Spaces and Fields

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Here are notes originally prepared for a talk to be given as part of Istanbul Model Theory Day, April 18, 2016, at the Istanbul Mathematical Sciences Center on the campus of Boğaziçi University. ¶ I submitted a long abstract, given here only as an appendix (Chapter A) because it is not a great summary of what I actually did talk about. In any case, the abstract was reported to be too long for a poster for the event, and so I submitted the following short abstract:

> A field acting on an abelian group produces a vector space. A Lie ring acting on an abelian group produces a differential field. Combining the two actions produces a Lie–Rinehart pair, and we shall look at this from the model-theoretic point of view.

The notes here cover more than this, and more than I expected to discuss in the hour and a half of my talk. ¶ As I said at the beginning of it, the talk continued one of May 13, 2013, called "Descartes as Model Theorist," whose notes I had just revised and expanded [16]. My theme would be model theory as self-conscious mathematics. ¶ I started with the puzzle in Figure 1.2. I compared modern and ancient proofs of Hero's formula, observing that the formula itself involved a product of four lengths. This disturbed Pappus, but apparently not Hero. Pappus was able to state the eight-line locus problem in terms not of a ratio of products of four lengths, but of a product of four *ratios* of lengths. In any case, the ancients could not solve even a five-line locus problem, but Descartes could. ¶ After a break, I sketched the derivation of the properties of a Lie–Rinehart pair, and especially the identities (2.4) and (2.12). I stated Theorem 1, mentioned Theorem 2, defined D-dependence, mentioned Theorems 3 and 4, and gave the short proof of Theorem 5.

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1. Analytic geometry

1.1. Axioms

To Euclidean plane geometry there are two approaches:

1. The so-called synthetic approach of Euclid and followers, whereby the properties of the plane are derived logically from axioms.
2. The "analytic" approach, in which the plane is given explicitly from the beginning, as the set of ordered pairs of real numbers.

In the latter case, there are two approaches to the real numbers:

(a) They may be taken as satisfying the axioms of a complete ordered field.
(b) They may be constructed from the counting numbers.

In the latter case, there are two approaches to the counting numbers:

[i] They may be taken as satisfying the Peano axioms.[1]
[ii] They may be constructed.

In the latter case, rigor and elegance would seem to require a construction like von Neumann's [23] on the basis of the Zermelo–Fraenkel axioms of set theory [25, 19].

[1] I call them Peano axioms by tradition. Peano stated them formally, symbolically, in 1889 [13]; but Dedekind understood their content earlier [1] and better [15].

Thus, be it "synthetic" or "analytic," every approach to geometry is based on axioms.

1.2. Analysis and synthesis

In Greek, **synthesis** (συνθέσις) is building up, or *composition,* in Anglicized Latin loan-translation; **analysis** (ἀνάλυσις) is cutting down, or *dissolution.*

1.2.1. Pappus of Alexandria

In geometry, analysis is assuming what you want to find, then working backwards to see how it can be reached. Synthesis is going forwards to reach it. This is how Pappus of Alexandria explained the terms in the fourth century CE,[2] at the beginning of Book VII of his *Collection* [11, p. 82]:[3]

> That which is called the *Domain of Analysis,* my son Hermodorus, is, taken as a whole, a special resource that was prepared, after the composition of the *Common Elements,*[4] for those who want to acquire a power in geometry that is capable of solving problems set to them; and it is useful for

[2] My handy reference for the dates of ancient mathematicians is the table on the inside back cover of Russo [17].

[3] The passage is also found in Thomas's Loeb anthology [21, p. 597]

[4] It is not clear why Jones capitalizes and italicizes this, as if it were the name of a particular book; for his note on the phrase reads, "That is, common to all branches of mathematics. Pappus sometimes calls Euclid's book the *First Elements* (πρώτα στοιχεῖα). Στοιχεῖα by itself sometimes means Euclid's *Elements* (and στοιχεῖον an individual proposition of the *Elements*), but the word often refers to other books, generally those that are not ends in themselves, but are intended to be used for other works . . . " [12, p. 380].

> this alone. It was written by three men: Euclid the Elementarist, Apollonius of Perge, and Aristaeus the elder, and its approach is by analysis and synthesis.
>
> Now, analysis is the path from what one is seeking, as if it were established, by way of its consequences, to something that is established by synthesis. That is to say, in analysis we assume what is sought as if it has been achieved, and look for the thing from which it follows, and again what comes before that, until by regressing in this way we come upon some one of the things that are already known, or that occupy the rank of a first principle. We call this kind of method 'analysis', as if to say *anapalin lysis* (reduction backward). In synthesis, by reversal, we assume what was obtained last in the analysis to have been achieved already, and, setting now in natural order, as precedents, what before were following, and fitting them to each other, we attain the end of the construction of what was sought. This is what we call 'synthesis'.

Today, in analytic geometry, what we want to find is real numbers, or rather a relation of two of them, which we call x and y; the relation will be expressed by an equation. This is how Descartes (at [2, p. 29] or [3, p. 11]) reformulates **locus problems,** after quoting Pappus on the subject.[5] We may state the general problem as follows.

If some even number $\ell_0, \dots, \ell_{2n-1}$ of straight lines are given in a plane, along with a ratio α, we want to find those points P in the plane such that, if the distance between P and ℓ_i be called d_i, then

$$\frac{d_0 \cdots d_{n-1}}{d_n \cdots d_{2n-1}} = \frac{\prod_{i<n} d_i}{\prod_{i<n} d_{n+i}} = \alpha. \tag{1.1}$$

[5]He quotes Pappus in Latin translation, rather than the original Greek, "afin que chacun l'entende plus aisément" [3, p. 7, n. 1].

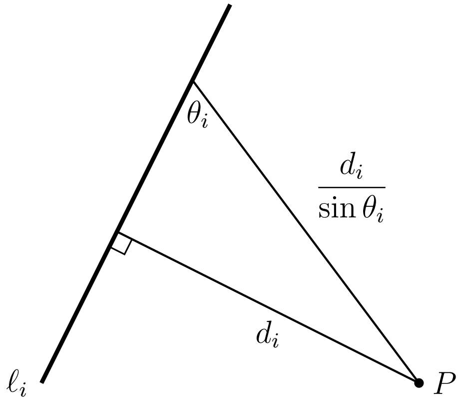


Figure 1.1.: From point to straight line at a given angle

If the number of given straight lines is odd, so that ℓ_{2n-1} is missing, we may replace d_{2n-1} with a given length, or with d_{2n-2}.

For Pappus, d_i might be the distance from P to ℓ_i *along a straight line that meets ℓ_i at a given angle.* This would mean adjusting α by the sine of the angle (Figure 1.1).

Also for Pappus, if $n = 3$, then the problem is to ensure that the ratio of the volumes of two parallelepipeds is α. If $n > 3$, then Pappus does not allow the product $\prod_{i<n} d_i$ to have a meaning. However, he makes two points:

1. Other mathematicians *do* find the product meaningful.
2. The equation (1.1) can be reformulated as

$$\frac{d_0}{d_n} \cdots \frac{d_{n-1}}{d_{2n-1}} = \prod_{i<n} \frac{d_i}{d_{n+i}} = \alpha,$$

   and the product here *does* have a meaning, since it is a product of *ratios,* which are dimensionless. Thus if lengths a, b, c, and d are given, then there is some e

such that $c/d = b/e$, and consequently

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a}{b} \cdot \frac{b}{e} = \frac{a}{e}.$$

Here are Pappus's own words [11, pp. 120–3], which Descartes quotes (the passage immediately preceding is quoted on page 19):

> (36) If three straight lines are given in position, and from some single point straight lines are drawn onto the three at given angles, and the ratio of the rectangle contained by two of the (lines) drawn onto (them) to the square of the remaining one is given, the point will touch a solid locus given in position, that is, one of the three conic curves [γραμμαί]. And if (straight lines) are drawn at given angles onto four straight lines given in positions, and the ratio of the (rectangle contained) by two of the (lines) that were drawn to the (rectangle contained) by the other two that were drawn is given, likewise the point will touch a section of a cone given in position.
>
> (37) Now if (they are drawn) onto only two (lines), the locus has been proved to be plane, but if onto more than four, the point will touch loci that are as yet unknown, but just called 'curves' [γραμμαί], and whose origins and properties are not yet (known). They have given a synthesis of not one, not even the first and seemingly the most obvious of them, or shown it to be useful. (38) The propositions of these (loci) are: If straight lines are drawn from some point at given angles onto five straight lines given in position, and the ratio is given of the rectangular parallelepiped solid contained by three of the (lines) that were drawn to the rectangular parallelepiped solid contained by the remaining two (lines) that were drawn and some given, the point will touch a curve given in position. And if onto six, and the

ratio of the aforesaid solid contained by the three to that by the remaining three is given, again the point will touch a (curve) given in position. If onto more than six, one can no longer say "the ratio is given of the something contained by four to that by the rest", since there is nothing contained by more than three dimensions.

