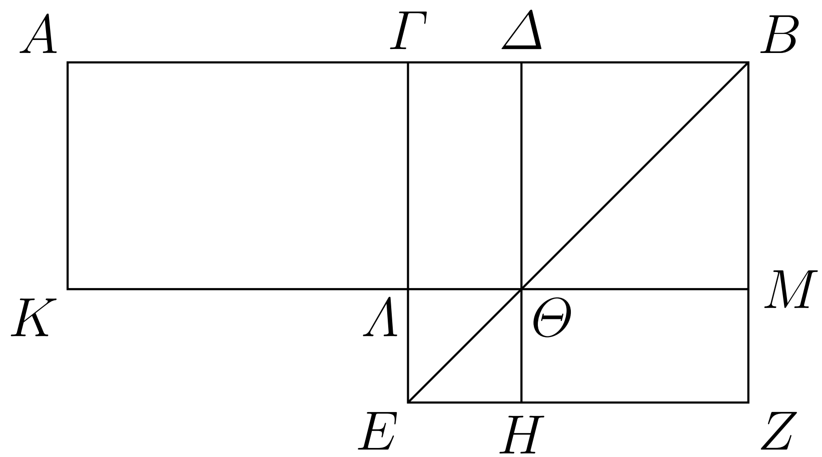
D. Pierce, **Vector-spaces over unspecified fields,** Nijmegen, 2006

From Euclid of Alexandria, *The Elements,* Proposition II.5:

Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω εἰς μὲν ἴσα κατὰ τὸ Γ, εἰς δὲ ἄνισα κατὰ τὸ Δ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΔ τετραγώνου ἴσον ἐστι τῷ ἀπὸ τῆς ΓΒ τετραγώνῳ.



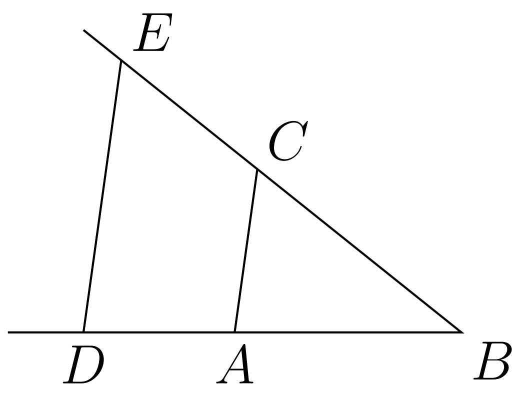
'For, let a straight [line] AB be cut into equal [segments] at Γ, and unequal at Δ; I say that the rectangle bounded by $A\Delta, \Delta B$, with the square on $\Gamma\Delta$, is equal to the square on ΓB.'

We might say

$$(x+y)(x-y)+y^2=x^2,$$

where x, y are the lengths of $A\Gamma$, $\Gamma\Delta$ respectively; maybe Euclid too (Descartes, *Rules for the Direction of the Mind,* 4).

From René Descartes, second page of *The Geometry:*



≪Soit par exemple AB l'vnité, & qu'il faille multiplier BD par BC, ie n'ay qu'a ioindre les poins A & C, puis tirer DE parallele a CA, & BE est le produit de cete Multiplication.≫

'For example, suppose AB is unity, and that one must multiply BD by BC; I need only join points A and C, then draw DE parallel to CA, and BE is the product of this multiplication.'

Take B as an origin of **vectors.** Then D is the multiple of A by a **scalar,** say $[D : A]$. Because $E - D \parallel C - A$, we have

$$[D : A] * C = E \qquad \& \qquad [D : A] = [E : C].$$

Precisely, a **vector-space** is a two-sorted structure $(V, K, *)$, where:

$$V \text{ is an abelian group}; \quad K \text{ is a field}; \quad * : K \times V \to V;$$
$$(t \mapsto (\boldsymbol{x} \mapsto (t * \boldsymbol{x}))) \in \operatorname{Hom}(K, (\operatorname{End}(V), \circ)).$$

So the signature of $(V, K, *)$ is $\{+, -, 0\} \amalg \{+, -, \cdot, 0, 1\} \amalg \{*\}$; there are variables $\boldsymbol{x}$ for vectors, and t for scalars.

