# Logics in general

David Pierce

August 30, 2014

Mimar Sinan Fine Arts University
http://mat.msgsu.edu.tr/~dpierce/

Here are notes made by way of investigating what a logic is. The original impetus was considerations of the topological meaning of the Compactness Theorem of first-order logic.

## Contents

## 1 Institutions

The question of what a logic is is taken up in, for example, [6], where the answer is sought in the **institution.** This consists of:

1) a category **Sgn** of **signatures;**
2) a functor from **Sgn** to **Set**, assigning
   a) to each signature $\mathscr{S}$ the set
   $$\mathrm{Sen}_{\mathscr{S}}$$
   of **sentences** in that signature, and
   b) to each morphism $f$ of signatures a **translation**
   $$\sigma \mapsto f(\sigma)$$
   of sentences;
3) a contravariant functor from **Sgn** to the quasi-category of all categories, assigning
   a) to each signature $\mathscr{S}$ the category
   $$\mathbf{Str}_{\mathscr{S}}$$
   of **structures** of $\mathscr{S}$, and
   b) to each morphism $f$ from $\mathscr{S}'$ to $\mathscr{S}$ in **Sgn** the **reduction**
   $$\mathfrak{A} \mapsto \mathfrak{A} \upharpoonright f$$
   from $|\mathbf{Str}_{\mathscr{S}}|$ to $|\mathbf{Str}_{\mathscr{S}'}|$;
4) On $|\mathbf{Sgn}|$, a function
   $$\mathscr{S} \mapsto \vDash_{\mathscr{S}},$$
   where $\vDash_{\mathscr{S}}$ is the **truth** relation from $|\mathbf{Str}_{\mathscr{S}}|$ to $\mathrm{Sen}_{\mathscr{S}}$, and if $\sigma \in \mathrm{Sen}_{\mathscr{S}'}$, and $f$ is a morphism from $\mathscr{S}'$ to $\mathscr{S}$ in **Sgn**, and $\mathfrak{A} \in \mathbf{Str}_{\mathscr{S}}$, then
   $$\mathfrak{A} \vDash_{\mathscr{S}} f(\sigma) \iff \mathfrak{A} \upharpoonright f \vDash_{\mathscr{S}'} \sigma.$$

Here I have adjusted the notation of [6], introduced the term *translation,* used *structure* in place of "model," and used *truth* in place of "satisfaction." We may require that the same sentences be true in isomorphic structures.

If $\sigma \in \mathrm{Sen}_{\mathscr{S}}$, I define

$$\mathbf{Mod}(\sigma) = \{\mathfrak{A} \in \mathbf{Str}_{\mathscr{S}} \colon \mathfrak{A} \vDash_{\mathscr{S}} \sigma\},$$

writing henceforth $\mathbf{Str}_{\mathscr{S}}$ instead of $|\mathbf{Str}_{\mathscr{S}}|$; $\mathbf{Mod}(\sigma)$ is the class of **models** of $\sigma$.

I am interested in the case where the model-classes are basic closed classes in a *topology* on $\mathbf{Str}_{\mathscr{S}}$. This will require an adjustment to the usual definition of topology, since we shall need to be able to talk about collections of closed classes. We can do this by using a relation as in §4 (page 10). The ideas of *pointless topology* seem pertinent here.

## 2 Pointless topology

By the usual definition, a **topology** on a set $X$ is a subset $\tau$ of $\mathscr{P}(\Omega)$ having the following closure properties:

1) $X \in \tau$,
2) $U \in \tau \And V \in \tau \implies U \cap V \in \tau$,
3) $\mathscr{X} \subseteq \tau \implies \bigcup \mathscr{X} \in \tau$.

By the last condition, since $\bigcup \varnothing = \varnothing$, this is in $\tau$. The pair $(X, \tau)$ is a **topological space.** The elements of $\tau$ are said to be **open subsets** of the space; complements in $X$ of elements of $\tau$ are **closed.**

If $(Y, \tau_1)$ is another topological space, a function $f$ from $X$ to $Y$ is **continuous** (with respect to $\tau$ and $\tau_1$) if

$$U \mapsto f^{-1}[U] \colon \tau_1 \to \tau.$$

In this case $f$ is a morphism from $(X, \tau)$ to $(Y, \tau_1)$ in the category **Top** of topological spaces. An isomorphism in this category is called a **homeomorphism.**

We investigate the possibility of recovering $(X, \tau)$ (up to homeomorphism) from the algebraic properties of $\tau$. References for this are Johnstone's article [5] and book [4], and also the notes of Gooding [2].