(39) Our immediate predecessors have allowed themselves to admit meaning to such things, though they express nothing at all coherent when they say "the (thing contained) by these", referring to the square of this (line) or the (rectangle contained) by these. But it was possible to enunciate and generally to prove these things by means of compound ratios, both for the propositions given above, and for the present ones, in this way:

(40) If straight lines are drawn from some point at given angles onto straight lines given in position, and there is given the ratio compounded of that which one drawn line has to one, and another to another, and a different one to a different one, and the remaining one to a given, if there are seven, but if eight, the remaining to the remaining one, the point will touch a curve given in position. And similarly for however many, even or odd in number. As I said, of not one of these that come after the locus on four lines have they made a synthesis so that they know the curve.

1.2.2. Hero of Alexandria

Mathematicians who have used products of four lengths include Hero of Alexandria (in the first century CE), in computing the area of a triangle from the lengths of its sides. Hero writes [21, pp. 471–7]:

There is a general method for finding, without drawing a perpendicular, the area (τὸ ἐμβαδόν) of any triangle whose three sides are given. For example, let the sides of the triangle be 7, 8 and 9.

1. Add together 7, 8 and 9; the result is 24.
2. Take half of this, which gives 12.
3. Take away 7; the remainder is 5.
4. Again, from 12 take away 8; the remainder is 4.
5. And again 9; the remainder is 3.
6. Multiply 12 by 5; the result is 60.
7. Multiply this by 4; the result is 240.
8. Multiply this by 3; the result is 720.
9. Take the square root of this and it will be the area of the triangle.[6]

Since 720 has not a rational square root, we shall make a close approximation . . .

The general rule is what we call **Hero's formula:** If a triangle have sides of lengths a, b, and c, so that the semiperimeter s of the triangle be half of $a+b+c$, then the area K of the triangle is given by

$$K=\sqrt{s(s-a)(s-b)(s-c)}.$$

Hero's proof will use the situation of the puzzle in Figure 1.2. What is the radius of the circle, inscribed in a right triangle, whose point of tangency with the hypotenuse divides this into segments of lengths 3 and 10 respectively?[7] If one calls the radius R, then by the Pythagorean Theorem,

$$(R+3)^2+(R+10)^2=13^2,$$

[6] The numbering of the steps is by me.

[7] "Maslanka Puzzles," *Guardian Weekly,* September 6–12, 2013. I had written the puzzle on a sheet of paper that I filed away with some notes on Descartes. Taking out the latter for present purposes, I found the puzzle too.

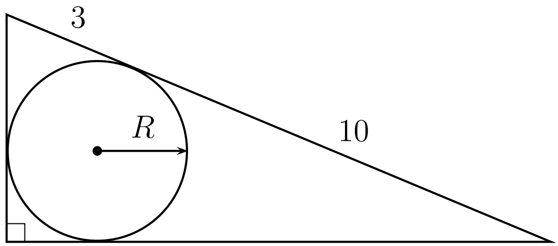


Figure 1.2.: What is the radius of the inscribed circle?

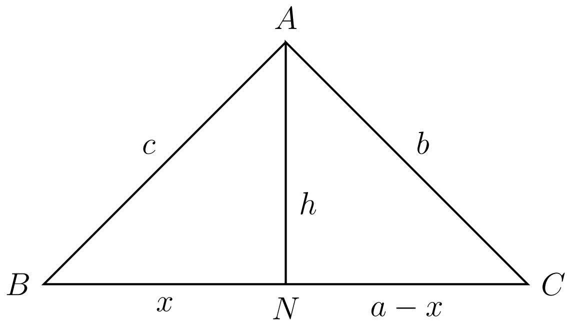


Figure 1.3.: Hero's formula, the modern way

$$2R^2 + 26R + 109 = 169,$$
$$R(R + 13) = 30.$$

Here $R + 13$ is the semiperimeter of the triangle, and so the area of the triangle must be 30.

We can obtain a modern "analytic" proof of Hero's formula, using the diagram of Figure 1.3, by expressing h^2 in terms of a, b, c, and x by means of the Pythagorean Theorem applied to two different triangles.[8] We solve for x and then h, plugging

[8] I follow here the Weeks–Adkins textbook [24, pp. 268–9] from which I was taught geometry in high school. The book was perhaps rigorous enough; but it made me wish we just read Euclid instead.

the result into $4K^2 = a^2h^2$. Details are as follows.

$$\begin{gathered} c^2 - x^2 = h^2 = b^2 - (a-x)^2, \\ c^2 = b^2 - a^2 + 2ax, \\ x = \frac{a^2 - b^2 + c^2}{2a}, \end{gathered}$$

$$\begin{aligned} 4K^2 &= a^2h^2 \\ &= a^2 \cdot (c^2 - x^2) \\ &= a^2 \cdot (c+x)(c-x) \\ &= a^2 \cdot \left(\frac{(a+c)^2 - b^2}{2a}\right)\left(\frac{b^2 - (a-c)^2}{2a}\right) \\ &= \frac{1}{4}(a+b+c)(a-b+c)(a+b-c)(-a+b+c) \\ &= 4s(s-b)(s-c)(s-a). \end{aligned}$$

Note that the sign of $a - x$ is irrelevant. The only "geometry" needed is the Pythagorean Theorem. A completely formulaic approach would be to note that

$$K = \frac{1}{2}ac\sin\beta,$$

while by the Law of Cosines

$$b^2 = a^2 + c^2 - 2ac\cos\beta,$$

so that

$$\begin{aligned} 4K^2 &= a^2c^2\sin^2\beta \\ &= a^2c^2 - a^2c^2\cos^2\beta \\ &= a^2c^2 - \left(\frac{b^2 - a^2 - c^2}{2}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&= \left(ac + \frac{b^2 - a^2 - c^2}{2}\right)\left(ac - \frac{b^2 - a^2 - c^2}{2}\right) \\
&= \frac{b^2 - (a-c)^2}{2} \cdot \frac{(a+c)^2 - b^2}{2} \\
&= 4(s-c)(s-a)s(s-c)
\end{aligned}
$$

as before.

By contrast, Hero's own proof[9] is thoroughly geometrical, in the sense of relying on the diagram, as in Figure 1.4.

1. Triangle $AB\Gamma$ is given.
2. H is the center of the inscribed circle.
3. Δ, E, and Z are the feet of the perpendiculars dropped from H to the sides of $AB\Gamma$.
4. The area of $AB\Gamma$ is twice the triangles $B\Gamma H$ and $A\Delta H$.
5. ΓB is extended to Θ so that $B\Theta = A\Delta$.
6. The area of $AB\Gamma$ is $\Gamma\Theta \cdot EH$.
7. Angles $\Gamma H\Lambda$ and $\Gamma B\Lambda$ are right.
8. $\Gamma HB\Lambda$ is a circle.
9. Angle $B\Lambda\Gamma$ is supplementary to $BH\Gamma$.
10. So is $AH\Delta$.
11. Thus angles $B\Lambda\Gamma$ and $AH\Delta$ are equal.
12. By similar triangles, one has the computations given in Figure 1.4.

