Lie-rings

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(From Plato's door, according to Tzetzes [12th c.]:
Μηδεὶς ἀγεωμέτρητος εἰσίτω μου τὴν στέγην.
Why did the AMS choose a version of this as its motto?)

E is an abelian group, in signature $\{+, -, 0\}$.

$\mathrm{End}(E)$ is its *abelian group* of endomorphisms.

$\mathrm{Mult}(E)$ is the abelian group of **multiplications** of E: bi-additive maps

$$(x, y) \mapsto \mathsf{m}(x, y) : E \times E \to E.$$

If $\mathsf{m} \in \mathrm{Mult}(E)$, then (E, m) is a **ring;** also, $\mathrm{Mult}(E)$ contains $\dot{\mathsf{m}}$, the **converse** of m, given by

$$\dot{\mathsf{m}}(x, y) = \mathsf{m}(y, x).$$

$\mathrm{Mult}(\mathrm{End}(E))$ contains **composition,** c, and so

$$\langle \mathsf{c}, \dot{\mathsf{c}} \rangle \leqslant \mathrm{Mult}(\mathrm{End}(E));$$

in particular, the **bracket,** $\mathsf{c} - \dot{\mathsf{c}}$ or b, is in $\mathrm{Mult}(\mathrm{End}(E))$.

A ring (E, m) is:

- **associative,** if $\mathsf{m}(\mathsf{m}(x, y), z) = \mathsf{m}(x, \mathsf{m}(y, z))$;
- **commutative,** if associative, and $\mathsf{m} - \dot{\mathsf{m}} = 0$;
- a **Lie-ring,** if $\mathsf{m} + \dot{\mathsf{m}} = 0$ and

$$\mathsf{m}(\mathsf{m}(x, y), z) = \mathsf{m}(x, \mathsf{m}(y, z)) - \mathsf{m}(y, \mathsf{m}(x, z))$$

(the **Jacobi identity**).

It is obvious that $(\mathrm{End}(E), \mathsf{c})$ is associative;

to see that $(\mathrm{End}(E), \mathsf{b})$ is a Lie-ring requires a computation.

Theorem A. *Among non-associative rings, Lie-rings are the most "natural" in a precise sense.*

Theorem B. *There are Lie-rings* (E, m) *with* t *in* $\mathrm{End}(E)$ *such that* $\mathrm{Th}(E, \mathsf{m}, t)$ *is model-complete and* ω*-stable.*

There is an isomorphism

$$\mathsf{m} \mapsto \lambda^{\mathsf{m}} : \mathrm{Mult}(E) \to \mathrm{Hom}(E, \mathrm{End}(E)),$$

- where λ^{m} is $x \mapsto \lambda^{\mathsf{m}}(x) : E \to \mathrm{End}(E)$,
- where $\lambda^{\mathsf{m}}(x)$ is $y \mapsto \mathsf{m}(x, y) : E \to E$.

We can now recast the Jacobi identity:

$$\mathsf{m}(\mathsf{m}(x, y), z) = \mathsf{m}(x, \mathsf{m}(y, z)) - \mathsf{m}(y, \mathsf{m}(x, z))$$

becomes

$$\begin{aligned} \lambda^{\mathsf{m}}(\mathsf{m}(x, y)) &= \lambda^{\mathsf{m}}(x) \circ \lambda^{\mathsf{m}}(y) - \lambda^{\mathsf{m}}(y) \circ \lambda^{\mathsf{m}}(x) \\ &= \mathsf{b}(\lambda^{\mathsf{m}}(x), \lambda^{\mathsf{m}}(y)), \end{aligned}$$

This says λ^{m} is a *ring*-homomorphism from (E, m) to $(\mathrm{End}(E), \mathsf{b})$.

Theorem A. *Let* $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, *and let* m *be* $p\mathsf{c} - q\dot{\mathsf{c}}$. *Call a ring* $(E, *)$ *an* m-ring *if* λ^* *is a ring-homomorphism from* $(E, *)$ *to* $(\mathrm{End}(E), \mathsf{m})$. *The following are equivalent:*

- $(\mathrm{End}(E), \mathsf{m})$ *is an* m*-ring for all* E.
- m *is* c *or* b *or* 0.

