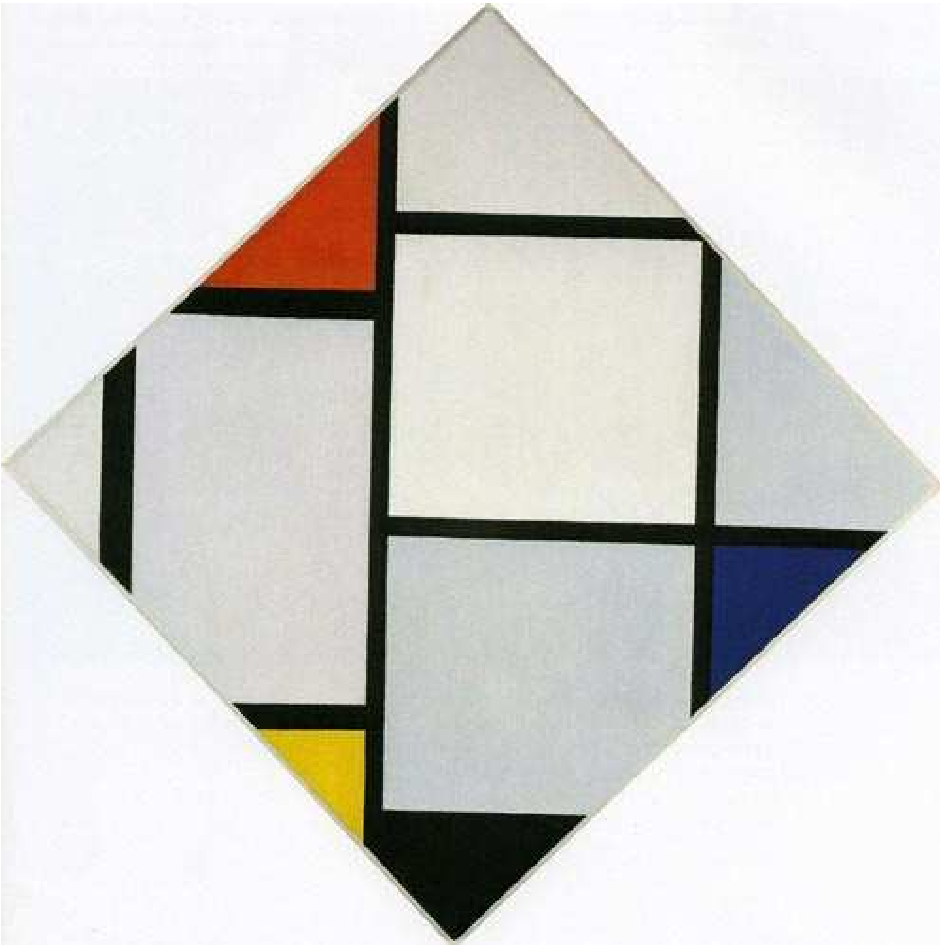

Piet Mondrian, *Tableau No. IV; Lozenge Composition with Red, Gray, Blue, Yellow, and Black*

Interacting Rings

David Pierce

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Lyon

The interacting rings in question arise from **differential fields:**

$$(K, \partial_0, \ldots, \partial_{m-1}),$$

where

1. K is a field—in particular, a **commutative ring;**
2. each ∂_i is a **derivation** of K: an endomorphism D of the abelian group of K that obeys the **Leibniz rule,**

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y);$$

3. $[\partial_i, \partial_j] = 0$ in each case, where $[\,\cdot\,,\,\cdot\,]$ is the **Lie bracket,** so

$$[x, y] = x \circ y - y \circ x.$$

A standard example is $(\mathbb{C}(x_0, \ldots, x_{m-1}), \frac{\partial}{\partial x_0}, \ldots, \frac{\partial}{\partial x_{m-1}})$.
In general, let

$$V = \operatorname{span}_K(\partial_i \colon i < m) \subseteq \operatorname{Der}(K);$$

then V is also a **Lie ring.**

Recall some notions due to Abraham Robinson:
The *quantifier-free* theory of $\mathfrak{A}_A$ is denoted by

$$\operatorname{diag}(\mathfrak{A}).$$

A theory T is **model complete** under any of three equivalent conditions:

1. whenever $\mathfrak{A}$ is a **model** of T, the theory

$$T \cup \operatorname{diag}(\mathfrak{A})$$

   is **complete;**

2. whenever $\mathfrak{A} \models T$,

$$T \cup \operatorname{diag}(\mathfrak{A}) \vdash \operatorname{Th}(\mathfrak{A}_A);$$

3. whenever $\mathfrak{A}, \mathfrak{B} \models T$,

$$\mathfrak{A} \subseteq \mathfrak{B} \implies \mathfrak{A} \preccurlyeq \mathfrak{B}.$$

Then T is complete if all models have a common submodel.

