Research statement

David Pierce

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Overview

My specific training, and my publications so far, are in the part of mathematical logic called *model theory,* and especially in the model theory of fields and differential fields. My interests and work also spread into foundations and history; these subjects are a way to understand mathematics as it is done *today,* and to add to it.

1 A primer on model theory

When model theorists address a general audience, they often feel the need to explain their subject from the very beginning. In that spirit, I compose the present section of this document.

Model theory is a specific area of mathematics, raising and answering its own questions. However, like category theory, it is also a point of view, a way to think about all of mathematics. Sometimes this point of view leads to new insights, proofs, and theorems.

To my knowledge, the first two textbooks (as opposed to treatises) of model theory are Bell and Slomson's *Models and Ultraproducts* of 1969 [7] and Chang and Keisler's *Model Theory* of 1973 [10]. According to Bell and Slomson,

> Model theory... can be described briefly as the study of the relationship between formal languages and abstract structures.

Chang and Keisler have a similar description:

> Model theory is the branch of mathematical logic which deals with the relation between a formal language and its interpretations, or models.

Chang and Keisler also suggest the equation

$$\text{universal algebra} + \text{logic} = \text{model theory};$$

but this is perhaps restrictive, simply because universal algebra is about sets with distinguished *operations,* and not relations in general; but model theory does study sets with arbitrary relations.

The emphasis on a formal language may be relaxed. According to Wilfrid Hodges in his encyclopedic volume *Model Theory* of 1993 [34, p. ix],

> Model theory is the study of the construction and classification of structures within specified classes of structures. A 'specified class of structures' is any class of structures that a mathematician might choose to name. For example, it might be the class of abelian groups, or of Banach algebras, or sets with groups which act on them primitively. Thirty or forty years ago the founding fathers of model theory were particularly interested in classes specified by some set of axioms in first-order predicate logic—this would include the abelian groups but not the Banach algebras or the primitive groups. Today we have more catholic tastes, though many of our techniques work best on the first-order axiomatisable classes.

This definition may misleadingly suppress the *logical* aspect of model theory. The classification of the finite simple groups is not really a model-theoretic project, although it has been an inspiration for the project of classifying the infinite simple groups of finite Morley rank, and Morley rank is a logical notion (see page 9).

I propose the following definition: **model theory** is the study of structures *quâ* models of theories. For *quâ,* one may read *in the capacity of* or *as.* The three terms *structure, model,* and *theory* must now be explained.

Structures appear in most mathematics. Groups, rings, ordered fields, partially ordered sets: all are examples of structures. A **structure** then is a set with some extra 'structure'. For us, this extra 'structure' consists of:

- some distinguished relations and operations on the set,
- some specific elements of the set.

None of the extra structure is actually *required* to be present; a bare set is an example of a structure.

Note that Hodges's example of Banach spaces does not exactly fit this definition of structure. Model-theorists have indeed developed a Banach-space logic for studying Banach spaces; but this subject is beyond the bounds of the present exposition.

The ordered field of real numbers is the structure consisting of:

- the set $\mathbb{R}$ of real numbers;
- the operations of addition, additive inversion, and multiplication on $\mathbb{R}$;
- the relation 'less than' on $\mathbb{R}$;
- the additive identity and the multiplicative identity in $\mathbb{R}$.

In this example, there are standard symbols for the distinguished operations, relations, and elements, and the structure can be denoted by

$$(\mathbb{R}, +, -, \cdot, <, 0, 1).$$

We can then speak of *reducts* of this structure, such as

$$(\mathbb{R}, +, -, \cdot, <).$$

This reduct is not much different from the original structure, since 0 and 1 are *definable* in the reduct, by the formulas

$$x + x = x, \qquad\qquad x \cdot x = x,$$

respectively; for example, the solution set of $x + x = x$ in $\mathbb{R}$ is just $\{0\}$. Also, the relation $<$, considered as the set $\{(x, y) \colon x < y\}$, is definable in $(\mathbb{R}, +, -, \cdot)$ by the formula

$$x \neq y \wedge \exists z\ x + z^2 = y$$

(where z^2 stands for $z \cdot z$). Finally, the operation $x \mapsto -x$, considered as the relation $\{(x, y) \colon y = -x\}$, is definable in $(\mathbb{R}, +, 0)$ by

$$x + y = 0;$$

hence it is definable in $(\mathbb{R}, +)$ by $(x + y) + (x + y) = x + y$.