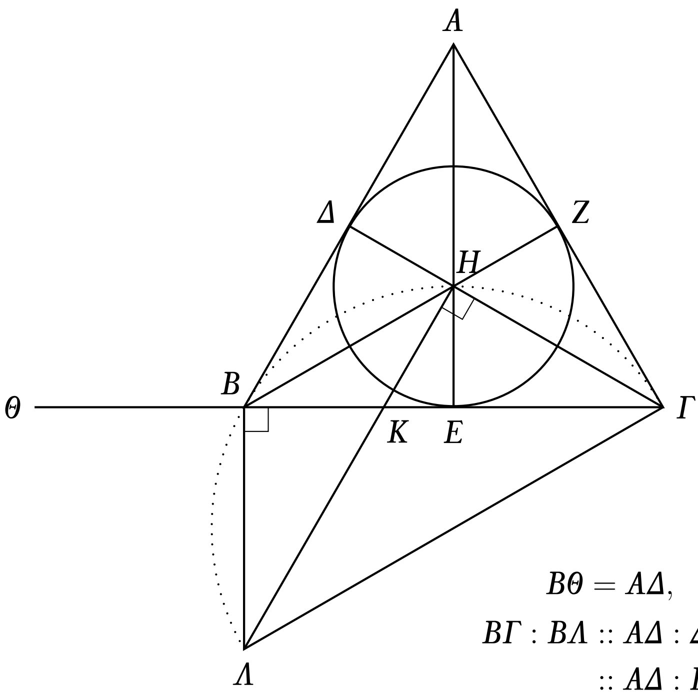
In Cartesian style, we might assign minuscule letters:

$$a = B\Gamma, \qquad b = A\Gamma, \qquad c = AB.$$

Letting $s = \frac{1}{2}(a+b+c)$ as before, we have

$$A\Delta = s - a, \qquad BE = s - b, \qquad E\Gamma = s - c,$$

[9] At least, the proof given by Hero; "Heron's formula . . . is now known from Arabian sources to have been discovered by Archimedes" [21, p. 477, n. *a*].



$$
\begin{aligned}
B\Theta &= A\Delta, \\
B\Gamma : B\Lambda &:: A\Delta : \Delta H \\
&:: A\Delta : EH, \\
B\Gamma : A\Delta &:: B\Lambda : EH \\
&:: BK : KE, \\
\Gamma\Theta : A\Delta &:: BE : KE, \\
\Gamma\Theta^2 : \Gamma\Theta \cdot A\Delta &:: BE \cdot E\Gamma : KE \cdot E\Gamma \\
&:: BE \cdot E\Gamma : EH^2, \\
\Gamma\Theta^2 \cdot EH^2 &= \Gamma\Theta \cdot A\Delta \cdot BE \cdot E\Gamma, \\
AB\Gamma &= \sqrt{\Gamma\Theta \cdot A\Delta \cdot BE \cdot E\Gamma}.
\end{aligned}
$$

Figure 1.4.: Hero's formula, his way

so that, if we define

$$d = B\Lambda, \qquad e = BK, \qquad f = KE, \qquad h = EH,$$

we can argue as in Figure 1.5. The manipulations are more transparent, though not of a kind that we are used to; and they require several references to the diagram. It may be that Hero's proof is better for somebody who can look at the diagram, but has no ready supply of paper for computations.

1.2.3. René Descartes

The **conic sections** are just that: the intersections of planes with cones. It turns out that every conic section, for some choice of rectangular coordinate system (including choice of unit length), for some positive ratio e, can be given by the equation

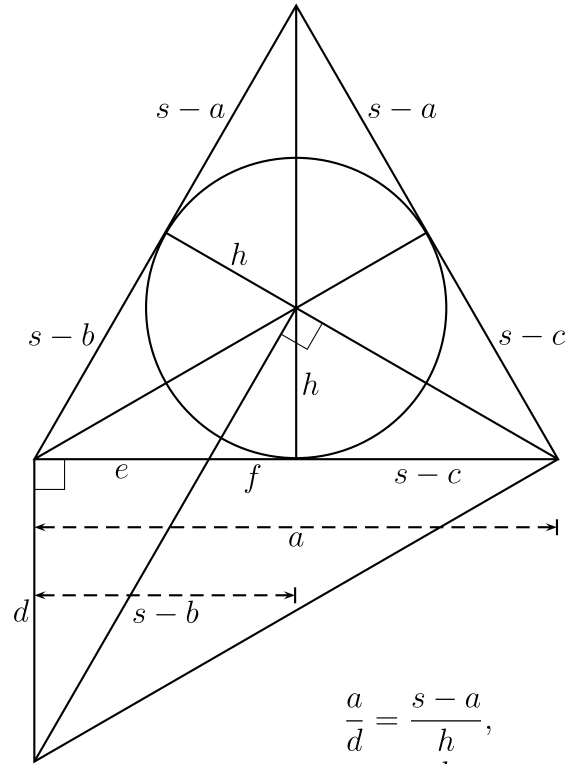
$$x^2 + y^2 = e^2 \cdot (x + 1)^2$$

(see Figure 1.6). The conic section is then the locus of points, the distances of each of which from the **focus** $(0, 0)$ and from the **directrix** $x + 1 = 0$ have the constant ratio e.

Instead of using three dimensions to define conic sections as such, some modern books use the focus-directrix property as a definition of what are still called the conic sections. An example is the textbook of Karakaş [4], the official reference for a course of analytic geometry that (with two senior colleagues) I was assigned to teach at METU in the fall semester of 2007–8.

We can infer from Pappus that Euclid knew of the focus-directrix property;[10] but in this case the property is a theorem, not a definition.

[10] See [20, pp. 492–503, esp. p. 495, n. *a*], [11, pp. 362–71], and [12, pp. 503–7 & 591–5].



$$\frac{a}{d} = \frac{s-a}{h},$$
$$\frac{a}{s-a} = \frac{d}{h} = \frac{e}{f},$$
$$\frac{s}{s-a} = \frac{e+f}{f} = \frac{s-b}{f},$$
$$\frac{s^2}{s \cdot (s-a)} = \frac{(s-b)(s-c)}{f \cdot (s-c)} = \frac{(s-b)(s-c)}{h^2},$$
$$s^2h^2 = s \cdot (s-a)(s-b)(s-c).$$

Figure 1.5.: Hero's formula, Cartesian style

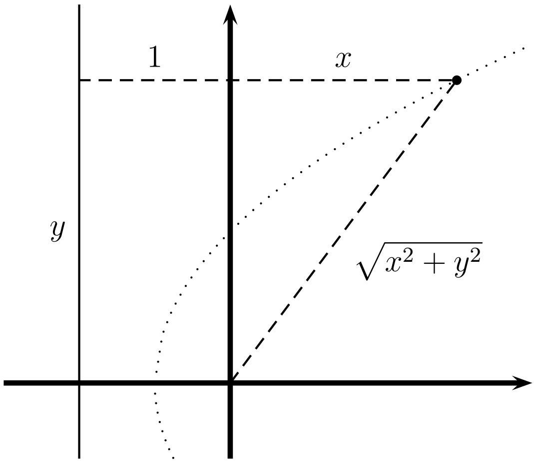


Figure 1.6.: A conic by the focus-directrix property

The conic sections are also the solutions of the four-line locus problem. Descartes works this out, though Pappus notes that it was known to Apollonius, and partially to Euclid before him. Just before the long passage on locus problems quoted earlier, Pappus castigates Apollonius because, instead of acknowledging the value of what Euclid did do on the locus problem, he boasts of having done better. Here is what Pappus says [11, pp. 116–21]:

> (32) . . . Apollonius says what the eight books of Conics that he wrote contain, placing a summary prospectus in the preface to the first, as follows: "The first contains the generation of the three sections and the opposite branches, and their fundamental *symptomata,*[11] more fully and more

[11] "σύμ-πτωμα, ατος, τό: *Anything that happens, a chance, occurence* . . . II. *property, attribute* . . . 2. Geom., *property,* of a curve, etc . . . " [8, p. 1686]. By transliterating the Greek word, and not simply translating it as "property," Jones seems to take it as a technical term for

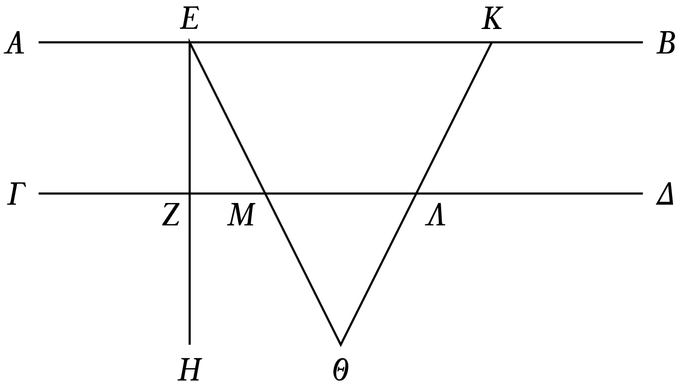


Figure 1.7.: A cutting-off of a ratio

thoroughly examined than in the writings of others . . . The third (has) many and various useful things, which are both for syntheses of solid loci, and for (their) diorisms;[12] and having found most of them both elegant and novel, we found that the synthesis of the locus on three and four lines was

what Pappus referred to earlier, in (30), as a συμβεβήκος of each of the ellipse, hyperbola, and parabola: namely that, respectively, "a certain area [χωρίον] applied to a certain line . . . falls short [ἐλλεῖπον] by a square, . . . exceeds [ὑπερβάλλον] by a square, . . . neither falls short nor exceeds."