Robinson's examples of model complete theories include the theories of

1. torsion-free divisible abelian groups (*i.e.* vector spaces over $\mathbb{Q}$),
2. algebraically closed fields,
3. real-closed fields.

Theorem (Robinson)**.** T is model complete, provided

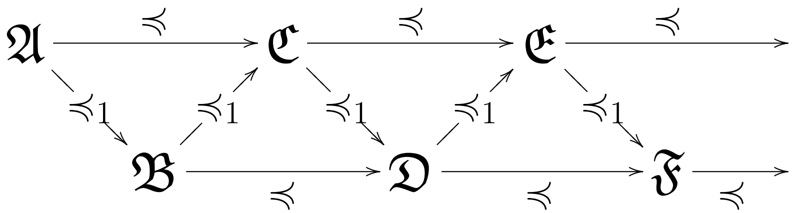
$$T \cup \operatorname{diag}(\mathfrak{A}) \vdash \operatorname{Th}(\mathfrak{A}_A)_\forall$$

whenever $\mathfrak{A} \models T$, that is,

$$\mathfrak{A} \subseteq \mathfrak{B} \implies \mathfrak{A} \preccurlyeq_1 \mathfrak{B}$$

whenever $\mathfrak{A}, \mathfrak{B} \models T$.

Proof. If $\mathfrak{A} \preccurlyeq_1 \mathfrak{B}$, then $\mathfrak{A} \preccurlyeq \mathfrak{C}$ for some $\mathfrak{C}$, where $\mathfrak{B} \subseteq \mathfrak{C}$; then $\mathfrak{B} \preccurlyeq_1 \mathfrak{C}$, so continue:



□

Let

$$\mathrm{DF}^m = \mathrm{Th}(\{\text{fields with } m \text{ commuting derivations}\}),$$
$$\mathrm{DF}^m_0 = \mathrm{DF}^m \cup \{p \neq 0 \colon p \text{ prime}\}.$$

Theorem (McGrail, 2000)**.** DF^m_0 has a **model companion,** DCF^m_0: that is,

$$(\mathrm{DF}^m_0)_\forall = (\mathrm{DCF}^m_0)_\forall$$

and DCF^m_0 is model complete.

Theorem (Yaffe, 2001)**.** The theory of fields of characteristic 0 with m derivations D_i, where

