack{(\sigma,\tau)\in\mathrm{Sym}(n)^2\\ \sigma\neq\tau}} \xi_\sigma \neq \xi_\tau \to \varphi_n(\boldsymbol{\xi}, \boldsymbol{x})\right),$$

where n ranges over $\mathbb{N}$, and $\varphi_n(\boldsymbol{\xi}, \boldsymbol{x})$ is the formula

$$\bigwedge_{i<n} \bigwedge_{\sigma \in \mathrm{Sym}(n)} \xi_\sigma \, x_i = x_{\sigma(i)}.$$

That is, GA^* is the theory of those models (P, A) of $\mathsf{GA}_\forall$ such that

   a) the functions on A induced by elements of P are invertible,
   b) A has at least two elements, and
   c) for each n in $\mathbb{N}$,
      (i) each set of n distinct elements of A is completely permuted by some $n!$ elements of P, and
      (ii) each set of $n!$ distinct elements of P completely permutes some n elements of A.

3. *GA^* does not admit full elimination of quantifiers, so $\mathsf{GA}_\forall$ has no model completion.*
4. *In the expanded signature with a symbol for the function*

$$(\xi, y) \mapsto \xi^{-1} \, y,$$

   the theory $\mathsf{GA}^\dagger$ axiomatized by GA^* with

$$\forall \xi \; \forall y \; \forall z \; (\xi \, y = z \leftrightarrow y = \xi^{-1} \, z)$$

   admits full elimination of quantifiers.
5. *$\mathsf{GA}^\dagger$ is complete, and therefore GA^* is complete.*
6. *GA^* has TP_2.*
7. *GA^* has NSOP_1 (by the sufficient condition of Chernikov and Ramsey [4, Prop. 5.3]).*
8. *GA^* has a prime model, in which the orbit of any finite set of points under any finite set of permutations is finite.*

9. GA^* *has no countable universal model.*
10. *There is a model* (P, ω) *of* GA^* *in which the model* $(\mathrm{Sym}(\omega), \omega)$ *of* FGA *embeds; thus every countable model of* GA^* *embeds (elementarily) in* (P, ω).

3 Fields with Automorphism and Valuation

A **difference field** is just a field with an automorphism. The theory of difference fields has a model companion, called ACFA [5, 3]. In the signature

$$\{+, -, 0, \times, 1, {}^{\sigma}, \in \mathfrak{O}, \in \mathfrak{M}\},$$

we axiomatize FAV by the field axioms, along with axioms

$$\forall x\ \forall y\ \big((x+y)^{\sigma} = x^{\sigma} + y^{\sigma} \wedge (x \cdot y)^{\sigma} = x^{\sigma} \cdot y^{\sigma}\big),$$
$$\forall x\ \exists y\ y^{\sigma} = x$$

for a surjective endomorphism (which for a field is an automorphism), and axioms

$$0 \in \mathfrak{O},$$
$$\forall x\ \forall y\ (x \in \mathfrak{O} \wedge y \in \mathfrak{O} \to -x \in \mathfrak{O} \wedge x + y \in \mathfrak{O} \wedge x \cdot y \in \mathfrak{O}),$$
$$\forall x\ \exists y\ (x \notin \mathfrak{O} \to x \cdot y = 1 \wedge y \in \mathfrak{O})$$

for a valuation ring, and (for convenience) the axiom

$$\forall x\ \Big(x \in \mathfrak{M} \leftrightarrow \exists y\ \big(x = 0 \vee (x \cdot y = 1 \wedge y \notin \mathfrak{O})\big)\Big),$$

or equivalently

$$\forall x\ \Big(x \notin \mathfrak{M} \leftrightarrow \exists y\ (x \cdot y = 1 \wedge y \in \mathfrak{O})\big),$$

for membership in the unique maximal ideal of the valuation ring. Because both the new predicate and its negation have existential definitions, the predicate does not affect the existence of a model-companion [7, Lem. 1.1, p. 427].

The theory ACVF of algebraically closed fields with proper valuation ring is the model companion of the theory of fields with valuation ring [10, §3.4, pp. 47 ff.]. This does not make it automatic that FAV has a model companion; for example, the theory of difference fields (of arbitrary characteristic) with a derivation has no model companion [6]. To obtain a model companion for FAV, it is enough to look at systems

$$\bigwedge_{f\in I_0} f=0 \wedge \bigwedge_{i<m} {X_i}^{\sigma}=X_{\tau(i)} \wedge \bigwedge_{k\in\kappa} X_k\in\mathfrak{M} \wedge \bigwedge_{\ell\in\lambda} X_\ell\in\mathfrak{O},$$

where

$$m\leqslant n<\omega,\quad I_0\subseteq_{\mathrm{fin}}\mathfrak{O}[X_j\colon j<n],\quad \tau\colon m\rightarrowtail n,\quad \kappa\subseteq\lambda\subseteq n.$$

Theorem 3 (Beyarslan, Hoffman, Onay, P.).

1. *The models of* ACFA *are precisely those difference fields such that*
   - *(a)* ***for all*** *m and n in ω such that $m\leqslant n$,*
   - *(b)* ***for all*** *injective functions τ from m into n,*
   - *(c)* ***for all*** *finite subsets I_0 of $K[X_j\colon j<n]$,*
   - *(d)* ***if***

     I_0 generates

     a prime ideal (I_0) of $K[X_j\colon j<n]$, (∗)

   - *(e)* ***and***

     $$\{f(X_{\tau(i)}\colon i<m)\colon f\in(I_0)\cap K[X_i\colon i<m]\} = (I_0)\cap K[X_{\tau(i)}\colon i<m], \quad (\dagger)$$

(f) ***then*** *the system*

$$\bigwedge_{f\in I_0} f=0 \wedge \bigwedge_{i<m} X_i{}^{\sigma}=X_{\tau(i)} \tag{‡}$$

has a solution in K.

2. FAV *has a model companion,* FAV*, *whose models are just those models* $(K,\sigma,\mathfrak{O})$ *of* FAV *in which*

$$\exists x\; x\notin\mathfrak{O}$$

and,

(a) ***for all*** m *and* n *in* ω *such that* $m\leqslant n$,
(b) ***for all*** *injective functions* τ *from* m *into* n,
(c) ***for all*** *finite subsets* I_0 *of* $\mathfrak{O}[X_j\colon j<n]$,
(d) ***for all*** *subsets* λ *of* n *and* κ *of* λ,
(e) ***if*** $(*)$, *and* $(\dagger)$, *and the set*

$$\mathfrak{M}\cup I_0\cup\{X_k\colon k\in\kappa\}$$

generates a proper *ideal of* $\mathfrak{O}\bigl[I_0\cup\{X_\ell\colon \ell\in\lambda\}\bigr]$,

(f) ***then*** K *contains a common solution to the system* (‡) *and the system*

$$\bigwedge_{\ell\in\lambda} X_\ell\in\mathfrak{O}\wedge\bigwedge_{k\in\kappa} X_k\in\mathfrak{M}.$$

3. FAV$^*\neq$ ACFA $\cup$ ACVF.

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