[12]Jones defines a diorism as, "the conditions of possibility and number of solutions of a problem" [11, p. 67]. He cites an example from *The Cutting off of a Ratio* of Apollonius [12, pp. 606–7]. See Figure 1.7, where it is given that $AB \parallel \Gamma\Delta$ and the point Θ lies within angle ΔZH. A ratio is given, and we are set the problem of finding K on EB so that the ratio of EK to $Z\Lambda$ is the given ratio. Since for any choice of K, $EK : Z\Lambda < E\Theta : \Theta M$, it must be that the given ratio is less than that of $E\Theta$ to ΘM: this is the diorism of the problem. A simpler example is found in Proposition I.22 of Euclid's *Elements,* which is the problem of constructing a triangle with given sides: the diorism is that any two of the sides must be greater than the third [12, p. 382].

not made by Euclid, but (merely) a fragment of it, nor this felicitously. For one cannot complete the synthesis without the things mentioned above . . . ”

(33) Thus Apollonius. The locus on three and four lines that he says, in (his account of) the third (book), was not completed by Euclid, neither he nor anyone would have been capable of; no, he could not have added the slightest thing to what was written by Euclid, using only the conics that had been proved up to Euclid's time, as he himself confesses when he says that it is impossible to complete it without what he was forced to write first. (34) But either Euclid, out of respect for Aristaeus as meritorious for the conics he had published already, did not anticipate him, or, because he did not desire to commit to writing the same matter as he (Aristaeus),—for he was the fairest of men, and kindly to everyone who was the slightest bit able to augment knowledge, as one should be, and he was not at all belligerent, and though exacting, not boastful, the way this man (Apollonius) was,—he wrote (only) as far as it was possible to demonstrate the locus by means of the other's *Conics,* without saying that the demonstration was complete. For had he done so, one would have had to convict him, but as things stand, not at all. And in any case, (Apollonius) himself is not castigated for leaving most things incomplete in his *Conics.* (35) He was able to add the missing part to the locus because he had Euclid's writings on the locus already before him in his mind, and had studied for a long time in Alexandria under the people who had been taught by Euclid, where he also acquired this so great condition (of mind), which was not without defect.

This locus on three and four lines that he boasts of having augmented instead of acknowledging his indebtedness to the first to have written on it, is like this . . .

Here follows the earlier quotation, where Pappus says in particular,

> of not one of these [locus problems] that come after the locus on four lines have they made a synthesis so that they know the curve.

Descartes does solve a particular five-line locus problem, as depicted in Figure 1.8, where GF, ED, AB, and IH are parallel and evenly spaced, and GI is perpendicular to them. We want to find C so that

$$CF \cdot CD \cdot CH = CB \cdot CM \cdot AI.$$

We let

$$CM = x, \qquad CB = y, \qquad AI = AE = GE = a,$$

so that

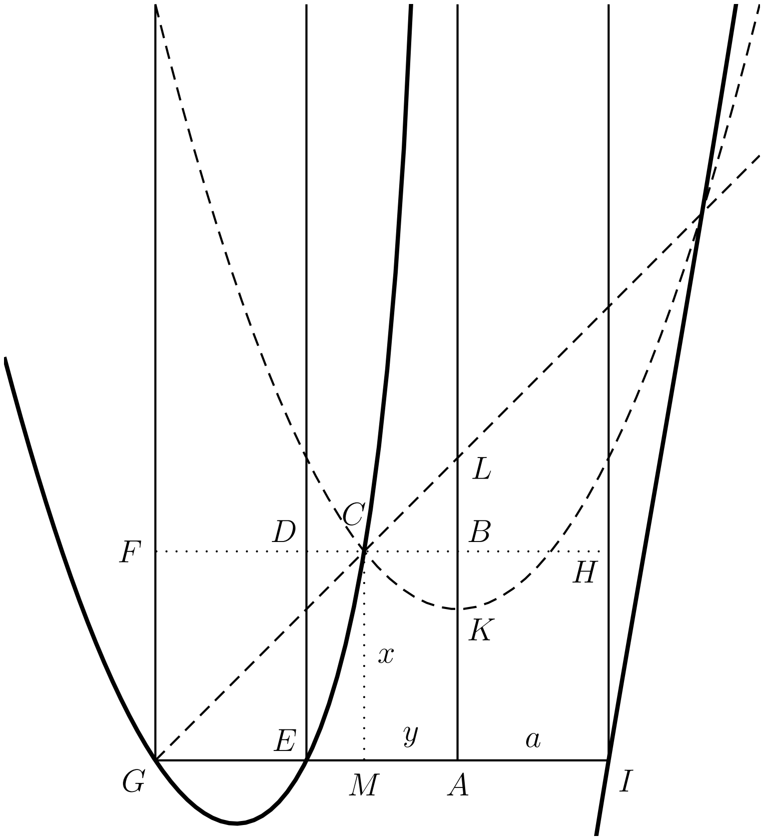
$$CF = 2a - y, \qquad CD = a - y, \qquad CH = y + a.$$

Then

$$\begin{aligned} axy &= (2a - y)(a - y)(y + a) \\ &= (y - 2a)(y^2 - a^2) \\ &= y^3 - 2ay^2 - a^2y + 2a^3. \end{aligned} \tag{1.2}$$

Does merely writing this equation mean that we have solved the locus problem? Note that we can solve the equation for x. In any case, following Descartes, we show that the equation defines the intersection of

- a parabola whose upright side (*latus rectum*) is a, whose axis is AB, and whose vertex is a point K, which slides along AB; and
- a straight line passing from G through the point L, which is on the axis of the parabola at a distance a from K.



$$x = \frac{y^2}{a} - 2y - a + \frac{2a^2}{y}$$

Figure 1.8.: A five-line locus problem

We shall show that the intersection of the parabola and the straight line does indeed solve the problem. This would seem to be the “synthesis” of the problem.

We establish two expressions for ***BK*** and show that their equation is just (1.2). By similar triangles,

$$GM : MC :: CB : BL,$$

that is,

$$\frac{2a-y}{x} = \frac{y}{BL},$$

and therefore

$$BL = \frac{xy}{2a-y}, \qquad \begin{aligned} BK &= KL - BL \\ &= a - \frac{xy}{2a-y} \\ &= \frac{2a^2 - ay - xy}{2a-y}. \end{aligned}$$

Also, by the *symptoma* of the parabola,

$$BK : BC :: BC : a, \qquad BK = \frac{y^2}{a}.$$

Equating the two forms of BK gives

$$\frac{2a^2 - ay - xy}{2a-y} = \frac{y^2}{a},$$
$$2a^3 - a^2y - axy = 2ay^2 - y^3,$$

which is equivalent to (1.2).

For Pappus, I think, just finding (1.2) does not solve the locus problem. For us, perhaps it does, since it gives us the tools for analyzing curves that we learn in calculus. In fact the equation already has geometric meaning, at least implicitly, since Descartes has shown that the Euclidean plane interprets the field operations given by the equation. In this way, field theory is geometry.

One can go the other way, using an ordered field to construct a Euclidean plane. Thus geometry is field theory.

Textbooks of today, such as the cited book of Karakaş [4], show some confusion over these points.

1.3. The Cantor–Dedekind Axiom

In a book from 1953 called *Fundamental Concepts of Algebra,* after reviewing the construction of the real numbers on the foundation of the Peano postulates, Bruce Meserve states the "Cantor–Dedekind axiom" [9, p. 32]:

> To each point of the line there corresponds one and only on real number and, conversely, to each real number there corresponds one and only one point of the line.