In short, everything that can be defined in $(\mathbb{R}, +, -, \cdot, <, 0, 1)$ can already be defined in $(\mathbb{R}, +, \cdot)$. This can be compared to the situation in propositional logic, where, because of De Morgan's law

$$\neg(P \vee Q) \iff \neg P \wedge \neg Q,$$

we have $P \vee Q \iff \neg(\neg P \wedge \neg Q)$, so that everything that can be said with 'or' can be said also with 'and' and 'not'.

However, not every proposition that can be expressed with 'and' and 'not' together can be expressed in terms of 'and' alone. Similarly, multiplication cannot be defined in $(\mathbb{R}, +)$. To prove this, all we need to know is that definable operations and relations of a structure are invariant under automorphisms of the structure. Since $(\mathbb{R}, +)$ is a torsion-free divisible abelian group, it can be understood as a vector-space over $\mathbb{Q}$. There is an automorphism of this space that takes the one-dimensional subspace $\mathbb{Q}$ to the subspace $\{x \cdot \sqrt{2} \colon x \in \mathbb{Q}\}$, and such an automorphism does not fix the subset $\{(x, y, z) \colon x \cdot y = z\}$ of $\mathbb{R}^3$. Therefore multiplication is not definable in $(\mathbb{R}, +)$.

The structure $(\mathbb{R}, +, -, \cdot, <, 0, 1)$ has the **signature** $\{+, -, \cdot, <, 0, 1\}$. In the signature, the symbols have no meaning; in the structure, they stand for particular operations, relations, or elements. Two structures can have the same signature. For example, all abelian groups can be understood to have the same signature $\{+, -, 0\}$; or the signature may be taken to be simply $\{+\}$, since the operation denoted by $-$ and the element denoted by 0 are definable in terms of $+$ by the same formulas in any abelian group.

When we are trying to make things precise, and the underlying set of a structure is A, then the structure itself might be denoted by $\mathfrak{A}$, if we really need to distinguish between the two. (Often we need not distinguish.) If S is a symbol in the signature of $\mathfrak{A}$, then the operation or relation or individual that it denotes in $\mathfrak{A}$ can be denoted also by $S^{\mathfrak{A}}$. For example, a function f from a group G to a group H is defined to be a homomorphism if

$$f(x \cdot^G y) = f(x) \cdot^H f(y);$$

here it is emphasized that the operations of multiplication on the left and right sides are actually distinct. However, often we do not feel the need to express this distinction notationally.

In the general situation, if $S^{\mathfrak{A}}$ is a relation, then it is considered to be one of the definable relations of $\mathfrak{A}$. If $S^{\mathfrak{A}}$ is an n-ary operation for some n in $\mathbb{N}$, then the relation $\{(\vec{x}, S^{\mathfrak{A}}(\vec{x})) \colon \vec{x} \in A^n\}$ is a definable $(n+1)$-ary relation. If $S^{\mathfrak{A}}$ is an element of A, then $\{S^{\mathfrak{A}}\}$ is a definable singulary relation. The collection of all definable relations of $\mathfrak{A}$ is built up by standard set-theoretic operations: binary intersection, binary union, complementation, and also the coordinate projections

$$X \mapsto \{(x_0, \dots, \hat{x}_i, \dots, x_{n-1}) \colon (x_0, \dots, x_{n-1}) \in X\}.$$

These are the operations expressed by the logical symbols $\wedge$, $\vee$, $\neg$, and $\exists x_i$. Also, the diagonal relation $\{(x, x) \colon x \in A\}$ is considered definable, and if some relation is definable, then so is its inverse image under a coordinate projection. Finally, $\{b\}$ is definable whenever $b \in A$.

Thus a **definable relation** of a structure is the solution set of a formula of *first-order logic* in the signature of the structure, possibly with *parameters* from the structure. The formula then **defines** its solution set. In high school algebra, when one studies graphs of equations like $y = mx + k$ or $x^2 + y^2 = r^2$, one is studying sets definable in $(\mathbb{R}, +, \cdot\,)$. But the formula defining a set may be more than an equation; it may involve Boolean connectives and quantifiers, as in the earlier examples. In $(\mathbb{R}, +, \cdot\,)$, the interval $[-1, 1]$ is defined by the formula $\exists y\ x^2 + y^2 = 1$.