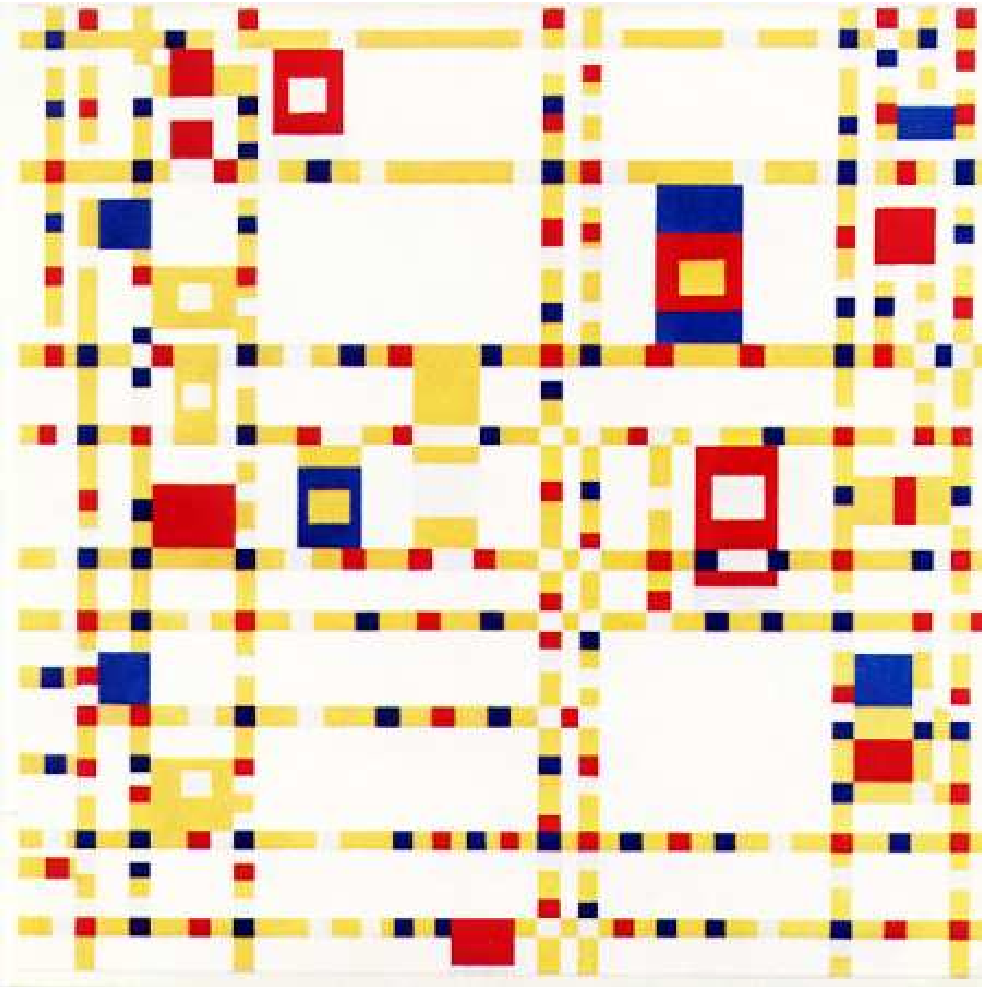
$$[D_i, D_j] = \sum a^k_{i\,j} D_k,$$

has a model companion.

Theorem (P, 2003; Singer, 2007)**.** The latter follows readily from the former.

Theorem (P, submitted March, 2008)**.** DF^m has a model companion, DCF^m, given in terms of varieties.

What is the model theory of V?



Piet Mondrian, *Broadway Boogie Woogie*

First consider rings in general.

In the most general sense, a **ring** is a structure

$$(E, \cdot),$$

where

1. E is an abelian group in $\{0, -, +\}$, and

2. the binary operation $\cdot$ distributes over $+$ in both senses: it is a **multiplication.**

Beyond this, there are axioms for:

commutative rings	**Lie rings**
$xy - yx = 0$	$x^2 = 0$
$(xy)z = x(yz)$	$(xy)z = x(yz) - y(xz)$

By itself, $(xy)z = x(yz)$ defines **associative rings;**
and $(xy)z = x(yz) - y(xz)$ is the **Jacobi identity.**

For rings, are there **representation theorems** like the following?

Theorem (Cayley)**.** Every abstract group $(G, 1, {}^{-1}, \cdot\,)$ embeds in the symmetry group

$$(\mathrm{Sym}(G), \mathrm{id}_G, {}^{-1}, \circ\,)$$

under $x \mapsto \lambda_x$, where

$$\lambda_g(y) = g \cdot y.$$

A ring is **Boolean** if it satisfies $x^2 = x$.

Theorem (Stone)**.** Every abstract Boolean ring $(R, 0, +, \cdot)$ or $\mathfrak{R}$ embeds in a Boolean ring of sets

$$(\mathscr{P}(\Omega), \varnothing, \triangle, \cap).$$

(Here $\Omega = \{\text{prime ideals of } \mathfrak{R}\}$, and the embedding is $x \mapsto \{\mathfrak{p} \colon x \notin \mathfrak{p}\}$.)

For associative rings and Lie rings *only,* there are such theorems.

I know no representation theorem for **abelian groups.** There are just **'prototypical'** abelian groups, like $\mathbb{Z}$. One might mention *Pontryagin duality:* Every (topological) abelian group G embeds in G^{**}, where $G^* = \mathrm{Hom}(G, \mathbb{R}/\mathbb{Z})$.

Prototypical **associative rings** include

1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$;

2. matrix rings.

