Geometry as made rigorous by Euclid and Descartes

David Pierce

October 31, 2013

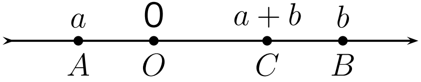
Contents

0 Introduction

According to one **textbook** of the subject,

> **analytic geometry** is based on the idea that a one-to-one correspondence can be established between the set of points of a straight line and the set of all real numbers.

- A straight line is an **ordered abelian group** in a geometrically natural way.



$$AO = CB$$

- This ordered group is **isomorphic** to $(\mathbb{R}, +, <)$.

The isomorphism from $(\mathbb{R}, +, <)$ to a straight line induces a **multiplication** on that straight line.

This multiplication has a **geometric meaning.**

This, if anything, is the "Fundamental Principle of Analytic Geometry."

Descartes establishes it.

Details can be worked out from Book I of Euclid's *Elements.*

1 Origins of geometry

Geometry comes from γεωμετρία, formed of γῆ (*land*) and μέτρον (*measure*).

According to Herodotus (b. *c.* 484 B.C.E.), in Egypt, land was taxed in proportion to size. If the Nile's annual flooding robbed you of land, the king sent **surveyors** to measure the loss.

> From this, to my thinking, the Greeks learned the **art of measuring land** (γεωμετρίη); the sunclock and the sundial, and the twelve divisions of the day, came to Hellas not from Egypt but from Babylonia. [2.109]

Plato (b. 427 B.C.E.) in the *Phaedrus* has Socrates say of the **Egyptian god Theuth,**

He it was who invented

- **numbers** (ἀριθμός) and
- **arithmetic** (λογισμός) and
- **geometry** (γεωμετρία) and
- **astronomy** (ἀστρονομία), also
- draughts and dice, and, most important of all,
- **letters** (γράμματα). [274c]

According to Aristotle (b. 384 B.C.E.),

> as more and more *skills* (τέχναι) were discovered, some relating to the *necessities* (ἀναγκαῖα) and some to the pastimes of life, the inventors of the latter were always considered wiser than those of the former, because their *sciences* (ἐπιστήμαι) did not aim at utility. Hence when all the discoveries of this kind were fully developed, the sciences concerning neither *pleasure* (ἡδονή) nor necessities were invented, and first in those places where men **had leisure** (σχολάζω).
>
> Thus **mathematics** (μαθηματικαί) originated in Egypt (Αἴγυπτος), because there the *priestly class* (ἱερέων ἔθνος) was allowed leisure. [*Metaphysics* I.i.16]

2 Euclid's geometry

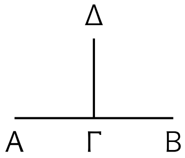
The *Elements* (Στοιχεῖα) of Euclid (fl. 300 B.C.E.) begins with five **Postulates** (Αἰτήματα "Demands").

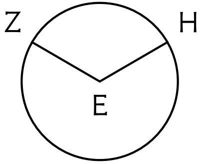
By the first four, we have three tools of a builder:

- a **ruler** or **chalk line,** (1) to draw a straight line from one point to another, or (2) to extend a given straight line;
- a **compass,** (3) to draw a circle with a given center, passing through a given point;
- a **set square,** whose mere existence ensures (4) that all right angles are equal to one another.

Actually these postulates allude to previous **Definitions** (Ὅροι "Boundaries"):

When a straight line set up on a straight line makes the adjacent angles *equal* (ἴσος) to one another, each of the equal angles is **right** (ὀρθός).

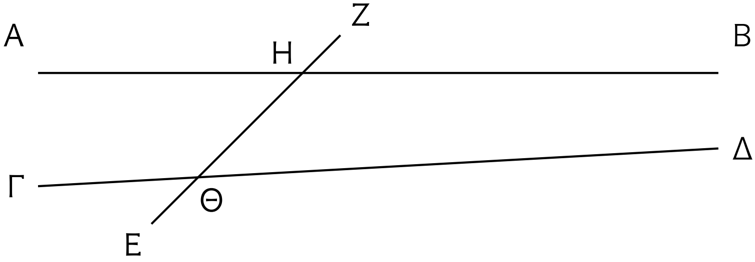




A **circle** (κύκλος) is. . . contained by one line such that all the straight lines falling upon it from one point [called the **center** (κέντρον)] are equal to one another.

