Löwnheim–Skolem Theorem

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These notes are part of a general investigation of the Compactness Theorem. They are in response to remarks of Dawson [1]. Dawson distinguishes and names four theorems (the typography is mine, the words are Dawson's):

The completeness theorem: Every valid sentence is provable.

The Skolem–Godel theorem: A set of sentences is (syntactically) consistent if and only if it is satisfiable in some model.

The compactness theorem: A set of sentences is satisfiable if and only if every finite subset of it is satisfiable (for short: if and only if the set of sentences is finitely satisfiable).

The (downward) Lowenheim–Skolem theorem: If a set of sentences is satisfiable at all, it is satisfiable in a structure whose cardinality is at most that of the number of symbols in the underlying language.

The first three theorems are, respectively, Theorems I, IX, and X of Gödel [3], except that the sets of sentences that Gödel considers are countable. Concerning these theorems, Dawson writes:

> Godel's proofs employed Skolem's methods; but, unlike Skolem, Godel carefully distinguished between syntactic and semantic notions. The relation between the works of the two men has been examined by Vaught (*1974,* 157–159) and, in great detail, by van Heijenoort and Dreben [7]. All three commentators agree that both the completeness and compactness theorems were implicit in Skolem [4], but that no one before Godel drew them as conclusions, not even *after* Hilbert and Ackermann, in their 1928 book *Grundzüge der theoretischen Logik*[,] singled out first-order logic for attention and explicitly posed the question of its completeness.

How is completeness implicit in Skolem's 1923 paper?[1]

The 1920 paper

I start with Skolem's 1920 paper [5]. Right away Skolem refers to Löwenheim's use of the term *first-order expressions.* Skolem will prefer *first-order proposition.* At the beginning he is not clear about free variables: are we talking about *sentences,* or arbitrary *formulas* in our sense? But the meaning of *first order* is the one that has come down to us, though the terminology used for describing the meaning is different:

> By a first-order expression Löwenheim understands an expression constructed from relative coefficients by means of the five fundamental logical operations, namely, in Schröder's terminology, identical multiplication and addition, negation, productation, and

[1] The paper is labelled with the year of 1922 in [6], presumably because the paper is based on an address to the Fifth Congress of Scandinavian Mathematicians, Helsinki, 4–7 August 1922. But publication was apparently not until 1923, as the bibliography of [6] indicates.

> summation. The five operations mentioned are denoted by a dot (or simply juxtaposition), the sign $+$, the bar $^{-}$, and the signs Π and Σ, respectively.

The examples that Skolem goes on to give allow construction of a dictionary:

$.$	$+$	$^{-}$	Π	Σ	$\Pi_x \Sigma_y R_{xy}$
$\wedge$	$\vee$	$\neg$	$\forall$	$\exists$	$\forall x\ \exists y\ Rxy$

Skolem's **normal form** is our *Skolem normal form:* an $\forall\exists$ formula. I quote his theorems verbatim.

Theorem 1. *If U is an arbitrary first-order proposition, there exists a first-order proposition U' in normal form with the property that U is satisfiable in a given domain whenever U' is, and conversely.*

Theorem 2. *Every proposition in normal form either is a contradiction or is already satisfiable in a finite or denumerably infinite domain.*

To prove this, we start with the simplest nontrivial case, a sentence $\forall x\ \exists y\ \varphi(x, y)$, where φ is an open (quantifier-free) formula. We assume the sentence has a model $\mathfrak{B}$. (Skolem says, "this proposition is satisfied in a given domain for certain values of the relatives.")

Using the Axiom of Choice (Skolem says "principle of choice"), we obtain an operation $x \mapsto x'$ on B such that

$$\mathfrak{B} \vDash \forall x\ \varphi(x, x').$$

Let $a \in B$, and let A be the intersection of the collection of all subsets of B that contain a and are closed under $x \mapsto x'$. By "Dedekind's theory of chains in *1888*" [2], A is countable, and

$$\mathfrak{A} \vDash \forall x\ \varphi(x, x'), \qquad \mathfrak{A} \vDash \forall x\ \exists y\ \varphi(x, y).$$