Meserve's purpose seems to be to establish what *continuity* of a line or curve is. Meserve will later give the ε–δ definition of continuity of real functions. In his *Fundamental Concepts of Geometry,* he says [10, p. 86]:

> In euclidean geometry we have the following theorem:
>
> > Cantor–Dedekind Theorem: *There exists an isomorphism between the set of real numbers and the set of points on a line in euclidean geometry.*
>
> This isomorphism enables us to use the set of real numbers as coordinates of the points on a euclidean line.

No proof is offered; but five pages later, the assertion is repeated as an axiom:

> Postulate of Continuity.

> P-14: *There exists a projective line m containing a set of points P_r isomorphic with the set of numbers in the extended real number system.*

This is the last axiom for a projective geometry in a geometry book that treats "synthetic" projective geometry first; only later does one see analytic projective geometry, and then affine and Euclidean geometry.

As of April 5, 2016, Meserve's two books were cited in the *Wikipedia* article, "Cantor–Dedekind axiom." According to this short article,

> This axiom is the cornerstone of analytic geometry. René Descartes implicitly assumes this axiom . . .

Similarly, Karakaş alludes to this "axiom" as "the fundamental principle of analytic geometry" [4, §1.5, pp. 15–6]. The isomorphism between real numbers and points in each case is, at best, an isomorphism of *ordered abelian groups.*

1.4. The real fundamental principle

Addition itself is fundamental to both geometry and arithmetic. To place two line segments end to end, and to combine two sets into one, are basic mathematical activities with an obvious correspondence. at least the two activities correspond, if each of the line segments is a *number* in the sense of Euclid, meaning it is a multitude of units.

If we conceive of a number as a row of dots, then we can multiply it by stacking new rows on top. This is like erecting a rectangle on a given line segment. See Figure 1.9. However, an array of dots, like a row of dots, is still a *set* of dots. But a rectangle is a completely different kind of thing from a straight line.

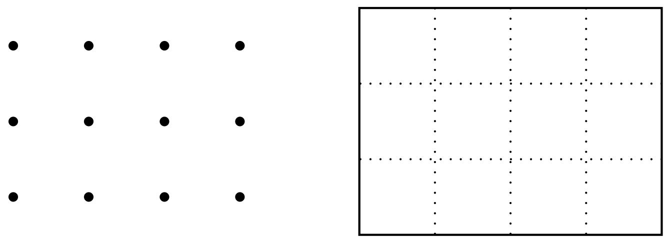


Figure 1.9.: Multiplication

The true fundamental principle of analytic geometry is that the fixing of two points on an infinite straight line can be shown *geometrically* to determine a field structure in which $\mathbb{Q}$ embeds.

If the line is continuous, then it will be isomorphic to $\mathbb{R}$; but this is not essential for geometry. What is essential for Cartesian geometry is that there be a multiplication on the line with a geometric meaning.

Descartes shows how this meaning arises from a theory of proportion, though he does not give details. In fact Proposition I.43 of Euclid's *Elements* is enough. This is the proposition according to which, if a parallelogram is divided into four, then two non-adjacent parts are equal if the diagonals of the other two are in a straight line. See "Descartes as Model Theorist" [16].

2. Lie–Rinehart pairs

By a **differential field,** let us understand a field K equipped with a set V of derivations that is closed with respect to

(1) addition and subtraction,
(2) scaling by elements of the field, and
(3) the Lie bracket.

In particular, V is a vector space over K; we shall assume further that it is nontrivial. We are going to consider the pair (K, V) as a two-sorted structure, called a **Lie–Rinehart pair.** In a typical example,

$$K = \mathbb{C}(t^0, \dots, t^{m-1}), \qquad V = \operatorname{span}_K \left(\frac{\partial}{\partial t^0}, \dots, \frac{\partial}{\partial t^{m-1}} \right),$$

where the t^i are algebraically independent over $\mathbb{C}$. We shall use minuscule Latin letters for elements of K; capitals, for V. Thus

$$a, b, x, y \in K, \qquad\qquad D, E, X, Y \in V.$$

We are going to require K to be a field. Others may not do so. They may assume simply that K is an associative algebra over some field, but that V is a Lie algebra over the field. We have such a field, namely the prime field of K.

2.1. Analysis

Algebraic properties of Lie–Rinehart pairs (K, V) are as follows.

2.1.1. Abelian groups, mutually acting

Both K and V are nontrivial additive abelian groups. Moreover, they act on each other as groups; that is, there are maps

$$(x, Y) \mapsto xY, \qquad (X, y) \mapsto Xy$$

from $K \times V$ to V and from $V \times K$ to K respectively that are additive in each argument, so

$$\left.\begin{aligned} (a+b)(D+E) &= aD + bD + aE + dE, \\ (D+E)(a+b) &= Da + Db + Ea + Eb. \end{aligned}\right\} \tag{2.1}$$

Since the endomorphisms of an abelian group compose another abelian group, we have homomorphisms

- $x \mapsto (Y \mapsto xY)$ from K to $\mathrm{End}(V)$,
- $X \mapsto (y \mapsto Xy)$ from V to $\mathrm{End}(K)$.

The latter map is literally an inclusion. All four structures here are to be understood implicitly as abelian groups,—and only that, so far.

2.1.2. Action of an associative ring

K is also equipped with a *multiplication,* denoted by $\times$ or $\cdot$ or juxtaposition. By the definition that I shall use, a **multiplication** on an abelian group is just a binary operation that distributes over addition (in both senses); a **ring** is then an abelian group with a multiplication. We have now a homomorphism

$$x \mapsto \widetilde{x}$$

from K to $\mathrm{End}(K)$, given by

$$\widetilde{a}b = a \cdot b. \tag{2.2}$$

The homomorphism is an embedding, since $\times$ has an identity, 1, which is not 0 (since K is nontrivial), and also $\times$ has inverses.

Since $\times$ is associative, we have

$$\widetilde{a \cdot b}x = (a \cdot b) \cdot x = a \cdot (b \cdot x) = \widetilde{a}(\widetilde{b}x) = (\widetilde{a} \circ \widetilde{b})x,$$

and thus the map $x \mapsto \widetilde{x}$ is an embedding of the associative ring $(K, \times)$ in the associative ring $(\mathrm{End}(K), \circ)$.

In fact $(K, \times)$ is a commutative ring:

$$a \cdot b = b \cdot a. \tag{2.3}$$

We shall not use this further, but it will be analogous to (2.8). We shall use that $(K, \times)$ is a division ring (hence a field).

K acts on V not only as an abelian group, but as an associative ring:

$$1D = D, \qquad (a \cdot b)D = a(bD).$$

That is, the map $x \mapsto (Y \mapsto xY)$ is a homomorphism from the associative ring $(K, \times)$ to the associative ring $(\mathrm{End}(V), \circ)$. Thus V is a module over K, in fact a vector space, since K is a field (or just a division ring). Since V is nontrivial, the homomorphism from K to $\mathrm{End}(V)$ is an embedding.

In short,

$$\begin{aligned} x \mapsto \widetilde{x} &: (K, \times) \rightarrowtail (\mathrm{End}(K), \circ), \\ x \mapsto x &: (K, \times) \rightarrowtail (\mathrm{End}(V), \circ). \end{aligned}$$

We may thus consider an element a of K as an element of $\mathrm{End}(V)$; but as an element of $\mathrm{End}(K)$, it is called $\widetilde{a}$.

2.1.3. Action of a space of derivations

V acts on K not only as an abelian group, but as a vector space over K. Thus

$$(aD)b = a \cdot (Db) = \widetilde{a}(Db).$$

We can write this more simply as

$$aD = \widetilde{a} \circ D. \tag{2.4}$$

This means the diagram of Figure 2.1 commutes.

K is acted on by V not only as an abelian group, but as a ring. In particular,

$$D(a \cdot b) = (Da) \cdot b + a \cdot Db, \tag{2.5}$$

that is, V acts on $(K, \times)$ not only as a space, but as a space of **derivations.** By (2.2), namely $\widetilde{a}b = a \cdot b$, we can write (2.5) as

$$D(\widetilde{a}b) = \widetilde{Da}b + \widetilde{a}(Db),$$

which means $D \circ \widetilde{a} = \widetilde{Da} + \widetilde{a} \circ D$ and hence

$$\widetilde{Da} = D \circ \widetilde{a} - \widetilde{a} \circ D. \tag{2.6}$$

This, along with being in $\mathrm{End}(K)$, is what it means for D to be a derivation of $(K, \times)$.