First-order logic is logic in which variables stand for individuals, not operations or sets. The Completeness Axiom for $(\mathbb{R}, <)$ is formulated in *second-order* logic, since the axiom is that every nonempty *subset* of $\mathbb{R}$ with an upper bound has a least upper bound. Similarly, the Induction Axiom for the structure $(\mathbb{N}, 1, x \mapsto x + 1)$ of the natural numbers is not first-order, but second-order.

Model theory is generally concerned with first-order logic. This restriction may seem mathematically unnatural; but it is no more unnatural than restricting one's attention to, say, groups or modules.

Formulas of first-order logic have finite length. In particular, the intersection of an infinite collection of definable sets of A^n is not necessarily definable. Such intersections are still of interest though; they are said to be *type-definable.*

The foregoing argument for why multiplication is not definable in $(\mathbb{R}, +)$ does not actually require that formulas be first-order. Model theory normally uses first-order logic because it makes available the *Compactness Theorem.* To talk about this, we should first work out the remaining two undefined terms in the definition of model theory.

A formula with n free variables in the signature of $\mathfrak{A}$ defines a subset of A^n. A formula with *no* free variables is a **sentence;** it defines a subset of A^0. It is convenient here to consider the non-negative integers as the set-theorist's natural numbers, as defined by von Neumann [67]: $0 = \varnothing$, and $n+1 = n \cup \{n\}$. Then A^n is the set of functions from $\{0, \ldots, n-1\}$ to A, and in particular A^0 is the set of functions with empty domain. There is only one such function, namely $\varnothing$ or 0; so $A^0 = \{0\} = 1$, and its subsets are 0 and 1. In the present context, we can consider these subsets as **falsehood** and **truth,** respectively. To say that a sentence is **true** in a structure is just to say that the nullary relation defined by the sentence is truth.

Suppose Δ is a set of first-order sentences in some signature. A **model** of Δ is a structure in which all sentences in Δ are true. A sentence σ is a **logical consequence** of Δ, and Δ **entails** σ, if σ is true in every model of Δ. A **theory** is a set of sentences that contains all of its own logical consequences. The set of logical consequences of Δ is then a theory: it is said to be the theory **axiomatized** by Δ. Often the distinction between a theory and the sentences that axiomatize it is blurred.

The **Compactness Theorem** is that, if every finite subset of some set of sentences has a model, then the whole set has a model.

For example, suppose T is the theory of finite fields, that is, T consists of all sentences in the signature $\{+, -, \cdot, 0, 1\}$ that are true in every finite field. If $n \in \mathbb{N}$, let σ_n be the sentence

$$\exists x_0 \cdots \exists x_{n-1} \bigwedge_{i<j<n} x_i \neq x_j,$$

which says that every model has at least n elements. Every finite subset of $T \cup \{\sigma_n \colon n \in \mathbb{N}\}$ has a model, namely a finite field that is sufficiently large. By the Compactness Theorem, the whole set has models, and these are infinite: they are the infinite models of the theory of finite fields. (They are called *pseudofinite fields* and are characterized by Ax [5].)

In algebraic geometry, a **constructible set** is a set defined by a *quantifier free* formula (with parameters) in an algebraically closed field. A theorem of Chevalley [28, p. 94] is that a coordinate projection of an constructible set is a constructible set. This is equivalent to Tarski's theorem that the theory of algebraically closed fields admits *elimination of quantifiers.*

As suggested in the example of pseudofinite fields, the **theory of** a class $\mathscr{K}$ of structures in some signature is the set of sentences in that signature that are true in every structure in $\mathscr{K}$. If $\mathscr{K}$ has a unique element $\mathfrak{A}$, then the theory of $\mathscr{K}$ is the **theory of** $\mathfrak{A}$. The theory of $\mathfrak{A}$ is a **complete theory:** that is, for every sentence σ of the signature of $\mathfrak{A}$, the theory of $\mathfrak{A}$ contains either σ or its negation $\neg\sigma$. For example, the structure $(\mathbb{N}, +, \cdot)$ has a complete theory. **Gödel's Incompleteness Theorem** [25] is that this theory is not *recursively* axiomatizable: there is no rule for writing down a set of sentences that axiomatize the theory.