But there are **non-associative rings:**

1. $(\mathbb{R}^3, \times)$ is a **Lie ring** (in fact, the *Lie algebra* of $\mathrm{SO}(3, \mathbb{R})$);

2. the **Cayley–Dickson algebras** $\mathbb{R}$, $\mathbb{R}'$, ... become non-associative after $\mathbb{R}''$ (which is $\mathbb{H}$):

Let $(E, \cdot)$ be a ring with an *involutive anti-automorphism* or **conjugation** $x \mapsto \bar{x}$. The abelian group $\mathrm{M}_2(E)$ is a ring under

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax + zb & ya + bw \\ xc + dz & cy + wd \end{pmatrix},$$

with conjugation

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} & \bar{z} \\ \bar{y} & \bar{w} \end{pmatrix}.$$

Let E' comprise the matrices

$$\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}.$$

Then E' is closed under the operations, and E embeds under

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}.$$

If E is an abelian group, then its multiplications compose an abelian group that has an involutory automorphism,

$$\mathsf{m} \mapsto \dot{\mathsf{m}},$$

where $\dot{\mathsf{m}}$ is the **opposite** of m:

$$\dot{\mathsf{m}}(x, y) = \mathsf{m}(y, x).$$

Let $\mathrm{End}(E)$ be the *abelian group* of endomorphisms of E. Then

1. $(\mathrm{End}(E), \circ)$ is an associative ring;
2. $(\mathrm{End}(E), \circ - \dot{\circ})$ is a Lie ring;
3. $(\mathrm{End}(E), \circ + \dot{\circ})$ is a **Jordan ring:**[1] a ring satisfying

$$xy = yx, \qquad (xy)x^2 = x(yx^2).$$

[1]Pascual Jordan, 1902–80.

If $(E, \cdot)$ is a ring, let

$$x \mapsto \lambda_x \colon E \longrightarrow \operatorname{End}(E),$$

where (as in the Cayley Theorem)

$$\lambda_a(y) = a \cdot y.$$

If p and q are in $\mathbb{Z}$, let $(E, \cdot)$ be called a (p, q)**-ring** if

$$x \mapsto \lambda_x \colon (E, \cdot) \longrightarrow (\operatorname{End}(E), p\circ - q\overset{\bullet}{\circ}).$$

Theorem. All associative rings are $(1, 0)$-rings; all Lie rings are $(1, 1)$-rings. In particular, $(\operatorname{End}(E), p\circ - q\overset{\bullet}{\circ})$ is a (p, q)-ring if

$$(p, q) \in \{(0, 0), (1, 0), (1, 1)\}.$$

Theorem (P)**.** The converse holds.

Proof. We have

$$x \mapsto \lambda_x : (\mathrm{End}(E), p\circ - q\dot{\circ}) \to (\mathrm{End}(\mathrm{End}(E)), p\circ - q\dot{\circ})$$

if and only if

$$\lambda_{xy} = \lambda_x \lambda_y,$$

that is,

$$\lambda_{px\circ y - qy\circ x}(z) = (p\lambda_x \circ \lambda_y - q\lambda_y \circ \lambda_x)(z),$$

that is,

$$\begin{aligned} p(px \circ y - qy \circ x) \circ z - qz \circ (px \circ y - qy \circ x) \\ = p\big(px \circ (py \circ z - qz \circ y) - q(py \circ z - qz \circ y) \circ x\big) \\ - q\big(py \circ (px \circ z - qz \circ x) - q(px \circ z - qz \circ x) \circ y\big), \end{aligned}$$

that is,

$$p^2 = p^3, \quad pq = p^2 q, \quad qp = q^3, \quad p^2 q = pq^2, \quad pq = pq^2$$

—assuming the 6 compositions $x \circ y \circ z$ *etc.* are independent in some example; and they are when $E = \mathbb{Z}^4$. □