The **Fifth Postulate** is that, if

$$\angle \mathsf{BH\Theta} + \angle \mathsf{H\Theta\Delta} < \mathsf{2} \text{ right angles,}$$



then AB and $\mathsf{\Gamma\Delta}$, extended, **meet.**

- This is **unambiguous** by the 4th postulate.
- It tells us what the 2nd postulate can **achieve.**

After the Postulates come the Axioms or **Common Notions** (Κοιναὶ ἔννοιαι):

1. **Equals** to the same are equal to one another.
2. If equals be **added** to equals, the wholes are equal.
3. If equals be **subtracted** from equals, the remainders are equal.
4. Things **congruent** with one another are equal to one another.
5. The whole is **greater** than the part.

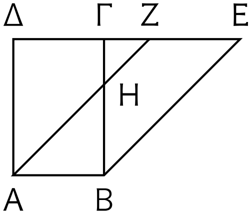
After the Common Notions come the **48 propositions** of Book I of the *Elements*, and then the remaining 12 books.

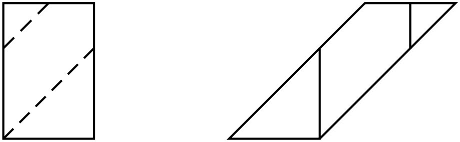
3 Equality and proportion

Equality in Euclid is:

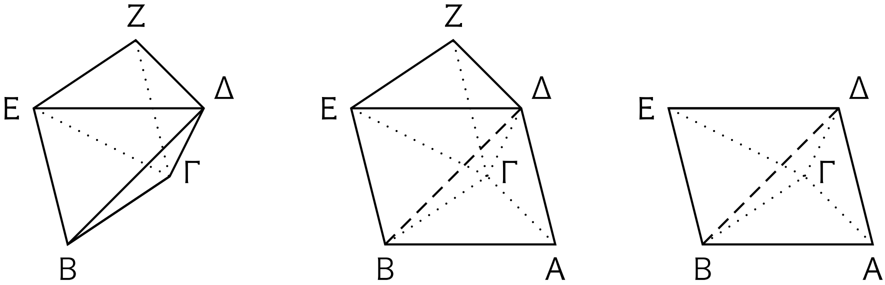
- not identity, by
  - the definitions of the circle and the right angle,
  - the 4th Postulate;
- symmetric (implicitly);
- transitive (Common Notion 1);
- implied by congruence (C.N. 4);
- implied by congruence of respective parts (C.N. 2);
- not universal (C.N. 5).

Equality is congruence of parts only in **Proposition I.35:** *Parallelograms on the same base and in the same parallels are equal.*





Equality is *not* congruence of parts in **Proposition XII.7:** *A triangular prism is divided into three equal triangular pyramids.*



This uses **Proposition XII.5:** *Triangular pyramids of the same height have to one another the* ***same ratio*** *as their bases.*

By **Book V,** a magnitude A has to B the **same ratio** (αὐτός λόγος) that C has to D if, for all positive integers k and n,

$$kA > nB \iff kC > nD.$$

Then the four magnitudes are **proportional** (ἀνάλογος), and today we write $A : B :: C : D$. The pair

$$\left(\left\{\frac{n}{k} : kA > nB\right\}, \left\{\frac{n}{k} : kA \leqslant nB\right\}\right)$$

is a **Dedekind cut.** Thus, for Dedekind (b. 1831), a ratio is a positive real number.

The theory of proportion is said to be due to **Eudoxus of Knidos** (b. 408 B.C.E.), a student of Plato.