I shall discuss the proof of Gödel's theorem in §§ 6 and 7. Meanwhile, there are many common examples of recursively axiomatizable complete theories. Abraham Robinson's *Complete Theories* [56] is all about them. One of them is ACF_p, the theory of algebraically closed fields of a given characteristic p. This completeness can be understood as the logical basis for the **Lefshetz Principle,** whereby certain statements proved in $\mathbb{C}$ by analytic methods automatically hold in every algebraically closed field of characteristic 0. Such statements certainly *do* hold generally, if they are first-order statements.[1]

In a lecture at the Mathematical Sciences Research Institute in Berkeley in 1998, Lou van den Dries [65] (citing Ehud Hrushovski as the source) described model theory as the 'geography of tame mathematics'. For present purposes, we can understand a structure to be **tame** if its complete theory is recursively axiomatizable. Model theory provides tools for identifying such structures, which may arise naturally in the study of **wild** (non-tame) structures. For example,

- $(\mathbb{N}, +, \cdot)$ is wild, by Gödel, as noted.
- $\mathbb{N}$ is definable in $(\mathbb{Z}, +, \cdot)$, by Lagrange's theorem that every positive integer is the sum of four squares; therefore $(\mathbb{Z}, +, \cdot)$ is wild.
- Therefore $(\mathbb{Q}, +, \cdot)$ is wild, by the theorem of Julia Robinson [58] that $\mathbb{Z}$ is definable in $(\mathbb{Q}, +, \cdot)$.

[1] Some loosening of the first-order requirement is possible; see Hodges [34, p. 700] for discussion and references.

- However, the Dedekind completion $(\mathbb{R}, +, \cdot)$ and the p-adic completions $(\mathbb{Q}_p, +, \cdot)$ of $(\mathbb{Q}, +, \cdot)$ are tame, by work of Tarski, Ax, Kochen, and Ershov.

Making use of the Compactness Theorem and its proof, one shows that a theory with one infinite model has a model of every infinite cardinality, at least if that cardinality is not less than the cardinality of the signature of the theory. Since $(\mathbb{R}, +, \cdot, <)$ is the only Dedekind-complete ordered field (up to isomorphism) of any cardinality, it follows that the Completeness Axiom cannot be recast in a first-order way.

The theory of $(\mathbb{C}, +, \cdot)$, which is the recursively axiomatizable theory ACF_0, has just one model (up to isomorphism) in every uncountable cardinality; in a word, the theory is **categorical** in every uncountable cardinality. Again, as noted, the complete first-order theory of $(\mathbb{R}, +, \cdot, <)$ is also recursively axiomatizable: it is the theory RCF of *real-closed ordered fields.* However, this theory is as far from categorical as possible: it has 2^κ nonisomorphic models of cardinality κ, for every infinite cardinal κ.

A model of ACF_p is determined up to isomorphism by its transcendence degree. The study of how theories can have such a property was pursued by Michael Morley, who showed [44] that a theory (in a countable signature) that is categorical in one uncountable cardinality must, like ACF_p, be categorical in every uncountable cardinality. In the argument, he used the notion now called *Morley rank* (see page 3). Saharon Shelah continued this work by identifying those complete first-order theories whose uncountable models could possibly be *classified:* classified by a cardinal invariant, as in the case of algebraically closed fields, or more generally by many cardinals, arranged in a *tree.* The specific possibilities for this classification were worked out by Hart and Laskowski [27, 26]. (Laskowski was my teacher.)

For an example of how such trees arise, let T be the theory of an equivalence relation E, all of whose classes are infinite. Then T turns out to be a complete theory, and each model is determined by:

1. the number of E-classes;
2. the size of each E-class.

This information can be arranged in a tree. To be precise, suppose $\mathfrak{M}$ is a model of T of cardinality κ. Then there is an injective function f or $x \mapsto (x_0, x_1)$ from M into $\kappa \times \kappa$ such that

$$x_0 = y_0 \iff x \mathrel{E} y.$$

That is, x_0 distinguishes an E-class, and x_1 distinguishes an element within a given E-class. Then we have a tree whose nodes are all of the form $(\ )$, (x_0), or (x_0, x_1). The number of isomorphism-classes of such trees is 2^κ.