A **derivation** of a ring (E, m) is an element D of $\mathrm{End}(E)$ such that

$$D(\mathsf{m}(x,y)) = \mathsf{m}(Dx,y) + \mathsf{m}(x,Dy);$$

—rearranged,

$$D(\mathsf{m}(x,y)) - \mathsf{m}(x,Dy) = \mathsf{m}(Dx,y);$$

—with y removed,

$$D \circ \lambda^{\mathsf{m}}(x) - \lambda^{\mathsf{m}}(x) \circ D = \lambda^{\mathsf{m}}(Dx),$$

that is,

$$\mathsf{b}(D, \lambda^{\mathsf{m}}(x)) = \lambda^{\mathsf{m}}(Dx);$$

—with x removed,

$$\lambda^{\mathsf{b}}(D) \circ \lambda^{\mathsf{m}} = \lambda^{\mathsf{m}} \circ D,$$

that is, the following commutes:

$$\begin{array}{ccc} E & \xrightarrow{\lambda^{\mathsf{m}}} & \mathrm{End}(E) \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle \lambda^{\mathsf{b}}(D)} \\ E & \xrightarrow[\lambda^{\mathsf{m}}]{} & \mathrm{End}(E) \end{array}$$

The derivations of (E, m) compose a subgroup

$$\mathrm{Der}(E, \mathsf{m})$$

of $\mathrm{End}(E)$; this subgroup is closed under b.

For any abelian group E, there is a commutative diagram

$$\begin{array}{ccc} (\mathrm{End}(E), \mathsf{b}) & \xrightarrow{\lambda^{\mathsf{b}}} & (\mathrm{Der}(\mathrm{End}(E), \mathsf{c}), \mathsf{b}) \\ & {\scriptstyle \lambda^{\mathsf{b}}} \searrow & \downarrow \subseteq \\ & & (\mathrm{End}(\mathrm{End}(E)), \mathsf{b}) \end{array}$$

Suppose now (E, m) is a Lie-ring. Then λ^{m} is the **adjoint representation:** there is a commutative diagram

$$\begin{array}{ccc} (E, \mathsf{m}) & \xrightarrow{\lambda^{\mathsf{m}}} & (\mathrm{Der}(E, \mathsf{m}), \mathsf{b}) \\ & {\scriptstyle \lambda^{\mathsf{m}}} \searrow & \downarrow \subseteq \\ & & (\mathrm{End}(E), \mathsf{b}) \end{array}$$

Combining gives

$$\begin{array}{ccc} (E, \mathsf{m}) & \xrightarrow{\qquad \lambda^{\mathsf{b}} \circ \lambda^{\mathsf{m}} \qquad} & \\ {\scriptstyle \lambda^{\mathsf{m}}} \searrow & & \searrow \\ & (\mathrm{End}(E), \mathsf{b}) \xrightarrow[\lambda^{\mathsf{b}}]{} & (\mathrm{Der}(\mathrm{End}(E), \mathsf{c}), \mathsf{b}) \end{array}$$

This means that every D in E is now a derivation $f \mapsto Df$ of $(\mathrm{End}(E), \mathsf{c})$ by the rule

$$(Df)x = \mathsf{m}(D, fx) - f(\mathsf{m}(D, x)).$$

Still (E, m) is a Lie-ring, mapping into $(\mathrm{Der}(\mathrm{End}(E), \mathsf{c}), \mathsf{b})$.

Let $t \in \mathrm{End}(E)$. Call the structure

$$(E, \mathsf{m}, t)$$

a **vector Lie-ring** if:

- $\{Dt : D \in E\}$ is the universe of a sub-*field* K of $(\mathrm{End}(E), \mathsf{c})$, and
- E acts on K as a vector-space (over K) of derivations:
$$(gD)f = g(Df).$$

Theorem B.

- *The class of vector Lie-rings is elementary.*
- *If $n < \omega$, then the theory of vector Lie-rings of dimension n is companionable.*
- *the model-companion of this theory becomes complete and ω-stable when characteristic 0 is specified.*

(Related results are being worked out independently by Martin Bays.)