By **Propositions V.9** and **16,**

$$\text{if } A : B :: C : C, \text{ then } A = B.$$

Proof. We use the so-called **Axiom of Archimedes** (b. 287 B.C.E.), found in Euclid's definition of **having a ratio** (λόγον ἔχω). Suppose

$$A > B.$$

Then for some n, we have $n(A - B) > B$. Consequently

$$nA > (n + 1)B, \qquad nC < (n + 1)C,$$

and therefore

$$A : B > C : C. \qquad \square$$

If the **Euclidean algorithm,**

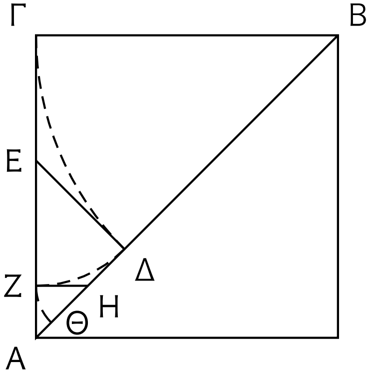
- applied to two **numbers,**
  - yields a *unit,* the numbers are *prime to one another* **(Proposition VII.1);**
  - yields a *number,* this is the *greatest common measure* of the original numbers **(VII.2);**
- applied to two **magnitudes,**
  - never ends, the two magnitudes are **incommensurable** (ασύμμετρος) **(X.2);**
  - yields a magnitude, this is the *greatest common measure* of the original magnitudes **(X.3).**

The Euclidean algorithm is **to subtract alternately** (ἀνθυφαιρέω).

This yields in the diagram

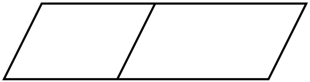
ΑΒ	ΑΓ
ΑΔ	ΑΖ
ΑΘ	...

and so the diagonal ΑΒ and side ΑΓ of the square are **incommensurable.**



The Euclidean algorithm is a remnant of an **earlier theory of proportion.**

According to Aristotle in the *Topics,*

> It would seem that in mathematics also some things are not easily proved by lack of a **definition,** such as that the straight line parallel to the side [of the parallelogram] divides *similarly* (ὁμοίως) both the line and the area. But
>
> 
>
> when the definition is stated, what was stated becomes immediately clear. For the areas and the lines have the same **antanaeresis** (ἀνταναίρεσις); and this is the *definition of the same ratio* (ὁρισμὸς τοῦ αὐτοῦ λόγου).

Alexander of Aphrodisias (fl. 200 C.E.) comments:

> For the *definition of proportions* (ὁρισμὸς τῶν ἀναλόγων) that the Ancients used is this:
>
> Magnitudes that have the same **anthyphaeresis** (ανθυφαίρεσις) are proportional.
>
> But [Aristotle] has called anthyphaeresis antanaeresis.

The connection between the Aristotle passage and the Euclidean algorithm was made by Oskar Becker in 1933.

Heath's second edition of the *Elements* is from 1925; his *History of Greek Mathematics*, 1921.

Anthyphaeresis yields continued fractions:

$$\begin{aligned}
\sqrt{3} &= 1 + (\sqrt{3} - 1), \\
\frac{1}{\sqrt{3}-1} &= \frac{\sqrt{3}+1}{2} = 1 + \frac{\sqrt{3}-1}{2}, \\
\frac{2}{\sqrt{3}-1} &= \sqrt{3}+1 = 2 + (\sqrt{3}-1),
\end{aligned}$$

and thus $\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}} = [1; \overline{1, 2}]$.

Likewise

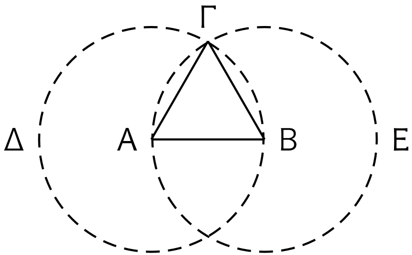
$$\begin{aligned} \sqrt{5} &= [2; \overline{4}], & \sqrt{13} &= [3; \overline{1, 1, 1, 1, 6}], \\ \sqrt{7} &= [2; \overline{1, 1, 1, 4}], & \sqrt{17} &= [4; \overline{8}], \\ \sqrt{11} &= [3; \overline{3, 6}], & \sqrt{19} &= [4; \overline{2, 1, 3, 1, 2, 8}]. \end{aligned}$$

Plato has Theaetetus say,

> Theodorus was proving to us a certain thing about square roots, I mean the square roots of 3 square feet and 5 square feet, namely, that these roots are not commensurable in length with the foot-length, and he proceeded in this way, taking each case in turn up to the root of 17 square feet; at this point for some reason he stopped.