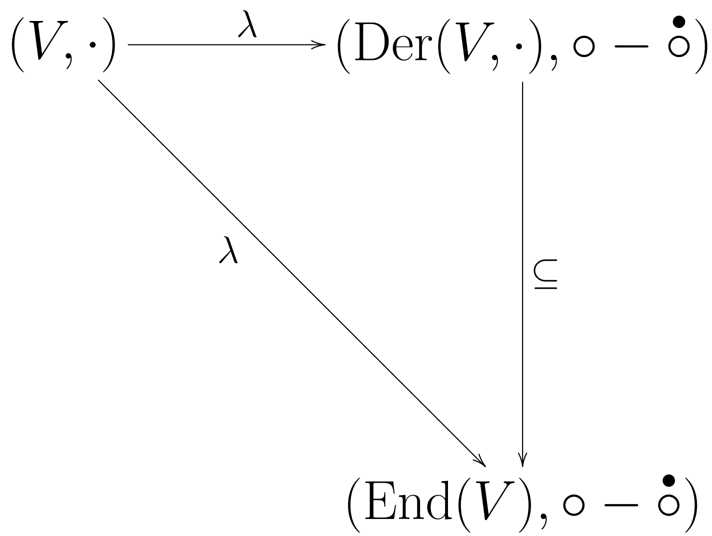
If $(V, \cdot)$ is a Lie ring, then each λ_x is a **derivation** of it: Write the Jacobi identity as

$$x(yz) = (xy)z + y(xz);$$

this means

$$\lambda_x(yz) = \lambda_x(y) \cdot z + y \cdot \lambda_x(z).$$

Thus λ factors:



For any abelian group V, the Lie ring $(\mathrm{End}(V), \circ - \overset{\bullet}{\circ})$ acts as a ring of derivations of the **associative** ring $(\mathrm{End}(V), \circ)$:

$$\begin{aligned}
[z, x \circ y] &= z \circ x \circ y \qquad\qquad\qquad\qquad\qquad\qquad - x \circ y \circ z \\
&= z \circ x \circ y - x \circ z \circ y + x \circ z \circ y - x \circ y \circ z \\
&= [z, x] \circ y \qquad\qquad\qquad\quad + x \circ [z, y].
\end{aligned}$$

$$\begin{array}{ccc}
(\mathrm{End}(V), \circ - \overset{\bullet}{\circ}) & \xrightarrow{\lambda} & (\mathrm{Der}(\mathrm{End}(V), \circ), \circ - \overset{\bullet}{\circ}) \\
& \searrow^{\lambda} & \downarrow \subseteq \\
& & (\mathrm{End}(\mathrm{End}(V)), \circ - \overset{\bullet}{\circ})
\end{array}$$

Combine the diagrams—again, $(V, \cdot)$ is a Lie ring:

$$\begin{array}{ccccc}
(V,\cdot) & \xrightarrow{\lambda} & (\mathrm{Der}(V,\cdot), \circ - \overset{\bullet}{\circ}) & & \\
& \searrow^{\lambda} & \downarrow \subseteq & & \\
& & (\mathrm{End}(V), \circ - \overset{\bullet}{\circ}) & \xrightarrow{\lambda} & (\mathrm{Der}(\mathrm{End}(V), \circ), \circ - \overset{\bullet}{\circ}) \\
& & & \searrow^{\lambda} & \downarrow \subseteq \\
& & & & (\mathrm{End}(\mathrm{End}(V)), \circ - \overset{\bullet}{\circ})
\end{array}$$

Each D in V determines the derivation

$$f \mapsto Df$$

of $(\mathrm{End}(V), \circ)$, where

$$Df = \lambda_{\lambda_D}(f) = [\lambda_D, f],$$

so that

$$Df(x) = D \cdot (f(x)) - f(D \cdot x).$$

If $(K, \partial_0, \dots, \partial_{m-1}) \models \mathrm{DF}^m$, and $V = \mathrm{span}_K(\partial_i \colon i < m)$, and t in K is not constant, then

$$K = \{Dt \colon D \in V\}.$$

Indeed, if $Dt = a \neq 0$, then

$$x = \frac{x}{a}(Dt) = \left(\frac{x}{a}D\right)t.$$

There is an *elementary* class consisting of all $(V, \cdot, t)$ such that

1. $(V, \cdot)$ is a Lie ring,
2. $t \in \mathrm{End}(V)$,
3. $(\{Dt \colon D \in V\}, \circ)$ is a **field** K,
4. for all f and g in K and D in V,

$$f \circ (Dg) = (f(D))g,$$

5. $\dim_K(V) \leqslant m$.

Let VL^m be the theory of this class. Then VL^m has $\forall\exists$ axioms.

Theorem (P)**.** The theory VL^m has a model companion, whose models are precisely those models $(V, \cdot, t)$ of VL^m such that, when we let

$$K = (\{Dt\colon D \in V\}, \circ),$$

then V has a commuting basis $(\partial_i\colon i < m)$ over K, and

$$(K, \partial_0, \dots, \partial_{m-1}) \models \mathrm{DCF}^m .$$

Here $\dim_C(V) = \infty$, where C is the constant field.