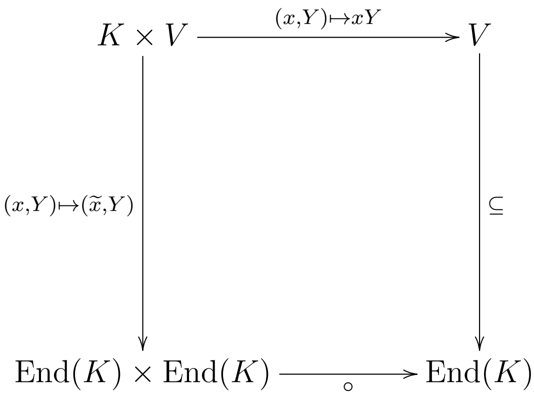


Figure 2.1.: Commutative diagram for $aD = \widetilde{a} \circ D$

2.1.4. Lie multiplication

The abelian group $\mathrm{End}(K)$ is closed not only under functional composition, but also under the bracket, $[\quad,\quad]$ or b, given by

$$[\varphi, \psi] = \varphi \,\mathsf{b}\, \psi = \varphi \circ \psi - \psi \circ \varphi.$$

Since composition is a multiplication on $\mathrm{End}(K)$, so is the bracket. Because of the **Jacobi identity** for the bracket on $\mathrm{End}(K)$, namely

$$\big[[\varphi, \psi], \chi\big] = \big[\varphi, [\psi, \chi]\big] - \big[\psi, [\varphi, \chi]\big], \tag{2.7}$$

as well as anticommutativity,

$$[\varphi, \psi] + [\psi, \varphi] = 0, \tag{2.8}$$

$(\mathrm{End}(K), \mathsf{b})$ is by definition a **Lie ring.**

Theorem 1. *For any abelian group* R*, on* $\operatorname{End}(R)$*, any integral combination*

$$(\varphi, \psi) \mapsto p\,\varphi \circ \psi - q\,\psi \circ \varphi$$

of composition and its converse is a multiplication, m*. The homomorphism*

$$\varphi \mapsto (\psi \mapsto \varphi \mathbin{\mathsf{m}} \psi)$$

from $\operatorname{End}(R)$ *to* $\operatorname{End}(\operatorname{End}(R))$ *preserves* m *for all* R*, that is,*

$$(\varphi \mathbin{\mathsf{m}} \psi) \mathbin{\mathsf{m}} \chi = p\,\varphi \mathbin{\mathsf{m}} (\psi \mathbin{\mathsf{m}} \chi) - q\,\psi \mathbin{\mathsf{m}} (\varphi \mathbin{\mathsf{m}} \chi),$$

if and only if m *is trivial, or composition, or the bracket, that is,*

$$(p, q) \in \{(0,0), (1,0), (1,1)\}.$$

The proof is a computation. I stated the result at the Logic Colloquium in Athens, 2005; I have not found it anywhere else.

2.1.5. Action of a Lie ring

We can now rewrite (2.6), namely $\widetilde{Da} = D \circ \widetilde{a} - \widetilde{a} \circ D$, as

$$\widetilde{Da} = [D, \widetilde{a}]. \tag{2.9}$$

This means the diagram in Figure 2.2 is commutative.

The bracket of any two derivations of K is a derivation of K. Indeed, in V we have

$$\begin{aligned}
\widetilde{[D,E]a} &= \widetilde{D(Ea)} - \widetilde{E(Da)} && [x \mapsto \widetilde{x} \text{ is additive}]\\
&= [D, \widetilde{Ea}] - [E, \widetilde{Da}] && [\text{by } (2.9)]\\
&= \big[D, [E, \widetilde{a}]\big] - \big[E, [D, \widetilde{a}]\big] && [\text{by } (2.9)]
\end{aligned}$$

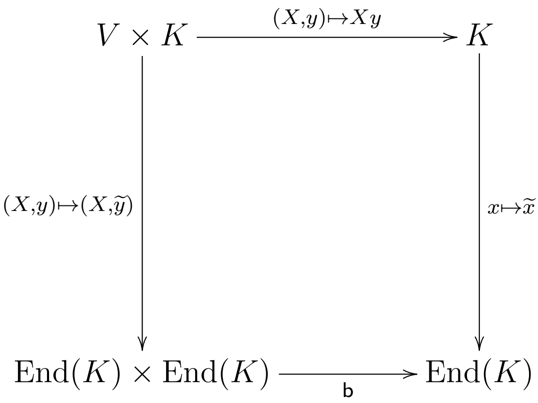


Figure 2.2.: Commutative diagram for $\widetilde{Da} = [D, \widetilde{a}]$

$$= \bigl[[D, E], \widetilde{a}\bigr], \qquad \text{[by the Jacobi identity]}$$

so (2.9) holds with $[D, E]$ for D. This proof is a bit shorter than checking

$$[D, E](a \cdot b) = ([D, E]a) \cdot b + a \cdot \bigl([D, E]b\bigr).$$

(The same number of equations is required, but the members are shorter.) Our example of V is closed under the bracket by (2.11) below, since the bracket of any two basis elements is 0, and so the bracket of any two elements of V must be a linear combination of basis elements.

Since V is closed under the bracket, there is a homomorphism

$$X \mapsto \widetilde{X}$$

from V to $\mathrm{End}(V)$, where

$$\widetilde{D}E = [D, E]. \tag{2.10}$$

The Jacobi identity (2.7) for the bracket on V can now be written as

$$\widetilde{[D,E]} = \left[\widetilde{D}, \widetilde{E}\right].$$

In particular, the map $X \mapsto \widetilde{X}$ is a homomorphism from (V, b) to $(\mathrm{End}(V), \mathsf{b})$.

In general,

$$\begin{aligned}
\widetilde{D}(aE) &= [D, aE] && \text{[by (2.10)]}\\
&= D \circ (aE) - (aE) \circ D\\
&= D \circ \tilde{a} \circ E - \widetilde{a} \circ E \circ D && \text{[by (2.4)]}\\
&= \widetilde{Da} \circ E + \widetilde{a} \circ D \circ E - \widetilde{a} \circ E \circ D && \text{[by (2.6)]}\\
&= \widetilde{Da} \circ E + \widetilde{a} \circ [D, E]\\
&= (Da)E + a[D, E]. && \text{[by (2.4)]}
\end{aligned}$$

Thus,

$$\widetilde{D}(aE) = (Da)E + a[D, E]. \tag{2.11}$$

In particular,

$$\widetilde{D}(aD) = (Da)D,$$

and hence if $D \neq 0$, so that $Da \neq 0$ for some a in K, then $\widetilde{D} \neq 0$. Thus the map $X \mapsto \widetilde{X}$ *embeds* (V, b) in $(\mathrm{End}(V), \mathsf{b})$.

In short,

$$\begin{aligned}
X \mapsto \widetilde{X} &\colon (V, \mathsf{b}) \rightarrowtail (\mathrm{End}(V), \mathsf{b}),\\
X \mapsto X &\colon (V, \mathsf{b}) \rightarrowtail (\mathrm{End}(K), \mathsf{b}).
\end{aligned}$$

2.1.6. Correlative identities

We can rewrite (2.11) as

$$(\widetilde{D} \circ a)E = (Da)E + (a \circ \widetilde{D})E$$

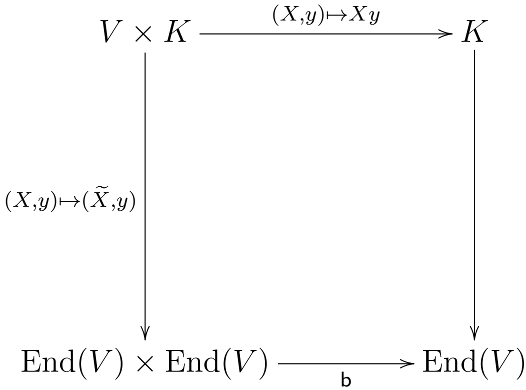


Figure 2.3.: Commutative diagram for $Da = [\widetilde{D}, a]$

and thus

$$Da = [\widetilde{D}, a]. \tag{2.12}$$

This means the diagram in Figure 2.3 commutates. We can also rewrite (2.12) as

$$\widetilde{D} \circ a - Da = a \circ \widetilde{D}. \tag{2.13}$$

We have now obtained the four correlative identities (2.12), (2.4), (2.9), and (2.13), namely

$$\begin{aligned} Da &= [\widetilde{D}, a], & aD &= \widetilde{a} \circ D, \\ \widetilde{Da} &= [D, \widetilde{a}], & \widetilde{D} \circ a - Da &= a \circ \widetilde{D}. \end{aligned}$$