4 Some propositions

Proposition I.1 of the *Elements* is the *problem* of constructing, on a given bounded straight line, an **equilateral triangle.**



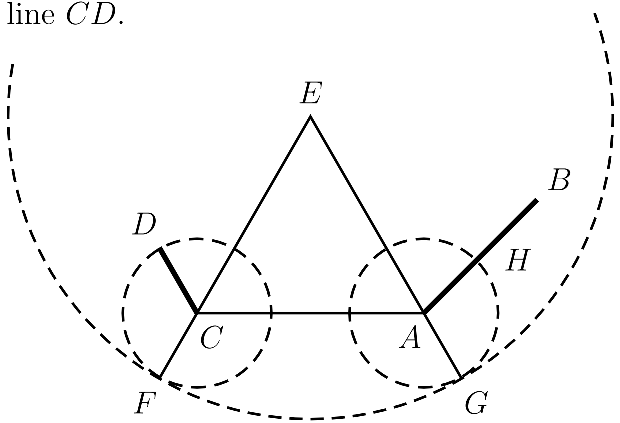
Does this need an **axiom of continuity?**

Proposition I.1 is a **problem** as opposed to a **theorem.**

Writes Pappus of Alexandria (fl. 320 C.E.):

> Those who wish to make more skilful distinctions in geometry find it worthwhile to call
>
> - a **problem** (πρόβλημα) that in which it is *proposed* (προβάλλεται) to do or construct something,
> - a **theorem** (θεώρημα) that in which the consequences and necessary implications of certain hypotheses *are investigated* (θεωρεῖται).
>
> But among the ancients some described them all as problems, some as theorems.

Propositions I.2 and **3** are the problem of **cutting off** from a given straight line AB a segment equal to a shorter straight line CD.



Thus our compass will **hold the gap** between its points.

5 Hilbert's geometry

In Euclid, transfer of lengths is **proved.**

So is transfer of angles **(Proposition I.23).**

These are **axioms** for David Hilbert (b. 1862) in *The Foundations of Geometry.*

But **drawing circles** is not.

In Hilbert's system, constructing an equilateral triangle takes a lot of work.

Hilbert's axioms for the plane:

I. Axiom(s) of connection: Two distinct points lie on a unique straight line.

II. Axioms of order:

   - The points of a straight line are densely linearly ordered without extrema.

   Pasch's Axiom: A straight line intersecting one side of a triangle intersects one of the other two.

III. Axiom of parallels (Euclid's Axiom): Through a given point, exactly one parallel to a given straight line can be drawn.

IV. Axioms of congruence.

- Every segment can be uniquely *laid off* upon a given side of a given point of a given straight line.
- Congruence of segments is transitive and additive.
- Every angle can be uniquely *laid off* upon a given side of a given half-ray.
- Congruence of angles is transitive.
- Side-Angle-Side.

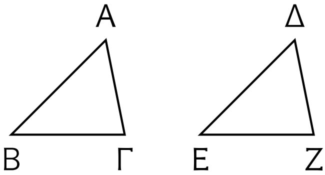
V. Axiom of continuity. (Archimedean axiom.)

Axiom of Completeness.

Side-Angle-Side is an axiom for David Hilbert (b. 1862).