The set $\tau$ is ordered by inclusion ($\subseteq$). With respect to this ordering, every subset $\mathscr{O}$ of $\tau$ has in $\tau$

- a supremum, namely $\bigcup \mathscr{O}$, and (therefore)
- an infimum, namely the *interior* of $\bigcap \mathscr{O}$:

$$\begin{aligned}\inf(\mathscr{O}) &= \bigcup \left\{U \in \tau \colon U \subseteq \bigcap \mathscr{O}\right\} \\ &= \sup\left(\left\{U \in \tau \colon U \subseteq \bigcap \mathscr{O}\right\}\right).\end{aligned}$$

Thus $\tau$ is a **complete lattice.** If $U$ and $V$ are in $\tau$, then

$$\inf(\{U, V\}) = U \cap V.$$

The lattice $\tau$ is **distributive** because

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W)$$

(and therefore also $U \cup (V \cap W) = (U \cup V) \cap (U \cup W)$). Since moreover

$$U \cap \bigcup \mathscr{O} = \bigcup \{U \cap V \colon V \in \mathscr{O}\},$$

the distributive lattice is called a **frame.**

Suppose $A$ is an arbitrary set with an ordering that makes it a frame. We may use the notation

$$\begin{aligned} \sup(\{x, y\}) &= x \vee y, & \inf(\{x, y\}) &= x \wedge y, \\ \sup(X) &= \bigvee X, & \inf(X) &= \bigwedge X, \\ \sup(\varnothing) &= \bot, & \inf(\varnothing) &= \top. \end{aligned}$$

In the category **Frm** of frames, the morphisms are the functions that preserve $\top$, $\bot$, $\wedge$, and $\bigvee$. We now have a contravariant functor $\Omega$ from **Top** to **Frm**, given by

$$\begin{aligned}\Omega(X,\tau) &= \tau,\\ \Omega(f) &= (U \mapsto f^{-1}[U]).\end{aligned}$$

The opposite category of **Frm** is **Loc**, the category of **locales;** so a locale is a frame, and a morphism of locales is a morphism of frames in the opposite direction. So $\Omega$ is a (covariant) functor from **Top** to **Loc**, and for this reason one may prefer to think in terms of locales rather than frames. But I shall avoid doing this, preferring morphisms to be actual functions.

Considering 0 as the empty set and 1 as $\{0\}$, we have a unique topology on 1; this topology is just $\mathscr{P}(1)$, which is $\{0, 1\}$ and can be considered as 2. So we can understand 2 as a frame.

Given the topological space $(X, \tau)$, we have a bijection $x \mapsto (0 \mapsto x)$ from $X$ to $\operatorname{Hom}((1, \mathscr{P}(1)), (X, \tau))$. Thus points $x$ of $X$ can be identified with morphisms $0 \mapsto x$. Given the morphism $0 \mapsto x$, we obtain the element $\Omega(0 \mapsto x)$ of $\operatorname{Hom}(\tau, 2)$ given by

$$\Omega(0 \mapsto x)(U) = \begin{cases} 1, & \text{if } x \in U,\\ 0, & \text{if } x \in X \smallsetminus U.\end{cases}$$

If $A$ is an arbitrary frame, we define

$$\operatorname{pt}(A) = \operatorname{Hom}(A, 2);$$

this is the set of **points** of $A$. Then

$$x \mapsto \Omega(0 \mapsto x)\colon X \to \operatorname{pt}(\tau).$$

If this map is bijective, the space $(X, \tau)$ is called **sober.**

The space $(X, \tau)$ is called **Tychonoff** (or $T_0$) if any two distinct points of it are topologically distinguishable, that is, some open set contains exactly one of them.

**1 Theorem.** *Assume $(X, \tau)$ is a topological space. Then the map*

$$x \mapsto \Omega(0 \mapsto x) \text{ from } X \text{ to } \operatorname{pt}(\tau)$$

*is injective if and only if $(X, \tau)$ is Tychonoff.*

*Proof.* If $\Omega(0 \mapsto x) \neq \Omega(0 \mapsto y)$, then by definition they differ at some open subset $U$ of $X$, and then this distinguishes $x$ from $y$. Conversely, if $x$ and $y$ are so distinguishable, then the original inequality holds. $\square$

An element $a$ of a lattice is called **prime** if

$$a \neq \top,$$
$$a = x \wedge y \implies a = x \text{ OR } a = y.$$

For example, if $p \in X$, then the element $\bigcup\{U \in \tau \colon p \notin U\}$ of $\tau$ is prime.