We have now seen two aspects of model-theoretic practice:

1. The study of structures of 'ordinary' mathematics, in order to understand their theories.
2. The creation of new structures whose theories have properties of interest.

Understanding a theory involves understanding the class of all (isomorphism classes of) models of the theory; it also involves understanding the definable relations of particular models.

2 Function fields

In algebraic geometry, as noted on page 7, a constructible set is defined by a quantifier-free formula in the signature of an algebraically closed field with parameters. Then an **algebraic set** is defined by a *positive* quantifier-free formula, that is, a formula built up from equations by conjunction and disjunction—by 'and' and 'or'—, but not negation. If this formula cannot be written in a nontrivial way as a disjunction, then the algebraic set that it defines is a **variety** (strictly, an *affine* variety). One project of algebraic geometry is to classify varieties up to *birational equivalence.* Birational equivalence corresponds to isomorphism of the *function fields* of the varieties.

Two structures with the same first-order theory are called **elementarily equivalent.** Then isomorphic structures, such as function fields, are elementarily equivalent. We have already observed that the converse fails, simply because a theory with infinite models has models of different cardinalities, and so these models cannot be isomorphic. Also, two elementarily equivalent structures of the same cardinality may fail to be isomorphic: an example is algebraically closed fields of the same characteristic, but distinct finite transcendence degrees.

We may ask whether elementarily equivalent function fields over the same algebraically closed field are elementarily equivalent. Then we are in the first aspect of model-theoretic practice mentioned at the end of the last section: studying known structures through consideration of their theories. Partial answers to our question are found in work of Jean-Louis Duret [16, 17] and then in work done by myself [47] and independently by Duret's student Xavier Vidaux [66].

The first thing to note is that function fields of different dimension are elementarily inequivalent: this can be established by means of the Tsen–Lang Theorem. Let us then restrict our attention to dimension one. It turns out that elementarily equivalent function fields of curves are isomorphic, *unless* both of the curves are elliptic curves with complex multiplication.

Suppose E_0 and E_1 are elliptic curves with complex multiplication over an algebraically closed field K. This means each endomorphism-ring $\mathrm{End}(E_i)$ is strictly larger than $\mathbb{Z}$. The condition that the rings $\mathrm{End}(E_i)$ be isomorphic to one another is strictly weaker than the condition that the function fields $K(E_i)$ be isomorphic. I obtained the result that, in case the characteristic of K is 0, the rings $\mathrm{End}(E_i)$ are isomorphic if and only if the fields $K(E_i)$ agree on all sentences of the form

$$\forall x_0 \cdots \forall x_{n-1} \, \exists y \, \varphi(x_0, \ldots, x_{n-1}, y),$$

where φ is quantifier-free. Vidaux and I have tried to determine whether this result can be generalized, even to arbitrary $\forall\exists$ sentences, that is, sentences of the form

$$\forall y_0 \cdots \forall y_{m-1} \, \exists x_0 \cdots \exists x_{n-1} \, \psi(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}).$$

where ψ is quantifier-free. If this problem can be solved, then the following should also be studied:

1) elliptic curves over finite fields,

2) arbitrary abelian varieties,
3) arbitrary varieties.

It may be that the truth lies deep. A long-standing question of Tarski was whether any two non-abelian free groups of finite rank are isomorphic. The question has supposedly been answered in the affirmative, by Kharlampovich and Myasnikov, and independently by Sela; but the work is apparently very difficult, and it is not clear (to me at least) whether others have understood it thoroughly. In fact the Istanbul Model Theory Seminar has spent some time studying this work. It may possibly illuminate the corresponding question for function fields of elliptic curves, namely, the question of whether two such non-isomorphic fields can be elementarily equivalent.

3 Differential fields

The field of complex numbers is algebraically closed, and every field has an algebraic closure. The field of real numbers is real-closed, and every ordered field has a real closure. In model-theoretic terms, what this means is that, of the *theory* of fields of characteristic 0 and the *theory* of ordered fields, each has a *model-companion.* A theory T^* is a **model companion** of a theory T (in the same signature) if:

- $T \subseteq T^*$, and every model of T *embeds* in a model of T^*—an **embedding** is a monomorphism, that is, an injective function that preserves structure;
- T^* is **model complete:** in other words all models of T^* are **existentially closed:** that is, if $\mathfrak{A}$ and $\mathfrak{B}$ are models of T^*, and $\mathfrak{A} \subseteq \mathfrak{B}$ (that is, $A \subseteq B$, and the inclusion of A in B is an embedding of $\mathfrak{A}$ in $\mathfrak{B}$), then every quantifier-free formula with parameters from $\mathfrak{A}$ that has a solution in $\mathfrak{B}$ has a solution in $\mathfrak{A}$.