For Euclid it is **Proposition I.4,** the first proper *theorem.* Suppose

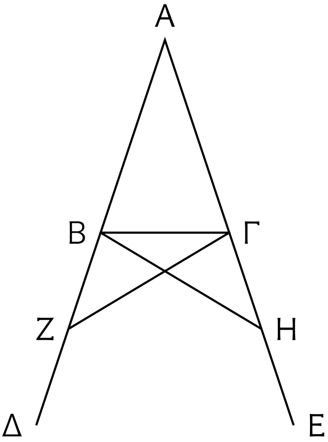
$$\mathsf{AB} = \mathsf{\Delta E},$$
$$\mathsf{A\Gamma} = \mathsf{\Delta Z},$$
$$\angle\mathsf{BA\Gamma} = \angle\mathsf{E\Delta Z}.$$



Then, by the meaning of equality:

1. AB can be applied exactly to $\mathsf{\Delta E}$.
2. At the same time, $\angle\mathsf{BA\Gamma}$ can be applied to $\angle\mathsf{E\Delta Z}$.
3. Then $\mathsf{A\Gamma}$ will be applied exactly to $\mathsf{\Delta Z}$.
4. Consequently $\mathsf{B\Gamma}$ will be applied exactly to EZ.

Proposition I.5 is that the base angles of an isosceles triangle are equal…



$$
\begin{aligned}
\because \quad & \mathsf{AB} = \mathsf{A\Gamma} \\
& \mathsf{AZ} = \mathsf{AH} \\
\therefore \quad & [\triangle \mathsf{AZ\Gamma} \cong \triangle \mathsf{AHB}] \\
& \mathsf{Z\Gamma} = \mathsf{HB} \\
& \angle \mathsf{A\Gamma Z} = \angle \mathsf{ABH} \\
& \angle \mathsf{AZ\Gamma} = \angle \mathsf{AHB} \\
& \mathsf{BZ} = \mathsf{\Gamma H} \\
& [\triangle \mathsf{BZ\Gamma} \cong \triangle \mathsf{\Gamma HB}] \\
& \angle \mathsf{B\Gamma Z} = \angle \mathsf{\Gamma BH} \\
& \angle \mathsf{AB\Gamma} = \angle \mathsf{A\Gamma B}
\end{aligned}
$$

Is Proposition I.5 the world's first theorem?

From *A Commentary on the First Book of Euclid's* Elements, by Proclus (of Byzantium, b. early 5th c. C.E.):

> We are endebted to old **Thales** [of Miletus, b. *c.* 624 B.C.E.] for the discovery of [Proposition I.5] and many other theorems. For he, it is said, was the first to notice and assert that in every isosceles triangle the angles at the base are equal, though in somewhat archaic fashion he called the equal angles similar (ὁμοίος). [p. 250]

Proposition I.5 is that the base angles of an isosceles triangle are equal.

Immanuel Kant (b. 1724) alludes to it in the *Critique of Pure Reason*:

> **Mathematics** has from the earliest times... travelled the secure path of a science. Yet it must not be thought that it was as easy for it as for logic... to find that royal path... its transformation is to be ascribed to a **revolution,** brought about by the happy inspiration of a single man... a new light broke upon the first person who demonstrated [Proposition I.5] (whether he was called "Thales" or had some other name). [B x–xi]

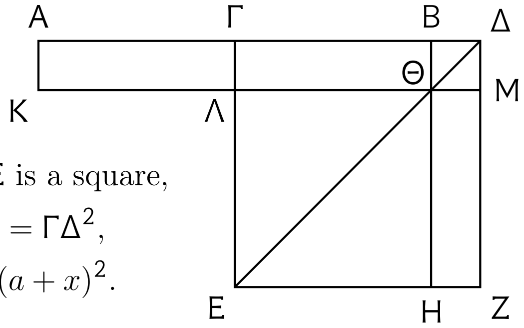
6 Analysis and synthesis

Again Pappus of Alexandria:

> The so-called *Treasury of Analysis*... is... the work of three men, Euclid the writer of the *Elements*, Apollonius of Perga, and Aristaeus the elder, and proceeds by the method of *analysis and synthesis.*
>
> Now **analysis** (ἀνάλυσις) is a way of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis...
>
> But in **synthesis** (συνθέσις) we proceed in the opposite way...