We have *not* got $\widetilde{aD} = a \circ \widetilde{D}$. Rather, using (2.11), we obtain

$$\begin{aligned} \widetilde{aD}E &= [aD, E] = -[E, aD] = -\widetilde{E}(aD) \\ &= -\big((Ea)D + a[E, D]\big) = a[D, E] - (Ea)D \end{aligned}$$

$$= (a \circ \widetilde{D})E - (Ea)D.$$

However, as we derived (2.13) from (2.12), so we can obtain (2.9) from (2.12) and (2.4), after observing that every element of K is a derivative. Indeed, if $Da \neq 0$, then

$$\left(\frac{b}{Da}D\right)a = b.$$

We now have

$$\begin{aligned}
\widetilde{Da} \circ E &= (Da)E \\
&= [\widetilde{D}, a]E && \text{[by (2.12)]} \\
&= [D, aE] - a[D, E] \\
&= [D, aE] - \widetilde{a} \circ [D, E] && \text{[by (2.4)]} \\
&= [D, aE] - \widetilde{a} \circ D \circ E + \widetilde{a} \circ E \circ D \\
&= D \circ (aE) - \tilde{a} \circ D \circ E && \text{[by (2.4)]} \\
&= [D, \widetilde{a}] \circ E,
\end{aligned}$$

and therefore (2.9).

2.2. Synthesis

We now observe that all of the foregoing properties of (K, V) can be expressed by $\forall\exists$ axioms in a two-sorted language. As before, we shall write minuscule letters for the variables of the **scalar** sort, K; capital letters, for the variables of **vector** sort, V. Each sort will have symbols for the standard operations on an abelian group; we can use the standard symbols $+$, $-$, and 0. There will be symbols for the left action of each sort on the other; we can use juxtaposition to denote these actions.

But we shall not use symbols for multiplication within the sorts. This means abD is unambiguously $a(bD)$, while DEa is $D(Ea)$.

1. Universal axioms make each sort an abelian group.
2. Now universal axioms ensure that these actions are actions of abelian groups on abelian groups. Thus we obtain the identities (2.1).
3. $\forall\exists$ axioms ensure that K is a field acting on V, which is non-trivial:

$$\begin{gathered}\exists X\ 0 \neq X,\\ aD = 0 \Rightarrow a = 0 \vee D = 0,\\ abD = baD,\\ \exists x\ abD = xD,\\ \exists x\ D = xD,\\ \exists x\ (D = xaD \vee a = 0).\end{gathered}$$

   Thus V becomes a vector space over K.
4. That the action of V on K is that of a vector space is given by (2.4), namely $aD = \widetilde{a} \circ D$, hence by

$$(aD)b = (\widetilde{a} \circ D)b.$$

   Since K now embeds in $\mathrm{End}(V)$, we express the identity by

$$\big((aD)b\big)E = a\big((Db)E\big).$$

5. Within the class of models of the axioms given so far, embeddings preserve multiplication of scalars. So, when we seek to identify the existentially closed members of the class, there is no harm in assuming we have a symbol for multiplication of scalars.

6. The same will not be true for the bracket of vectors. We express the faithfulness of the action of the vectors by an $\forall\exists$ axiom:

$$\forall X\ \exists y\ (Xy \neq 0 \vee X = 0).$$

7. As (2.12), namely $Da = [\widetilde{D}, a]$ is expressed by

$$(Da)E = [\widetilde{D}, a]E,$$

we can now express it in our restricted language by computing

$$\begin{aligned}\big((Da)E\big)b &= \big([\widetilde{D}, a]E\big)b \\ &= \big(\widetilde{D}(aE) - a(\widetilde{D}E)\big)b \\ &= \big([D, aE] - a[D, E]\big)b \\ &= \big([D, \widetilde{a} \circ E] - \widetilde{a} \circ [D, E]\big)b \\ &= (D \circ \widetilde{a} \circ E - \widetilde{a} \circ D \circ E)b \\ &= \big(D \circ (aE) - (aD) \circ E\big)b,\end{aligned}$$

and so

$$\big((Da)E\big)b = D\big((aE)b\big) - (aD)(Eb).$$

2.3. Theory

2.3.1. Existentially closed models

Let the theory of Lie-Rinehart pairs (in our sense) of characteristic 0 be LR_0. In any existentially closed model of LR_0, the constant field, defined by the formula

$$\forall Y\ (Yx = 0),$$

is the separable closure of the constant field. Therefore LR_0 is not companionable.

Nonetheless, from Özcan Kasal's 2010 doctoral dissertation at METU [5] we have the next theorem. Given a model (K, V) of LR_0, we shall use the notational convention whereby

- an element $\boldsymbol{a}$ of K^m is a row vector $\begin{pmatrix} a_0 & \cdots & a_{m-1} \end{pmatrix}$,
- an element $\boldsymbol{D}$ of V^m is a column vector $\begin{pmatrix} D^0 \\ \vdots \\ D^{m-1} \end{pmatrix}$,
- application of tuples of derivations to tuples of scalars is evaluated like matrix multiplication, so that, for example, $\boldsymbol{Da}$ is the matrix $(D^i a_j)^{i<m}_{j<m}$.

Theorem 2 (Kasal, 2010)**.** *A model (K, V) of LR_0 is existentially closed if and only if*

(a) for all m in $\mathbb{N}$, for all algebraically independent $\boldsymbol{a}$ in K^m, for all $\boldsymbol{b}$ in K^m, for some D in V,

$$D\boldsymbol{a} = \boldsymbol{b}$$

—this condition is not first-order;

(b) K is of infinite transcendence degree: this is implied by—and by the previous condition it implies—that for all m in $\mathbb{N}$, for some $\boldsymbol{a} \in K^m$, for some $\boldsymbol{D}$ in V^m,

$$\boldsymbol{Da} = I_m, \tag{2.14}$$

so that $(a_j : j < m)$ is algebraically independent;

(c) for all m in $\mathbb{N}$, for all linearly independent $\boldsymbol{D}$ in V^m, the structure $(K, \boldsymbol{D})$ is an existentially closed model of the theory of fields of characteristic 0 with m derivations having no required interaction; those models compose an elementary class, which can be axiomatized on the pattern for DCF_0 given in [14].

2.3.2. Differential dependence

In a Lie–Rinehart pair (K, V), let us say that a scalar b is **differentially dependent** or **D-dependent** on a set A of scalars if, for some m in ω, for some $\boldsymbol{a}$ in A^m,

$$\forall Y\ (Y\boldsymbol{a} = \mathbf{0} \Rightarrow Yb = 0).$$

Then algebraic dependence implies D-dependence.

Theorem 3 (Kasal, 2010)**.** *D-dependence makes the scalar field of a Lie–Rinehart pair a pregeometry. In particular, the closure of every subset of K under D-dependence has a basis, and all bases are equipollent, so that the subset has a **D-dimension.***

Proof. D-dependence is

finitary: dependence on A means dependence on a finite subset of A, by definition;

increasing: every element of A depends on A;

monotone: if $A \subseteq B$ and c depends on A, then it depends on B;

idempotent: if everything in B depends on A, and c depends on B, then c depends on A;

Finally, D-dependence allows **exchange.** For suppose c depends on $A \cup \{b\}$, but not A. We show b depends on $A \cup \{c\}$. Under the two hypotheses:

(1) for some m in ω, for some $\boldsymbol{a}$ in A^m, for all derivations D,

$$D\boldsymbol{a} = \mathbf{0} \wedge Db = 0 \Rightarrow Dc = 0; \qquad (2.15)$$

(2) but then for *some* derivation E,

$$E\boldsymbol{a} = \mathbf{0} \wedge Ec \neq 0.$$

In particular, $Eb \neq 0$. Suppose if possible b is independent of $A \cup \{c\}$. Then for some derivation F,

$$F\boldsymbol{a} = \mathbf{0} \wedge Fc = 0 \wedge Fb \neq 0.$$

But in this case the derivation $E - (Eb/Fb)F$ is a counterexample to (2.15). □

Given the Lie–Rinehart pair (K, V), suppose $\boldsymbol{a}$ in K^m is D-independent. Then there is $\boldsymbol{D}$ in V^m such that

$$\boldsymbol{Da} = I_m,$$

as in (2.14).