The question then arises of which other theories have model companions. Tame complete theories are often model complete, as for example each theory ACF_p is. However, the theory of fields of any characteristic also has a model-companion, namely the incomplete theory ACF of algebraically closed fields of any characteristic.

An example developed by Abraham Robinson [55] is the theory DF of **differential fields,** namely fields with an additional operation δ that is an additive endomorphism and obeys the Leibniz rule

$$\delta(x \cdot y) = x \cdot \delta y + y \cdot \delta x.$$

Again a required characteristic on models can be indicated by a subscript. Robinson showed that DF_0 has a model companion; his student Carol Wood [68] showed the same for DF_p when p is positive.

But one does not want to know just that there is a model companion; one wants to understand its models. Robinson's axioms for ${\mathrm{DF}_0}^*$, and then Wood's for ${\mathrm{DF}_p}^*$, were complicated. In an algebraically closed field, every consistent system of polynomial equations and inequations in any number of variables has a solution; a **'crude'** axiomatization of ACF says this; but in fact the axioms need only say that every non-constant polynomial in *one variable* has a root. Lenore Blum [8] observed this and found a similarly simple axiomatization for ${\mathrm{DF}_0}^*$; Wood [69] adapted this to ${\mathrm{DF}_p}^*$. An alternative, 'geometric' style of simplification was published by Anand Pillay and me [53] for ${\mathrm{DF}_0}^*$; Piotr Kowalski [38] carried out this work for ${\mathrm{DF}_p}^*$ (he also placed it in a more general setting: that of 'derivations of the Frobenius map'). I found a slightly simpler form of 'geometric' axiomatization and used it in [49] to give a model-companion of DF (with no specified characteristic): the models of DF^* are those differential fields (K, δ) such that

1. K is separably closed,
2. (K, δ) is **differentially perfect:** the kernel of δ is K^p, if K has characteristic p, which is positive;
3. for every affine variety V over K, if there are K-rational maps φ and ψ from V into affine n-space, and φ is dominant and separable, then V has a K-rational point P where φ and ψ are regular and $\delta(\varphi(P)) = \psi(P)$.

A model of DF^* is a 'universal domain' where all consistent systems of ordinary differential equations and inequations have solutions (although the solutions are algebraic or 'formal'; they are not given as functions). One may ask about partial differential equations: does the theory DF^m of fields with m commuting derivations for some positive integer m have a model-companion? Wood's student Tracey McGrail [43] showed that it does in characteristic 0; but its axioms were 'crude' in the sense above. Similar work was carried out independently by Yaffe [70].

In fact Yaffe's work was in an apparently more general setting: the m derivations need not commute, but the Lie bracket of any two of them is a fixed linear combination of all of them. I observed in [48] that the generalization was only apparent, in the sense that axioms for the model companion of a theory of Yaffe's 'Lie differential fields' (of characteristic 0) could be easily derived from axioms for $(\mathrm{DF}_0^m)^*$. Singer [60] made the observation more explicit.

The main point of my paper [48] was to give an alternative, 'geometric' axiomatization of $(\mathrm{DF}_0^m)^*$; but this supposed axiomatization turned out to be wrong. Correcting the problem required a whole new approach, presented in [50]. The main idea is that, for every insoluble system of differential equations of a given order, there is a bound on the number of times the equations must be differentiated to establish the insolubility. I carried out the work in arbitrary characteristic; in particular, I showed the existence of a model-companion of DF_p^m even when $p > 0$: this result is apparently new.

On a field, a derivation is not the only interesting singulary operation: one can consider also an endomorphism σ, or equivalently the **difference-operator** $x \mapsto x^\sigma - x$. Derivations and difference-operators are such that their behavior at sums and products is determined by polynomials, and they are 0 at 0 and 1. Buium [9] showed that they were the *only* examples of such operations.