However, for an infinite field K, the theory of Lie algebras over K apparently has no model-companion (Macintyre, announced 1973).

Is there a model-complete theory of infinite-dimensional Lie algebras with no extra structure?

Adolph Gottlieb, *Centrifugal*

We can also consider (V, K) as a two-sorted structure.

Suppose first (V, K) is just a vector space, in the signature comprising

1. the signature of abelian groups, for the vectors;

2. the signature of rings, for the scalars;

3. a symbol $*$ for the (right) action $(v, x) \mapsto v * x$ of K on V.

Let the theory of such structures of dimension n be

$$T_n,$$

where $n \in \{1, 2, 3, \dots, \infty\}$.

Theorem (Kuzichev, 1992)**.** T_n admits elimination of quantified vector-variables.

A theory is **inductive** if unions of chains of models are models.

Theorem (Łoś & Suszko 1957, Chang 1959)**.** A theory T is inductive if and only if

$$T = T_{\forall\exists}.$$

Hence all model complete theories have $\forall\exists$ axioms.
Of an arbitrary T, a model $\mathfrak{A}$ is **existentially closed** if

$$\mathfrak{A} \subseteq \mathfrak{B} \implies \mathfrak{A} \preccurlyeq_1 \mathfrak{B}$$

for all models $\mathfrak{B}$ of T.

Theorem (Eklof & Sabbagh, 1970)**.** Suppose T is inductive. Then T has a model companion if and only if the class of its existentially closed models is elementary. In this case, the theory of this class is the model companion.

Again, T_n is the theory of vector spaces of dimension n.

If $n > 1$, then no completion ${T_n}^*$ of T_n can be model complete, because it cannot be $\forall\exists$ axiomatizable:

There is a chain

$$(V, K) \subseteq (V', K') \subseteq \cdots \subseteq (V^{(s)}, K^{(s)}) \subseteq \cdots$$

of models of ${T_n}^*$, where

1. $(V^{(s)}, K^{(s)})$ has basis $(v_s, \dots, v_{s+n-1})$, but

2. $v_s = v_{s+1} * x_s$ for some x_s in $K^{(s+1)} \smallsetminus K^{(s)}$, so

3. the union of the chain has dimension 1.

The situation changes if there are *predicates* for linear dependence.

Let VS_n (where n is a positive integer) be the theory of vector spaces with a new n-ary predicate P^n for linear dependence. So P^n is defined by

$$\exists x^0 \cdots \exists x^{n-1} \Big(\sum_{i<n} v_i * x^i = 0 \;\&\; \bigvee_{i<n} x^i \neq 0\Big).$$

Let VS_∞ be the union of the VS_n.

Theorem (P)**.**

1. VS_n has a model companion, the theory of n-dimensional spaces over algebraically closed fields.

2. VS_∞ has a model companion (even, model *completion*), the theory if infinite-dimensional spaces over algebraically closed fields.

The key is lowering dimension to n.
Given a field-extension L/K, where where

$$[L : K] \geqslant n + 1,$$

we can embed (K^{n+1}, K) in (L^n, L), *as models of* VS_n, under

$$\begin{pmatrix} x^0 \\ \vdots \\ x^{n-1} \\ x^n \end{pmatrix} \mapsto \begin{pmatrix} 1 & & 0 & -a^0 \\ & \ddots & & \vdots \\ 0 & & 1 & -a^{n-1} \end{pmatrix} \begin{pmatrix} x^0 \\ \vdots \\ x^{n-1} \\ x^n \end{pmatrix},$$

that is,

$$\boldsymbol{x} \mapsto \left(I \,\middle|\, -\boldsymbol{a} \right) \boldsymbol{x},$$

where the a^i are chosen from L so that the tuple

$$(a^0, \dots, a^{n-1}, 1)$$

is linearly independent over K.