2.3.3. A model companion

For all m in $\mathbb{N}$, let us introduce a new predicate for the relation of m-ary relation of D-dependence of scalars. One formula that defines this relation is

$$\forall \boldsymbol{Y} \ \det(\boldsymbol{Yx}) = 0,$$

where $\boldsymbol{Y}$ and $\boldsymbol{x}$ are m-tuples of vector and scalar variables respectively. Let LR_0' be LR_0 together with axioms defining the new predicates.

Theorem 4 (Kasal, 2010)**.** *The existentially closed models of* LR_0' *are precisely those models* (V, K) *meeting the following conditions.*

I. *The D-dimension of K is infinite, that is, for all m in $\mathbb{N}$, for some $\boldsymbol{a}$ in K^m, for some $\boldsymbol{D}$ in V^m,*

$$\boldsymbol{Da} = I_m.$$

II. For all k, ℓ, m, n, and s in ω, for all D-independent $\boldsymbol{a}$ in K^k, for all $\boldsymbol{b}$ in K^ℓ, whenever

a) $\boldsymbol{x} \in K^m$, $\boldsymbol{y} \in K^n$, $\boldsymbol{z} \in K^s$, $(\boldsymbol{x}, \boldsymbol{y})$ is algebraically independent, and U is an open subset of an affine variety over $\mathbb{Q}(\boldsymbol{a}, \boldsymbol{b})$ that has generic point $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$,

b) $\boldsymbol{D} \in V^k$, $\boldsymbol{E} \in V^m$, $\boldsymbol{D}\boldsymbol{a} = I_k$, $\boldsymbol{E}\boldsymbol{a} = 0$, but $(\boldsymbol{D}, \boldsymbol{E})$ is linearly independent;

c) $\left(\begin{array}{c|c} F & G \\ \hline I_m & H \end{array}\right)$ is a $(k+m) \times (m+n)$ matrix with entries from the coordinate ring $\mathbb{Q}(\boldsymbol{a}, \boldsymbol{b})[U]$,

then U contains $(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e})$ such that

a) $\left(\begin{array}{c|c} F & G \\ \hline I_m & H \end{array}\right)$ evaluated at $(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e})$ gives

$$\begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{E} \end{pmatrix} \left(\begin{array}{c|c} \boldsymbol{c} & \boldsymbol{d} \end{array} \right), \quad \text{that is,} \quad \left(\begin{array}{c|c} \boldsymbol{Dc} & \boldsymbol{Dd} \\ \hline \boldsymbol{Ec} & \boldsymbol{Ed} \end{array}\right);$$

b) each entry of $(\boldsymbol{d}, \boldsymbol{e})$ is D-dependent on $(\boldsymbol{a}, \boldsymbol{c})$.

The existentially closed models of LR_0' therefore compose an elementary class, whose theory is thus model-complete. The theory is moreover complete, but unstable.

2.3.4. The tree property

The **tree property** [18, Defn 0.1], of a complete theory that has it, is that for some formula $\varphi(\boldsymbol{x}; \boldsymbol{y})$, for some tree $(\boldsymbol{a}_\sigma : \sigma \in \omega^{<\omega})$ of tuples of parameters from some model of the theory,

- for all σ in $\omega^{<\omega}$, whenever $i < j < \omega$, the conjunction

  $$\varphi(\boldsymbol{x}, \boldsymbol{a}_{\sigma\, i}) \wedge \varphi(\boldsymbol{x}, \boldsymbol{a}_{\sigma\, j})$$

  is inconsistent with the model, but

- for each τ in ω^ω, the type

$$\{\varphi(\boldsymbol{x}, \boldsymbol{a}_{\tau\restriction n}) : n \in \omega\}$$

is consistent with the model.

For example, in $(\mathbb{Q}, <)$, we may let φ be $y_0 < x < y_1$, and we may require

$$a^0_\sigma \leqslant \ldots \leqslant a^0_{\sigma\ i} < a^1_{\sigma\ i} \leqslant a^0_{\sigma\ i+1} < \cdots < a^1_\sigma,$$

so that, in particular, each open interval (a^0_σ, a^1_σ) has disjoint subintervals $(a^0_{\sigma\ i}, a^1_{\sigma\ i+1})$.

Having the tree property means precisely not being simple [6, Prop. 2.3.7, p. 25].

In the example, whenever ρ and σ in $\omega^{<\omega}$ are incomparable, that is, $\sigma(i) \neq \rho(i)$ for some i in the domain of each, the formulas $\varphi(x, a_\rho)$ and $\varphi(x, a_\sigma)$ are inconsistent. This means the theory of $(\mathbb{Q}, <)$ has what may be called TP_1 [7].

There is another form, TP_2, of the tree property; having the tree property means having TP_1 or TP_2 [18, Thm 0.2]. A complete theory T has TP_2 if, for some formula $\varphi(\boldsymbol{x}; \boldsymbol{y})$, for some infinite matrix $(\boldsymbol{a}^i_j)^{i<\omega}_{j<\omega}$ of tuples of parameters from some model of T, for each i in ω, no two specializations $\varphi(\boldsymbol{x}, \boldsymbol{a}^i_j)$ are consistent with one another, but for every τ in ω^ω, the type $\{\varphi(\boldsymbol{x}, a^i_{\sigma(i)} : i < \omega\}$ is consistent.

Theorem 5. LR'_0 *has* TP_2.

Proof. We use $\varphi(X, y^0, y^1)$, which is $Xy^0 = y^1$. We let $\boldsymbol{a}^i_j$ be (t^i, b_j), where $(t^i : i < \omega)$ is D-independent, and the b_j are all distinct. See Figure 2.4. Evidently $Xt^i = b_i \wedge Xt^i = b_j$ is inconsistent if $i \neq j$, but since the t^i are D-independent, for each σ we can define D so that, for each i, $Dt^i = b_{\sigma(i)}$. $\square$

	b_0	b_1	$\cdots$	b_i	$\cdots$	b_j	$\cdots$
t^0							
t^1							
$\vdots$							
t^i				$Xt^i = b_i$		$Xt^i = b_j$	
$\vdots$							
t^j				$Xt^j = b_i$		$Xt^j = b_j$	

Figure 2.4.: TP_2 for LR_0'

A. Long abstract

Using areas, Euclid proved results that today we consider as algebraic. We consider them so, because Descartes justified algebra by showing how it could be considered as geometry.

Such observations can be understood as resulting from the equivalence of certain categories.

Models of a given first-order theory T are the objects of two different categories:

- **Mod**(T), in which the morphisms are embeddings, and
- **Mod**$^*(T)$, in which the morphisms are elementary embeddings.

The latter category is closed under direct limits; if the former is likewise closed, then T has universal-existential axioms (and conversely). T is called model-complete if the two categories are the same. T is called companionable if it is included in[1] a model-complete theory, called the model companion of T, in a model of which each model of T embeds.

A vector space here is a pair (K, V), where K is a field, V is an abelian group, and K acts on V. The theory of vector spaces in this sense has a model companion, which is theory of one-dimensional vector spaces.

If T is the theory of vector spaces of dimension at least two,

[1] The original had "includes" by mistake, perhaps because T is included in T^* if and only if the class of models of T includes the class of models of T^*.

and U is the theory of abelian groups with an appropriate notion of parallelism, then $\mathbf{Mod}(T)$ and $\mathbf{Mod}(U)$ are equivalent. If S is field theory, and T_n is the theory of n-dimensional vector spaces (where $n > 0$), with a basis named, then $\mathbf{Mod}(S)$ and $\mathbf{Mod}(T_n)$ are equivalent.

In a vector space (K, V), V may also act on K as a Lie ring of derivations; then (K, V) becomes a Lie–Rinehart pair. Such pairs can be given universal-existential axioms, using only the signature of abelian groups for each of K and V, along with a symbol for the action of each on the other. In his 2010 dissertation, Özcan Kasal showed that the resulting theory is not companionable, although if predicates for certain definable relations are introduced, the theory becomes companionable, and the model companion is not stable. It turns out that like the theory of the integers as a group, the model companion even has the so-called tree property.

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