Before the paper [53] of Pillay and me, Hrushovski had shown the the theory of **difference fields**—fields with an endomorphism—has a model companion (see [42] and [12]). Then it is easy to show that the theory of fields with a singulary operation that is either a derivation or a difference-operator has a model-companion; but I did this 'geometrically' in [49] without distinguishing the two cases in the axioms. More interesting results in that paper occur when the theory of fields with *both* a derivation and a difference-operator is considered. In characteristic 0, there is a model-companion; but there is not, in positive characteristic. Briefly, the problem is that, in a positive characteristic p, if $\delta(x^{\sigma^n}) = 0$ for *all* nonnegative integers n, then x should have a pth root in a model of the model-companion; but this condition cannot be made first-order, so there is no model-companion.

4 Vector spaces

Structures as defined in the 'Primer on model theory' (§1 above) are more precisely **one-sorted** structures, because each of them is based on only one set, and we may call that set a **sort.** However, sometimes one wants to work with more than one sort. For example, a vector space has a sort of scalars and a sort of vectors, so it is a two-sorted structure.

In his *Geometry* [14] of 1637, René Descartes observed that all arithmetic operations on numbers could be mimicked by manipulations of line segments in a Euclidean plane. In fact it is enough for the plane to have the structure of a vector space. Before Descartes, it had perhaps been felt that the only rigorous form of mathematics was geometric; presumably it was because of this feeling that Euclid's *Elements* [21] expressed, in geometric language, theorems of what we today would call algebra or number theory. Descartes observed that, because algebra (or more precisely field theory) *can* be put in geometric terms, there is no need actually to do so. In short, algebra can be done with the rigor of geometry.

We may observe conversely that, if field theory can be expressed geometrically, there is no real need for fields as distinct structures. More precisely, in a vector space of dimension at least 2, the sort of scalars is not needed, as long as the sort of vectors has the relation of parallelism.

I worked this out in [51]. The result is that there is a certain *equivalence of categories.* In one of the categories, the objects are vector spaces of dimension 2 or more, considered as quadruples $(V, K, *, \parallel)$, where

1) V is an abelian group in the signature $\{+, -, \mathbf{0}\}$;
2) K is a field in the signature $\{+, -, \cdot, 0, 1\}$;
3) $*$ is the action of K on V, that is, a certain function from $K \times V$ to V;
4) $\parallel$ is the binary relation of parallelism on V, defined by the formula

$$\exists x\ \exists y\ \ (x * \boldsymbol{u} + y * \boldsymbol{v} = \mathbf{0} \wedge (x \neq 0 \wedge y \neq 0))\,.$$

The arrows in this category are just embeddings (as defined in the last section). In the other category, the objects are the reducts $(V, \|)$ of the objects $(V, K, *, \|)$ in the first category; the arrows are still embeddings. Then these two categories are equivalent. In fact, if we are given an object $(V, \|)$ from the second category, then we can define a set K of certain equivalence-classes of pairs of parallel vectors, and we can define an addition and a multiplication on K, and an action of K on V, so that $(V, K, *, \|)$ is an object of the first category. In particular then, the objects of the second category are just the models of a certain theory.

There is some subtlety in the choice of arrows for these categories. For example, even though parallelism is definable in a vector space of dimension at least two, the categories of vector spaces with and without a symbol for parallelism are *not* equivalent, if the arrows are just embeddings. For example, the identity on the ring $\mathbb{H}$ of quaternions induces an embedding of the vector space $(\mathbb{H}, \mathbb{R}, *)$ in $(\mathbb{H}, \mathbb{C}, *)$; but the same function is *not* an embedding of $(\mathbb{H}, \mathbb{R}, *, \|)$ in $(\mathbb{H}, \mathbb{C}, *, \|)$. Indeed, 1 and i (as vectors in $\mathbb{H}$) are not parallel with respect to the scalar field $\mathbb{R}$, but they are parallel with respect to $\mathbb{C}$.

The two categories of vector spaces with and without parallelism become equivalent if the arrows are *elementary* embeddings. An embedding of structures is just a function that preserves the truth of *quantifier-free* sentences with parameters; an **elementary embedding** preserves the truth of *all* sentences. In particular, the inclusion of $(\mathbb{H}, \mathbb{R}, *)$ in $(\mathbb{H}, \mathbb{C}, *)$ is not an elementary embedding.