Euclid's **Proposition II.6** is a *synthesis*:

> If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.



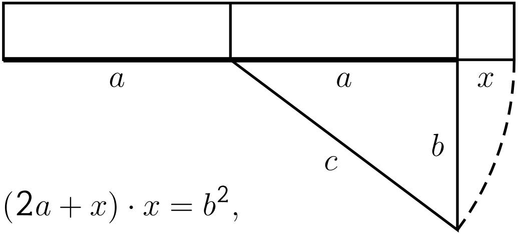
$\because \quad \mathsf{A\Gamma} = \mathsf{\Gamma B}$ and $\mathsf{\Gamma\Delta ZE}$ is a square,

$$\therefore \quad \mathsf{A\Delta} \cdot \mathsf{\Delta B} + \mathsf{\Gamma B}^2 = \mathsf{\Gamma\Delta}^2,$$

$$(2a + x) \cdot x + a^2 = (a + x)^2.$$

Proposition II.6 results from an analysis of a special case of **Proposition VI.29:**

> To a given straight line $[2a]$ to *apply* (παραβάλλω) a [rectangle] equal to a given [square b^2] and **exceeding** (ὑπερβάλλω) by a [square].



We want x such that

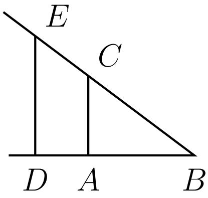
$$(2a + x) \cdot x = b^2,$$

$$(2a + x) \cdot x + a^2 = a^2 + b^2 = c^2.$$

By II.6 it suffices if $a + x = c$.

7 Descartes's geometry

Euclid's products are **areas.** As Descartes (b. 1596) observes, they can be **lengths,** if a unit length is chosen.



If AB is the unit, and $DE \parallel AC$, then

$$BE = BD \cdot BC.$$

Thus any number of lengths can be **multiplied.**

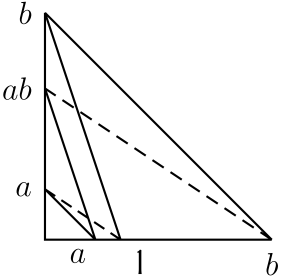
Descartes quotes Pappus (fl. 320 C.E.) as noting that any number of **ratios** can be multiplied:

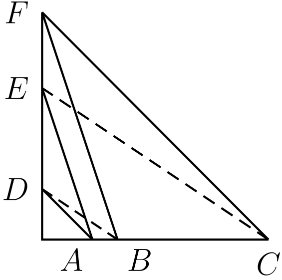
$$A : B \,\&\, B : C \,\&\, \ldots \,\&\, Y : Z :: A : Z.$$

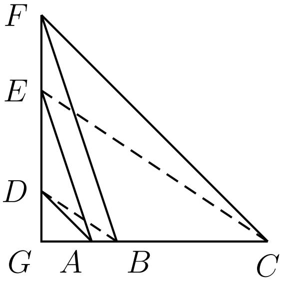
As Hilbert shows, multiplication is commutative by a version of **Pappus's Hexagon Theorem.**

Let $AD \parallel CF$ and $AE \parallel BF$. Then

$$ab = ba \iff BD \parallel CE,$$
$$BD \parallel CE.$$







We have assumed $AD \parallel CF$ and $AE \parallel BF$. By Euclid's **Proposition VI.2,**

$$GA : GC :: GD : GF,$$
$$GB : GA :: GF : GE.$$

Then by **V.23,**

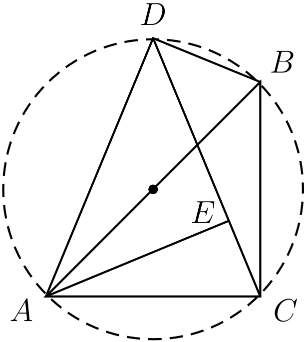
$$GB : GC :: GD : GE,$$

so $CE \parallel BD$. But Euclid's proof of V.23 relies on **V.8,**

$$A > B \implies A : C > B : C,$$

and the proof of this uses the **Archimedean Axiom.**

Hilbert avoids the Archimedean Axiom with a lemma:



$$
\begin{aligned}
AE &\perp CD, \\
\angle ADB &= 90^\circ, && \textbf{[III.31]} \\
\angle ACB &= 90^\circ, \\
\angle DAB &= \angle DCB && \textbf{[III.27]} \\
&= \angle EAC, && \textbf{[I.32]} \\
\angle DAE &= \angle BAC.
\end{aligned}
$$