Abraham Robinson (*Complete Theories,* Amsterdam, 1956) showed model-completeness of the theory of non-trivial vector-spaces over a *fixed* scalar field: ‘partial’ model-completeness.

If $n \in \{1, 2, 3, \dots, \infty\}$, let T_n be the theory of n-dimensional vector-spaces. Andrey Kuzichev (*Z. Math. Logik Grundlag. Math.,* 1992) showed elimination of quantified *vector*-variables in T_n. Hence, if U is a complete field-theory, then $T_n \cup U$ is complete.

T_n has $\exists\forall$ axioms for raising dimension. ‘Hence:’

$T_n \cup U$ is not inductive $(\forall\exists)$; much less model-complete (unless $n = 1$).

We can remedy this ‘problem’ with a predicate for parallelism.

The vector-space $(V, K, *)$ expands to $(V, K, *, \|)$, where

$$\boldsymbol{x} \parallel \boldsymbol{y} \iff \exists s \, \exists t \, (s * \boldsymbol{x} + t * \boldsymbol{y} = \boldsymbol{0} \land (s \neq 0 \lor t \neq 0)).$$

If $\dim_K V > 1$, then $(V, K, *, \|)$ is interpretable uniformly in the reduct $(V, \|)$, and

$$(V, K, *, \|) \subseteq (W, L, *, \|) \iff (V, \|) \subseteq (W, \|).$$

(By contrast, three-sorted structures $(G, N, G/N)$, where $N \lhd G$, are interpretable in the reducts (G, N); but

$$(\mathbb{Z}, 4\mathbb{Z}, \mathbb{Z}_4) \not\subseteq (\mathbb{Z}, 2\mathbb{Z}, \mathbb{Z}_2) \quad \& \quad (\mathbb{Z}, 4\mathbb{Z}) \subseteq (\mathbb{Z}, 2\mathbb{Z}).)$$

The theory VS_2 of the expansions $(V, K, *, \|)$ is inductive.

Hence the theory of the reducts $(V, \|)$ is inductive.

The model-companion of VS_2 is the theory ${\mathrm{VS}_2}^*$ of **two-dimensional** vector-spaces over **algebraically closed** fields. In particular, ${\mathrm{VS}_2}^*$ is model-complete:

Replace the binary $\|$ with an n-ary predicate $\|^n$ for linear dependence; in the new signature, let VS_n be the theory of vector-spaces.

Theorem. *The existentially closed models of* VS_n *are* n*-dimensional (over algebraically closed fields; since these models compose an elementary class, their theory is the model-companion of* VS_n*).*

Proof. Say $K \subset L$ (both fields), and $\dim_K L \geqslant n+1$.

I say that $(K^{n+1}, K, \|^n)$ embeds in $(L^n, L, \|^n)$:

Suppose $\{a_0, \dots, a_{n-1}, 1\}$ from L is linearly independent over K. Let

$$A = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{pmatrix} = \left(\frac{I_n}{-\boldsymbol{a}}\right).$$

The embedding is $\boldsymbol{x} \mapsto \boldsymbol{x} \cdot A$, taking the rows of I_{n+1} to the rows of A:

Indeed, write n elements of K^{n+1} as the rows of $\left(U \middle| \boldsymbol{v} \right)$. These rows are dependent if and only if

$$0 = \det \begin{pmatrix} U & \boldsymbol{v} \\ \hline \boldsymbol{a} & 1 \end{pmatrix} = \det(U - \boldsymbol{v} \cdot \boldsymbol{a}).$$

But $U - \boldsymbol{v} \cdot \boldsymbol{a} = \left(U \middle| \boldsymbol{v} \right) \cdot \begin{pmatrix} I_n \\ \hline -\boldsymbol{a} \end{pmatrix}$, whose rows are the images in L^n of the rows of $\left(U \middle| \boldsymbol{v} \right)$. □

Compare with structures (L, K, P^n), where P^n is n-ary algebraic dependence. From a standard counterexample:

$$(K(a_0, \ldots, a_n), K, P^n) \subset (K(a_0, \ldots, a_n, b, c), K(b, c), P^n),$$

where $\dim(a_0 \cdots a_n / K) = n + 1$, and (b, c) is a generic solution to $\sum_{i=0}^{n} a_i x^i = y$, so that $\dim(a_0 \cdots a_n / Kbc) = n$.