A subset $I$ of a lattice is an **ideal** if

$$\bot \in I, \qquad \top \notin I,$$
$$x \in I \;\&\; y \in I \implies x \wedge y \in I,$$
$$y \in I \;\&\; x \leqslant y \implies x \in I.$$

The ideal $I$ is **prime** if

$$x \wedge y \in I \implies x \in I \text{ OR } y \in I.$$

If $a$ belongs to a lattice $L$, we define

$$(a) = \{x \in L \colon x \leqslant a\};$$

this is the **principal ideal** generated by $a$.

**2 Theorem.** *Principal ideals are indeed ideals. A principal ideal is prime if and only if its generator is prime. If $A$ is a frame, then the map*

$$f \mapsto f^{-1}(0)$$

*on* $\operatorname{Hom}(A, 2)$ *is a bijection onto the set of principal prime ideals of $A$.*

*Proof.* Suppose $f \in \operatorname{Hom}(A, 2)$, and let $a = \bigvee f^{-1}(0)$. Then

$$f(a) = \bigvee\{0\} = 0,$$

and in general

$$f(x) = \begin{cases} 0, & \text{if } x \leqslant a, \\ 1, & \text{if } x \nleqslant a. \end{cases}$$

Thus $f^{-1}(0) = (a)$. □

The space $(X, \tau)$ is called **Hausdorff** (or $T_2$) if any two distinct points of $X$ belong to disjoint open sets.

**3 Theorem.** *Hausdorff spaces are sober.*

In general, given the frame $A$, we can understand $\operatorname{pt}(A)$ as a topological space as follows. If $a \in A$, let

$$[a] = \{f \in \operatorname{pt}(A) \colon f(a) = 1\}.$$

**4 Theorem.** *If $A$ is a frame, then $\{[x] : x \in A\}$ is a topology on* $\operatorname{pt}(A)$.

*Proof.* Just note

$$\begin{gathered} \operatorname{pt}(A) = [\top], \\ [a] \cap [b] = [a \wedge b], \\ \bigcup\{[x] : x \in X\} = \left[\bigvee X\right]. \end{gathered}$$ □

Thus we can understand

$$\Omega(\mathrm{pt}(A)) = \{[x] : x \in A\}.$$

We now have

$$x \mapsto [x] : A \to \Omega(\mathrm{pt}(A));$$

evidently this is surjective, though it need not be injective. If $A$ and $B$ are frames, and $g \in \mathrm{Hom}(B, A)$, we define

$$\mathrm{pt}(g) = (f \mapsto f \circ g),$$

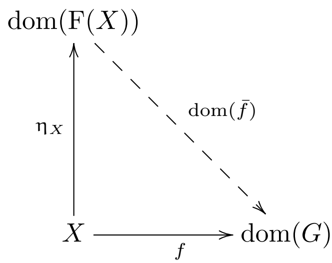
which is in $\mathrm{Hom}(\mathrm{pt}(A), \mathrm{pt}(B))$. So pt is a contravariant functor from **Frm** to **Top**, hence a covariant functor from **Loc** to **Top**.

## 3 Adjoints

Let **Grp** be the category of groups; **Set**, of sets. Following Hodges [3, p. 2], let us denote the *forgetful functor* $(G, \times) \mapsto G$ from **Grp** to **Set** by dom. This has a **left adjoint,** namely a functor F from **Set** to **Grp** such that, for all sets $X$, there is a map $\eta_X$ from $X$ to $\mathrm{dom}(\mathrm{F}(X))$ such that, for all groups $G$, if $f\colon X \to \mathrm{dom}(G)$, then there is a homomorphism $\bar{f}$ from $\mathrm{F}(X)$ to $G$ such that

$$\mathrm{dom}(\bar{f}) \circ \eta_X = f,$$

that is, the following diagram commutes.

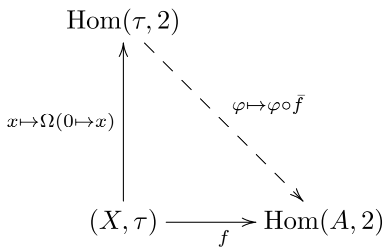


The function $X \mapsto \eta_X$ is the **unit** of the adjunction.