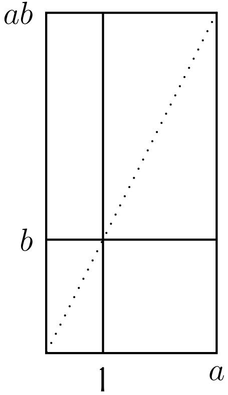
Thus, if $AB = c$, $\angle BAC = \beta$, and $\angle CAE = \alpha$, then

$$(c\cos\beta)\cos\alpha = (c\cos\alpha)\cos\beta.$$

Alternatively, we can establish Hilbert's **algebra of segments** on the basis of Book I of the *Elements* alone.

Multiplication as in the diagram is commutive, given that:

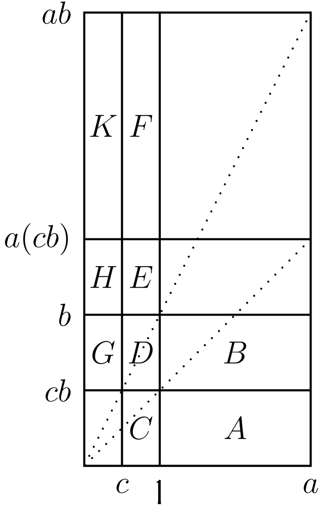
- the rectangles about the diagonal are equal **(I.43)**,
- all rectangles of equal dimensions are congruent **(I.8, 33)**.



For associativity, we use I.43 and its converse:

By definition of ab, cb, and $a(cb)$,

$$A + B = E + F + H + K,$$
$$C = G,$$
$$A = D + E + G + H.$$



Also $a(cb) = c(ab)$ if and only if

$$C + D + E = K.$$

We compute

$$D + C + B = F + K.$$

We finish by noting

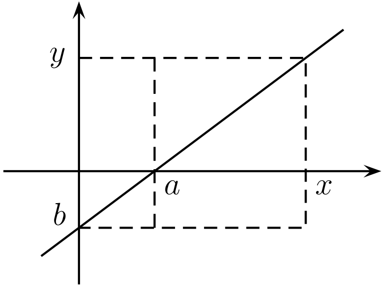
$$B = E + F.$$

8 Conclusion

We thus **interpret** an ordered field in the Euclidean plane.

The **positive elements** of this ordered field are congruence-classes of line-segments.

We impose a **rectangular coordinate system** as usual.



Straight lines are now given by **linear equations:**

$$a \cdot (y - b) = -b \cdot x,$$
$$bx + ay = ab.$$

Conversely, let an **ordered field** K be given.

In $K \times K$, obtain the **Cauchy–Schwartz Inequality,** and then the **Triangle Inequality.** Define

- **line segments:** $\boldsymbol{ab}$ is the set
$$\{\boldsymbol{x} \colon |\boldsymbol{b}-\boldsymbol{a}| = |\boldsymbol{b}-\boldsymbol{x}| + |\boldsymbol{x}-\boldsymbol{a}|\};$$
- their **congruence:** $\boldsymbol{ab} \cong \boldsymbol{cd}$ means
$$|\boldsymbol{b}-\boldsymbol{a}| = |\boldsymbol{d}-\boldsymbol{c}|;$$
- **angle congruence:** $\angle\boldsymbol{bac} \cong \angle\boldsymbol{edf}$ means
$$\frac{(\boldsymbol{c}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a})}{|\boldsymbol{c}-\boldsymbol{a}|\cdot|\boldsymbol{b}-\boldsymbol{a}|} = \frac{(\boldsymbol{f}-\boldsymbol{d})\cdot(\boldsymbol{e}-\boldsymbol{d})}{|\boldsymbol{f}-\boldsymbol{d}|\cdot|\boldsymbol{e}-\boldsymbol{d}|}.$$