More generally, we may have a sentence

$$(*)\qquad \forall x_0 \cdots \forall x_{m-1}\, \exists y_0 \cdots \exists y_{n-1}\, \varphi(x_0, \dots, x_{m-1}, y_0, \dots, y_{n-1}),$$

again assumed to have a model $\mathfrak{B}$. Again by the Axiom of Choice we obtain a function $\boldsymbol{x} \mapsto \boldsymbol{x}'$ from B^m to B^n such that

$$\mathfrak{B} \vDash \forall \boldsymbol{x}\ \varphi(\boldsymbol{x}, \boldsymbol{x}').$$

(Skolem's indices in tuples start with 1, and there is no notation for sets of tuples of given length.) Again if we start with a and close under $\boldsymbol{x} \mapsto \boldsymbol{x}'$, we obtain a countable set A such that $\mathfrak{A} \vDash \forall \boldsymbol{x}\ \exists \boldsymbol{y}\ \varphi(\boldsymbol{x}, \boldsymbol{y})$. Skolem takes to lemmas to show that A is countable.

> Theorem 2 admits generalizations of high order. Thus, it is not difficult to prove the following: Either it is contradictory to suppose that a simply infinite sequence of first-order propositions in normal form is simultaneously satisfiable or the sequence is already simultaneously satisfiable in a denumerably infinite domain.

Skolem gives the proof for sentences in normal form in which only two variables occur, and he considers the "logical product" or "propositional product"

$$\mathit{\Pi}_{x_1} \mathit{\Sigma}_{y_1} U^1_{x_1 y_1} \mathit{\Pi}_{x_2} \mathit{\Sigma}_{y_2} U^2_{x_2 y_2} \cdots \text{ad infinitum}$$

The formal statement is,

Theorem 3. *If a proposition can be represented as a product of a denumerable set of first-order propositions, it either is a contradiction or is already satisfiable in a denumerable domain.*

Skolem continues in this vein, with infinite "sums" (disjunctions), and then repeated taking of sums and products. He also considers propositions

$$\mathit{\Pi}_{x_1} \mathit{\Pi}_{x_2} \cdots \mathit{\Pi}_{x_m} \mathit{\Sigma}_{y_1} \mathit{\Sigma}_{y_2} \cdots \text{ad inf. } U_{x_1 \cdots x_m y_1 y_2 \cdots \text{ad inf.}}$$

and countably infinite products of these (Theorem 8). There are no examples or concluding remarks.

The 1922 paper

Skolem's 1922 paper [4] addresses eight points about Zermelo's set-theory. First,

> If we adopt Zermelo's axiomatization, we must, strictly speaking, have a general notion of domains in order to be able to provide a foundation for set theory. The entire content of this theory is, after all, as follows: for every domain in which the axioms hold, the further theorems of set theory also hold. But clearly it is somehow circular to reduce the notion of set to a general notion of domain.

The basic concern seems to be that you cannot do set theory without already knowing what sets are. He seems to be mistaken: We start out with a notion of (as I say) *collection,* and then in particular *class*: a "domain" for set theory is a class, not a set.

Skolem's second point is that Zermelo is deficient for not defining "definite proposition." The allusion is to

> Axiom III. (Axiom of separation.) Whenever the propositional function $\mathfrak{E}(x)$ is definite for all elements of a set M, M possesses a subset $M_{\mathfrak{E}}$ containing as elements precisely those elements x of M for which $\mathfrak{E}(x)$ is true. [8, p. 202]

Here $\mathfrak{E}(x)$ ought to be, in our terms, a first-order proposition in the signature $\{\in\}$ (with equality).

Skolem continues with the **Skolem paradox:**

> This third point is the most important: *If the axioms are consistent, there exists a domain B in which the axioms hold and whose elements can all be enumerated by means of the positive finite integers.*

> ... So far as I know, no one has called attention to this peculiar and apparently paradoxical state of affairs.