Why? Given an $(n+1)\times n$ matrix U over K, we want to show

$$\operatorname{rank}(U) = n \iff \det\Big(\big(\, I \,\big|\, -\boldsymbol{a} \,\big)\, U\Big) \neq 0.$$

Write U as $\left(\begin{array}{c} X \\ \hline \boldsymbol{y}^{\mathrm{t}} \end{array}\right)$. Then

$$\operatorname{rank}(U) = n \iff \det\left(\begin{array}{c|c} X & \boldsymbol{a} \\ \hline \boldsymbol{y}^{\mathrm{t}} & 1 \end{array}\right) \neq 0.$$

Moreover,

$$\det\left(\begin{array}{c|c} X & \boldsymbol{a} \\ \hline \boldsymbol{y}^{\mathrm{t}} & 1 \end{array}\right) = \det(X - \boldsymbol{a}\boldsymbol{y}^{\mathrm{t}}),$$

$$X - \boldsymbol{a}\boldsymbol{y}^{\mathrm{t}} = \big(\, I \,\big|\, -\boldsymbol{a} \,\big) \left(\begin{array}{c} X \\ \hline \boldsymbol{y}^{\mathrm{t}} \end{array}\right) = \big(\, I \,\big|\, -\boldsymbol{a} \,\big)\, U.$$

That does it.

Compare:
Let T be the theory of fields with an algebraically closed subfield. The existentially closed models of T have transcendence-degree 1, because of

Theorem (Robinson)**.** We have an inclusion

$$K(x, y) \subseteq L(y)$$

of pure transcendental extensions, where

$$K(x, y) \cap L = K,$$

provided

$$L = K(\alpha, \beta),$$

where

$$\alpha \notin K(x, y)^{\mathrm{alg}}, \qquad \beta = \alpha x + y.$$

(Hence T has no model companion.)

A **Lie–Rinehart pair** can be defined as any (V, K), where:

1. V and K are abelian groups, *each* acting on the other, from the left and right respectively, by

$$(x, y) \mapsto x\, D\, y, \qquad\qquad x * y \leftarrow\!\shortmid (x, y).$$

2. The actions are faithful:

$$\exists y\ (x\, D\, y = 0 \Rightarrow x = 0), \qquad \exists x\ (x * y = 0 \Rightarrow y = 0).$$

3. Multiplications are induced,
   (i) on V, by the bracket;
   (ii) on K, by (opposite) composition:

$$[x, y]\, D\, z = x\, D(y\, D\, z) - y\, D(x\, D\, z), \quad x * (y \cdot z) = (x * y) * z.$$

4. These multiplications are compatible with the actions:

$$(x * y)\, D\, z = (x\, D\, z) \cdot y, \quad x * (y\, D\, z) = [y, x * z] - [y, x] * z.$$

Then V does act on K as a Lie ring **of derivations;** that is,

$$x\,D(y \cdot z) = (x\,D\,y) \cdot z + y \cdot (x\,D\,z).$$

Indeed,

$$\begin{aligned}
&w * (x\,D(y \cdot z)) \\
&= [x, w * (y \cdot z)] - [x, w] * (y \cdot z) \\
&= [x, (w * y) * z] - ([x, w] * y) * z \\
&= (w * y) * (x\,D\,z) + [x, w * y] * z \\
&\qquad\qquad - [x, w * y] * z + (w * (x\,D\,y)) * z \\
&= (w * y) * (x\,D\,z) + (w * (x\,D\,y)) * z \\
&= w * (y \cdot (x\,D\,z)) + w * ((x\,D\,y) \cdot z) \\
&= w * (y \cdot (x\,D\,z) + (x\,D\,y) \cdot z).
\end{aligned}$$

We may (asymmetrically!) make K commutative, and make V torsion-free as a K-module, so K is an integral domain.

The multiplications are **definable.**
Indeed, let V and K act mutually as abelian groups, as before. Then K becomes a sub-ring of $(\mathrm{End}(V), \circ)$ and an integral domain when we require

$$\begin{gathered}\exists w\ (x * y) * z = x * w,\\ x * y = 0 \Rightarrow x = 0 \vee y = 0,\\ (x * y) * z = x * w \Rightarrow x = 0 \vee (u * y) * z = u * w,\\ (x * y) * z = (x * z) * y\end{gathered}$$