Alternatively, define figures to be **congruent** when they can be transformed into one another by a composition of a **translation**

$$\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{a}$$

and a **rotation**

$$\boldsymbol{x} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \boldsymbol{x},$$

where $a^2 + b^2 = 1$.

K should be **Euclidean** or at least **Pythagorean.**

One just ought to be clear what one is doing.

References

[1] Aristotle. *The Metaphysics, Books I–IX*, volume XVII of *Loeb Classical Library*. Harvard University Press and William Heinemann Ltd., Cambridge, Massachusetts, and London, 1980. with an English translation by Hugh Tredennick. First printed 1933.

[2] Euclid. *Euclidis Elementa*, volume I of *Euclidis Opera Omnia*. Teubner, 1883. Edidit et Latine interpretatvs est I. L. Heiberg.

[3] Euclid. *The thirteen books of Euclid's Elements translated from the text of Heiberg. Vol. I: Introduction and Books I, II. Vol. II: Books III–IX. Vol. III: Books X–XIII and Appendix*. Dover Publications Inc., New York, 1956. Translated with introduction and commentary by Thomas L. Heath, 2nd ed.

[4] Euclid. *The Bones*. Green Lion Press, Santa Fe, NM, 2002. A handy where-to-find-it pocket reference companion to Euclid's *Elements*.

[5] Euclid. *Euclid's Elements*. Green Lion Press, Santa Fe, NM, 2002. All thirteen books complete in one volume, the Thomas L. Heath translation, edited by Dana Densmore.

[6] Thomas Heath. *A history of Greek mathematics. Vol. I.* Dover Publications Inc., New York, 1981. From Thales to Euclid, Corrected reprint of the 1921 original.

[7] Thomas Heath. *A history of Greek mathematics. Vol. II.* Dover Publications Inc., New York, 1981. From Aristarchus to Diophantus, Corrected reprint of the 1921 original.

[8] Herodotus. *The Persian Wars, Books I–II*, volume 117 of *Loeb Classical Library.* Harvard University Press, Cambridge, Massachusetts and London, England, 2004. Translation by A. D. Godley; first published 1920; revised, 1926.

[9] David Hilbert. *The foundations of geometry.* Authorized translation by E. J. Townsend. Reprint edition. The Open Court Publishing Co., La Salle, Ill., 1959.

[10] Immanuel Kant. *Critique of Pure Reason.* The Cambridge Edition of the Works of Kant. Cambridge University Press, Cambridge, paperback edition, 1999. Translated and edited by Paul Guyer and Allen W. Wood; first published, 1998.

[11] Plato. *Euthyphro, Apology, Crito, Phaedo, Phaedrus.* Loeb Classical Library. Harvard University Press and William Heinemann Ltd., Cambridge, Massachusetts, and London, 1982. with an English Translation by Harold North Fowler, first printed 1914.

[12] Proclus. *A commentary on the first book of Euclid's* Elements. Princeton Paperbacks. Princeton University Press, Princeton, NJ, 1992. Translated from the Greek and with an introduction and notes by Glenn R. Morrow, reprint of the 1970 edition, with a foreword by Ian Mueller.

[13] Ivor Thomas, editor. *Selections illustrating the history of Greek mathematics. Vol. I. From Thales to Euclid*, volume 335 of *Loeb Classical Library*. Harvard University Press, Cambridge, Mass., 1951. With an English translation by the editor.

[14] Ivor Thomas, editor. *Selections illustrating the history of Greek mathematics. Vol. II. From Aristarchus to Pappus*, volume 362 of *Loeb Classical Library*. Harvard University Press, Cambridge, Mass, 1951. With an English translation by the editor.