Skolem immediately resolves the paradox. Meanwhile, he has proved it using Löwenheim's theorem, proved now without the Axiom of Choice: We are again considering a satisfiable sentence in Skolem normal form as in $(*)$. There are some "solutions"—models—of the sentence

$$\exists \boldsymbol{y}\ \varphi(0, \dots, 0, \boldsymbol{y}).$$

Moreover, some such model will have the universe $1 + n$, that is, $\{0, \dots, n\}$, because this set is just large enough to allow the y_j to be distinct from one another and from 0. We designate each such model by $L_{j,1}$ for some j. Then there are models $L_{j,2}$ of

$$\bigwedge_{\boldsymbol{x} \in^m (1+n)} \varphi(\boldsymbol{x}, \boldsymbol{y})$$

that have universe $1 + n + n \cdot (n+1)^m$. We continue thus: the $L_{j,3}$ will be models of

$$\bigwedge_{\boldsymbol{x} \in^m (1+n+n\cdot(n+1)^m)} \varphi(\boldsymbol{x}, \boldsymbol{y})$$

with universe $1 + n + n \cdot (n+1)^m + n \cdot (1 + n + n \cdot (n+1)^m)^m$, and so on.

Let the universe of $L_{j,n}$ be A_n. Thus

$$A_0 = 1, \qquad\qquad A_{k+1} = A_k + n \cdot {A_k}^m,$$

but this is of little importance. Now we order the $L_{j,n}$ for fixed n. First we order linearly the relation symbols occurring in φ.[2] Then

[2] At one point Skolem calls these, "class and relation symbols (class and relative coefficients in the sense of Schröder)".

$L < L'$ if and only if, when m is minimal such that both L and L' include *no* common $L_{j,m}$, then, for the least R that has different interpretations in L and L', for the least tuple of elements of A_n (in the lexicographic ordering) where the interpretations differ, R fails for that tuple in L, but not L'. Then the $L_{j,n}$ are numbered so that

$$L_{0,n} < L_{1,n} < \cdots$$

If $k < n$, there is a_k^n such that

$$L_{a_k^n,k} \subset L_{1,n}.$$

As n grows (while k is fixed), a_k^n can only increase: this is because

$$L_{i,n} \subset L_{i^*,n+1} \ \& \ L_{j,n} \subset L_{j^*,n+1} \ \& \ L_{i^*,n+1} < L_{j^*,n+1} \implies L_{i,n} \leqslant L_{j,n}.$$

The point is that the various $L_{i,n}$ compose a *tree* when ordered by $\subset$; and we can refine this into a linear ordering with the desired property. Then the $L_{1,n}$ have a "limit," which is a model of the original sentence.

Now we generalize to a countably infinite list of sentences. Skolem hardly even alludes to the details; his main point is to observe that when Zermelo's "Axiom III (axiom of separation) can be replaced by an infinite sequence of simpler axioms"...[3]

References

[1] John W. Dawson, Jr. The compactness of first-order logic: from Gödel to Lindström. *Hist. Philos. Logic*, 14(1):15–37, 1993.

[3] Editing this document on June 18, 2015, I have added the ellipsis and suppressed ("commented out") the ensuing section, called "Old," whose material, including a derivation of the Skolem normal form, seems to have been transferred to (or rewritten in) my notes on Gödel's Completeness Theorem.

[2] Richard Dedekind. *Essays on the Theory of Numbers. I: Continuity and Irrational Numbers. II: The Nature and Meaning of Numbers.* authorized translation by Wooster Woodruff Beman. Dover Publications Inc., New York, 1963.

[3] Kurt Gödel. The completeness of the axioms of the functional calculus of logic. In van Heijenoort [6], pages 582–91. First published 1930.

[4] Thoralf Skolem. Some remarks on axiomatized set theory. In van Heijenoort [6], pages 290–301. First published 1923.

[5] Thoralf Skolem. Logico-combinatorial investigations in the satisfiability or provability of mathematical propositions: A simplified proof of a theorem by L. Löwenheim and generalizations of the theorem. In van Heijenoort [6], pages 252–63. First published 1920.

[6] Jean van Heijenoort, editor. *From Frege to Gödel: A source book in mathematical logic, 1879–1931.* Harvard University Press, Cambridge, MA, 2002.

[7] Jean van Heijenoort and Burton Dreben. Introductory note to *1929, 1930*, and *1930a.* In Solomon Feferman et al., editors, *Kurt Gödel. Collected works. Vol. I*, pages 44–59. The Clarendon Press, Oxford University Press, New York, 1986. Publications 1929–1936.

[8] Ernst Zermelo. Investigations in the foundations of set theory I. In van Heijenoort [6], pages 199–215. First published 1908.