**5 Theorem.** $\Omega$ *is a left adjoint of* pt *whose unit* $\eta$ *is given by*

$$\eta_{(X,\tau)}(x) = \Omega(0 \mapsto x).$$

*Proof.* Given a topological space $(X,\tau)$, a frame $A$, and a continuous function $f$ from $X$ to $\mathrm{pt}(A)$, we have to find a homomorphism $\bar{f}$ from $A$ to $\tau$ such that the following diagram commutes.




Thus we want

$$f(x)(a) = \Omega(0 \mapsto x)(\bar{f}(a)) = \begin{cases} 1, & \text{if } x \in \bar{f}(a), \\ 0, & \text{if } x \in X \smallsetminus \bar{f}(a), \end{cases}$$

which we have, provided

$$\begin{aligned} \bar{f}(a) &= \{x \in X \colon f(x)(a) = 1\} \\ &= \{x \in X \colon f(x) \in [a]\} \\ &= f^{-1}\big[[a]\big]. \end{aligned}$$

□

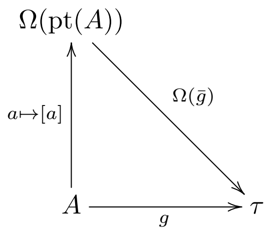
The adjunction of the theorem has the **co-unit** $\epsilon$ given by

$$\epsilon_A(a) = [a].$$

This means that for all frames $A$, for all topological spaces $(X, \tau)$, if $g \in \mathrm{Hom}(A, \tau)$, then there is a continuous function $\bar{g}$ from $X$ to $\mathrm{pt}(A)$ such that

$$\Omega(\bar{g})([a]) = g(a),$$

that is, the following diagram commutes.




What we want is

$$g(a) = \Omega(\bar{g})([a]) = \bar{g}^{-1}\left[[a]\right],$$

so if $\bar{g}$ exists at all as a map from $X$ to $\mathrm{pt}(A)$, it is continuous. We want also

$$\bar{g}(x) \in [a] \iff x \in g(a),$$

that is,

$$\bar{g}(x)(a) = \begin{cases} 1, & \text{if } x \in g(a), \\ 0, & \text{if } x \notin g(a). \end{cases}$$

Since $g(a) \in \tau$, the function $\bar{g}(a)$ so defined is indeed a morphism from $A$ to 2.

A reference besides [4] for category theory is Barr & Wells [1].

## 4 Topologies on classes

All of the foregoing work with topologies was in terms of *open sets,* because these are the terms of the references. Henceforth it will be more convenient to use closed sets. Note that the closed subsets of a topological space $A$ compose a set $\kappa$ such that

1) $\varnothing \in \kappa$,
2) $X \in \kappa \ \& \ Y \in \kappa \implies X \cup Y \in \kappa$,
3) $\mathscr{X} \subseteq \kappa \implies \bigcap \mathscr{X} \in \kappa$.

In the last condition, we understand $\bigcap \varnothing$ to be $A$, so this is in $\kappa$.

We are going to want to topologize a proper class $\boldsymbol{A}$ so that closed sub*classes* may indeed be proper classes. Then $\kappa$ will not *contain* the closed classes; but it will *index* them. We proceed as follows.

**6 Definition.** A **topology** on a class $\boldsymbol{A}$ is a relation $\vDash$ from $\boldsymbol{A}$ to a set $\kappa$ such that

1) for some $x$ in $\kappa$, for all $a$ in $\boldsymbol{A}$,
$$a \nvDash x;$$
2) for all $x$ and $y$ in $\kappa$, there is $z$ in $\kappa$ such that, for all $a$ in $\boldsymbol{A}$,
$$a \vDash z \iff a \vDash x \text{ OR } a \vDash y;$$
3) for every subset $\mathscr{X}$ of $\kappa$, there is $y$ in $\kappa$ such that, for all $a$ in $\boldsymbol{A}$,
$$a \vDash y \iff \forall x\ (x \in \mathscr{X} \implies a \vDash x).$$

Given a relation $\vDash$ from $\boldsymbol{A}$ to $\kappa$, for each $x$ in $\kappa$ we define

$$\mathbf{Mod}(x) = \{a \in \boldsymbol{A} \colon a \vDash x\}.$$

(Strictly this should be $\mathbf{Mod}_{\vDash}(x)$.) Then we define the relation $\sim$ on $\kappa$ by

$$x \sim y \iff \mathbf{Mod}(x) = \mathbf{Mod}(y).$$

The three conditions in the foregoing definition are equivalent respectively to the following.

