Geometry as made rigorous by Euclid and Descartes

Seminar at Mimar Sinan

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Abstract: For Immanuel Kant (born 1724), the discovery of mathematical proof by Thales of Miletus (born around 624 B.C.E.) is a revolution in human thought. Modern textbooks of analytic geometry often seem to represent a return to pre-revolutionary times. The counterrevolution is attributed to René Descartes (born 1596). But Descartes understands ancient Greek geometry and adds to it. He makes algebra rigorous by interpreting its operations geometrically.

The definition of the real numbers by Richard Dedekind (born 1831) makes a rigorous converse possible. David Hilbert (born 1862) spells it out: geometry can be interpreted in the ordered field of real numbers, and even in certain countable ordered fields.

In modern textbooks, these ideas are often entangled, making the notion of proof practically meaningless. I propose to disentangle the ideas by means of Book I of Euclid's *Elements* and Descartes's *Geometry.*

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Mathematicians and commentators

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Richard Dedekind	b.	1831
David Hilbert	b.	1862

Introduction

Rigor in mathematics is *ability to stand up under questioning.*

Rigor in **education** has an extra component: *teaching what questions should be asked.*

This talk is inspired or rather provoked by **two books** of analytic geometry that fail to be rigorous.

One is an old book [11] used by my mother in college. When **young** I used this book in order to sketch the graphs of conic sections and of trigonometric and logarithmic functions.

But this book is not a book that one can sit and read for **pleasure.** I think Spivak's *Calculus* [12] is such a book. But the analytic geometry book begins with uninspiring exercises about coordinates, with no motivation.

Possible motivation can be found in the problem of **duplicating the cube,** as solved by Menaechmus.

The problem is to find two **mean proportionals** to a unit length and its double. In modern symbolic terms, this is to solve the system

$$\frac{1}{x} = \frac{x}{y} = \frac{y}{2}.$$

From this system we obtain

$$\frac{1}{x^3} = \frac{1}{2}.$$

Geometrically then, x is the side of a cube that has twice the volume of the unit cube.

Many solutions of this problem are reported by Eutocius (flourished around 500 C.E.) in his commentary [3] on Archimedes. This commentary has been revised by Isidore of Miletus, who, with Anthemius of Tralles, is one of the master-builders of the **Ayasofya.** Menaechmus was a student of Eudoxus of Knidos and a contemporary of Plato [1]. Eudoxus invented the theory of proportion found in Euclid's *Elements* [7, 6]; we shall talk about this later.

Menaechmus's solution to the problem above—in fact one of his two solutions—can be understood as follows. We obtain two equations

$$xy = 2, \qquad 2x = y^2.$$

It is known that these are the equations of certain **conic sections,** which Apollonius [2] would later call the hyperbola and the parabola. The point is that Menaechmus knows that the curves really can be obtained by slicing a cone. The hyperbola can be given asymptotes as in Figure 1. Then the axis of the parabola will be the horizontal asymptote of the hyperbola. The **coordinates** of the intersection of the two conic sections solve the original problem.

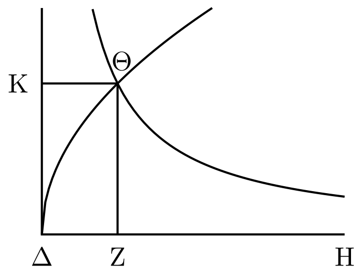

Figure 1: Menaechmus's finding of two mean proportionals

Was Menaechmus doing analytic geometry as we understand it? Perhaps not. Today we would just calculate the solution to the original system as

$$(x, y) = (\sqrt[3]{2}, \sqrt[3]{4}).$$

But a point with these coordinates cannot be found with the usual tools of straightedge and compass. Menaechmus gives us reason to believe that this point **exists anyway.** The reason he gives is geometric.

Two thousand years later, René Descartes [5] seems to share the view that solutions to equations should be **understood geometrically.** For example, in Figure 2, assuming $GE = EA = AI = a$, suppose we want

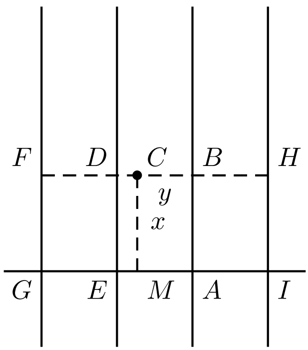

Figure 2: Descartes's locus problem

the locus of points C such that

$$CF \cdot CD \cdot CH = CB \cdot CM \cdot AI,$$

that is,

$$(2a - y)(a - y)(a + y) = yxa.$$

Given any value of y, we can compute x and thus sketch the curve as in Figure 3. But Descartes finds it worthwhile to do more. He shows that

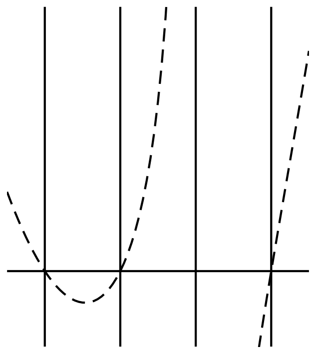

Figure 3: The locus itself

the point C lies on the intersection, shown in Figure 4, of:

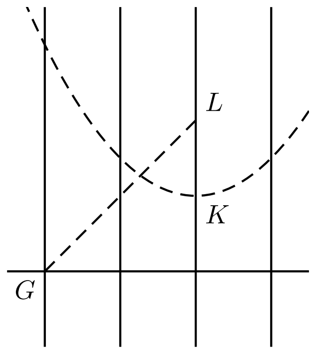

Figure 4: Descartes's geometrical solution

- a parabola with axis AB and *latus rectum* a whose vertex K slides along AB,
- the straight line through GL, where $KL = a$.

Thus the curve given by the cubic equation above becomes geometrically meaningful.

Again, we think the problem of duplicating the cube is solved simply by taking the cube root of 2. But how is this taken? There is an algorithm for finding **decimal approximations.** But why do we think these approximations have a limit? We can just declare that $\sqrt[3]{2}$ is some infinite decimal expansion. But why do we think that infinite decimal expansions like this compose a field?

Richard Dedekind [4] claims that, before he gave a rigorous definition of the rational numbers, the theorem

$$\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$$

had not been proved. David Fowler (author of *The Mathematics of Plato's Academy* [8]) seems to be correct that Dedekind is correct.

There is **no algorithm** for computing with infinite decimals. For example, what is the following sum?

$$3.1415926535\ldots + 0.8584073464\ldots$$

It is either $3.9\ldots$ or $4.0\ldots$, but we cannot specify a number of digits that are sufficient to tell us which. Fowler gives the example

$$1.222\ldots \times 0.818181\ldots$$

which is

$$\left(1 + \frac{2}{9}\right) \times \frac{81}{99} = \frac{11}{9} \times \frac{81}{99} = 1;$$

but no amount of multiplying finite decimal approximations tells us that the product is not required to begin as 0.9.

Dedekind's definition of the real numbers explicitly avoids making use of geometric notions. Therefore we can use the set of ordered pairs of real numbers as a *model* for geometric axioms, thus showing that these axioms are consistent. David Hilbert [10] does this.

Conversely, the Euclidean plane can be used to turn a straight line into a model of axioms for an ordered field. Descartes suggests this. David Hilbert[10] fills in the missing details. More recently, Robin Hartshorne [9] does the same, using theorems about circles from Book III of Euclid's *Elements.* In fact Book I of the *Elements* is enough.

Thus there are two complementary approaches to analytic geometry. Either geometry or algebra can be taken as fundamental. But textbooks assume both of these foundations. I think this is a defect of rigor.

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