Then we can require V to act on K as a **module** (over K) of **derivations:**

$$\begin{aligned}&(x * y) * z = x * w\\ &\qquad \Rightarrow x * (v\, D\, w) = (x * y) * (v\, D\, z) + (x * (v\, D\, y)) * z\end{aligned}$$

$$x * ((y * z)\, D\, w) = (x * (y\, D\, w)) * z.$$

However, with no symbol for the bracket on V, the theory of Lie–Rinehart pairs is not inductive.
Indeed, the union of the chain

$$(V_0, K_0) \subseteq (V_1, K_1) \subseteq \cdots$$

of Lie–Rinehart pairs is not a Lie–Rinehart pair when

$$K_n = \mathbb{Q}(t^i \colon i < n), \qquad V_n = \operatorname{span}_{K_n}(D_i \restriction K_n \colon i < n),$$

where

$$D_0 = \sum_{i<\omega} \partial_i, \quad D_1 = \sum_{i<\omega} (i+1) t^i \partial_{i+1}, \quad D_n = \partial_n \text{ if } 1 < n < \omega,$$

where

$$\partial_i t^j = \delta_i^j.$$

For,

$$[D_0, D_1] = \sum_{i<\omega} (i+1) \partial_{i+1} \notin V.$$

Let T be the theory of pairs (V, K), where K is a field of characteristic 0, and V acts on K as a vector space of derivations. Let $\mathrm{DCF}_0^{(m)}$ be the model-companion of the theory of fields of characteristic 0 with m derivations with no required interaction.

Theorem (Özcan Kasal)**.** The existentially closed models of T are just those such that

1. $\text{tr-deg}(K/\mathbb{Q}) = \infty$;
2. $(K, v_0, \dots, v_{m-1}) \models \mathrm{DCF}_0^{(m)}$ whenever $(v_0, \dots, v_{m-1})$ is linearly independent over K;
3. if $(x^0, \dots, x^{n-1})$ is algebraically independent, and $(y^0, \dots, y^{n-1})$ is arbitrary, then for some v in V,

$$\bigwedge_{i<n} v\,D\,x^i = y^i.$$

These are not first-order conditions: they require the constant field to be $\mathbb{Q}^{\mathrm{alg}}$.

The picture changes when (for each n) a predicate Q_n is introduced for the n-ary relation on scalars defined by

$$\bigvee_{i<n} \forall v \left(\bigwedge_{j \neq i} v\, D\, x^j = 0 \Rightarrow v\, D\, x^i = 0 \right).$$

Let the new theory be

$$T',$$

so

$$T' \vdash \forall \boldsymbol{x} \left(\neg Q_n \boldsymbol{x} \Leftrightarrow \exists \boldsymbol{v} \bigwedge_{\substack{i<n \\ j<n}} v_i\, D\, x^j = \delta_i^j \right).$$

Say $(a^0, \ldots, a^{n-1})$ from K is **D-dependent** if

$$(V, K) \models Q_n a^0 \cdots a^{n-1}.$$

So algebraic dependence implies D-dependence.
Also, D-dependence also makes K a pregeometry.

Theorem (Özcan Kasal)**.** The existentially closed models of T' are those (V, K) such that $D\text{-dim}(K) = \infty$ and whenever

1. $(v_0, \dots, v_{k+\ell-1})$ is linearly independent, and
$$\bigwedge_{\substack{i<k+\ell \\ j<k}} v_i \, D \, a^j = \delta_i^j,$$
2. U is a quasi-affine variety over $\mathbb{Q}(\boldsymbol{a}, \boldsymbol{b})$ with a generic point
$$(x^0, \dots, x^{\ell-1}, y^0, \dots, y^{m-1}, \boldsymbol{z}),$$
where $(\boldsymbol{x}, \boldsymbol{y})$ is algebraically independent over $\mathbb{Q}(\boldsymbol{a}, \boldsymbol{b})$,
3. $g_i^j \in \mathbb{Q}(\boldsymbol{a}, \boldsymbol{b})[U]$, where $i < k + \ell$ and $j < m$;

then U contains $(a^k, \dots, a^{k+\ell-1}, \boldsymbol{c}, \boldsymbol{d})$ such that

1. each c^j and d^j is D-dependent on $(a^0, \dots, a^{k+\ell-1})$,
2. $\displaystyle\bigwedge_{\substack{i<k+\ell \\ j<k+\ell}} v_i \, D \, a^j = \delta_i^j \;\&\; \bigwedge_{\substack{i<k+\ell \\ j<m}} v_i \, D \, c^j = g_i^j(a^k, \dots, a^{k+\ell-1}, \boldsymbol{c}, \boldsymbol{d})$.

Franz Kline, *Palladio*

FIN