1) For some $x$,
$$\mathbf{Mod}(x) = \varnothing.$$

2) For all $x$ and $y$, for some $z$,
$$\mathbf{Mod}(z) = \mathbf{Mod}(x) \cup \mathbf{Mod}(y).$$

3) For every set $\mathscr{X}$, for some $y$,
$$\mathbf{Mod}(y) = \bigcap_{x \in \mathscr{X}} \mathbf{Mod}(x).$$

By replacing $\kappa$ with $\kappa/{\sim}$, we may assume $\sim$ is just equality, and then $\kappa$ will have the usual algebraic properties of a topology of closed sets: it will be a "dual frame." But we do not want to simplify things that much.

**7 Definition.** Suppose the relation $\vDash$ from $\boldsymbol{A}$ to $\kappa$ is a topology on $\boldsymbol{A}$. A subset $\mathscr{B}$ of $\kappa$ is a **basis** of this topology if for every $x$ in $\kappa$ there is a subset $Y$ of $\mathscr{B}$ such that
$$\mathbf{Mod}(x) = \bigcap_{y \in Y} \mathbf{Mod}(y).$$

**8 Theorem.** *Suppose* $\vDash$ *is a relation from* $\boldsymbol{A}$ *to a set* $\mathscr{B}$, *and* $\mathscr{B}$ *has an element* $\perp$ *and a binary operation* $\vee$ *such that*
$$\mathbf{Mod}(\perp) = \varnothing, \qquad \mathbf{Mod}(x \vee y) = \mathbf{Mod}(x) \cup \mathbf{Mod}(y).$$
*Then the relation from* $\boldsymbol{A}$ *to* $\mathscr{P}(\mathscr{B})$ *given by*
$$\mathbf{Mod}(\mathscr{X}) = \bigcap_{x \in \mathscr{X}} \mathbf{Mod}(x)$$
*is a topology on* $\boldsymbol{A}$ *for which (the image in* $\mathscr{P}(\mathscr{B})$ *under* $x \mapsto \{x\}$ *of)* $\mathscr{B}$ *is a basis.*

**9 Definition.** A **logic** consists of:

1) a class **Str**,
2) a set Sen,
3) a relation $\vDash$ from **Str** to Sen, and
4) an element $\bot$ and a binary operation $\vee$ on Sen

such that, for all $\mathfrak{A}$ in **Str**, for all $\sigma$ and $\tau$ in Sen,

$$\left.\begin{array}{c}\mathfrak{A} \nvDash \bot, \\ \mathfrak{A} \vDash \sigma \vee \tau \iff \mathfrak{A} \vDash \sigma \text{ OR } \mathfrak{A} \vDash \tau.\end{array}\right\} \qquad (*)$$

The class **Str** consists of the **structures** of the logic; the set Sen consists of the **sentences** of the logic. The relation $\vDash$ is **truth,** and if $\mathfrak{A} \in$ **Str** and $\sigma \in$ Sen and $\mathfrak{A} \vDash \sigma$, then $\mathfrak{A}$ is a **model** of $\sigma$.

**10 Example.** In what would appear to be the simplest nontrivial example:

1. Sen is the absolutely free algebra—understood as a set of written formulas—on an infinite set $V$ of "propositional variables."
2. The class **Str** is $\mathscr{P}(V)$, the class of subsets of $V$.
3. If $\mathfrak{A} \in \mathscr{P}(V)$ and $P \in V$, then

   $$\mathfrak{A} \vDash P \iff P \in \mathfrak{A};$$

   this and the rules $(*)$ determine $\vDash$.

**11 Definition.** In an arbitrary logic, sentences having the same models are **logically equivalent.**

**12 Theorem.** *On the set of sentences of an arbitrary logic, logical equivalence is a congruence relation with respect to the Boolean operations, and the quotient of the algebra of sentences with respect to this relation is an idempotent commutative monoid.*

## References

[1] Michael Barr and Charles Wells. *Category theory for computing science.* Les Publications CRM, Montréal, third edition, 1999.

[2] Joshua Gooding. Pointless topology. `http://www.math.ucdavis.edu/~gooding/talks`, May 2008. slides of a talk, 18 pages.

[3] Wilfrid Hodges. *Model theory*, volume 42 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1993.

[4] Peter T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1982.

[5] Peter T. Johnstone. The point of pointless topology. *Bull. Amer. Math. Soc. (N.S.)*, 8(1):41–53, 1983.

[6] Till Mossakowski, Joseph Goguen, Răzvan Diaconescu, and Andrzej Tarlecki. What is a logic? In *Logica universalis*, pages 113–133. Birkhäuser, Basel, 2005.