Suppose a theory T is such that every embedding of its models is elementary. Then by the definition on page 12, T is model complete. The converse of this observation is a theorem of Abraham Robinson [56, 2.3.1].

Supposing a theory T has a model companion T^* (as defined on page 12), we have that all models of T^* are existentially closed models of T^*; therefore they are also existentially closed models of T. More is true. First of all, every existentially closed model of T will be a model of T^*, by work of Eklof and Sabbagh [19, Prop. 7.10]. Now suppose T is **inductive,** that is, it meets either of the following two conditions, which are equivalent by a theorem of Chang [11] and of Łoś and Suszko [40]:

1. The union of an ascending chain of models of T is a model of T.
2. T has $\forall\exists$ axioms (in the sense of §2).

In this case, if there is a theory whose models are precisely the existentially closed models of T, then this theory is a model companion of T^* [19, Cor. 7.13].

Loosely, in an existentially closed model of a theory, everything that *can* happen *does* happen. When the model is a vector space, it may seem that two conflicting things can happen:

1. The dimension can always be made higher by addition of new vectors.
2. Linearly independent vectors can be made dependent by addition of new scalars.

Because of the precise definition of existential closedness, it is the latter tendency that wins out: the new scalars satisfy a quantifier-free formula; the new vectors, only a universal formula. In an existentially closed vector space in the usual signature, the scalar field is algebraically closed; but the dimension of the space is simply 1. However, in the signature with a binary symbol for parallelism, the existentially closed vector spaces have dimension 2. More generally, in the signature with an n-ary symbol for linear dependence, the existentially closed vector spaces have dimension n. Again, this is worked out in [51].

5 Interacting rings

In a vector space, the vectors may act as derivations of the scalar field. The theory of such structures has no model companion, unless one adds some new symbols to the signature. These matters were worked out by my student Özcan Kasal [37].

There is a remarkable symmetry in this situation. We are considering quadruples $(V, K, *, D)$, where $(V, K, *)$ is a vector space (in the notation on page 15), and in particular $*$ is the action of K as a scalar field on V, but now also D is an action of V as a space of derivations on K. Also, as K has a multiplication, so we require V to have a multiplication: the 'bracket' operation $(\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{u} \circ \boldsymbol{v} - \boldsymbol{v} \circ \boldsymbol{u}$. Thus, V is a Lie ring, while K is an associative, commutative ring. Then axioms for these structures $(V, K, *, D)$ come in dual pairs.

There seems to be some precedent for referring to $(V, K, *, D)$ as a **Lie–Rinehart pair.** The set $\mathrm{Der}(K)$ of derivations of a field K can be given the structure of both a vector-space over K and a Lie ring. Let V be a subspace and sub-ring of $\mathrm{Der}(K)$, and let k be the constant field of V. Then V is what is termed by Rinehart [54] a (k, K)-*Lie algebra.* Other terms include *pseudo-algèbre de Lie* [32] and *Lie d-ring* [45], as one may learn from Stasheff [62, p. 228], in whose own terminology (V, K) is a *Lie–Rinehart pair over k.* This term applies more generally to the situation where k is just a (commutative associative) ring and K is a commutative algebra over k; but I shall require K to be a field. In any case, reference to k may distract us from seeing the symmetry or dualism present in the pair (V, K), or rather the quadruple $(V, K, *, D)$ discussed above. But it is just this dualism that I want now to emphasize.

Probably the theory of Lie–Rinehart pairs (in the present sense) has no model companion, by a result of Macintyre that was announced [41], but not published: namely, the theory of Lie algebras (over a given field) has no model companion. In fact, just recovering this result would be a worthwhile exercise.

The dualism between associative rings and Lie rings can be brought out in a way that I presented first at Logicum Colloquium 2005 in Athens, in a contributed talk. Suppose R is an abelian group. Then the endomorphisms of R compose an abelian group, $\mathrm{End}(R)$. If $\circ$ is composition in $\mathrm{End}(R)$, let $\circ'$ be reverse composition:

$$f \circ' g = g \circ f.$$

Then for any pair (p, q) of integers, there is an operation $p\circ - q\circ'$ on $\mathrm{End}(R)$, and it is a **multiplication** on $\mathrm{End}(R)$ in the most general sense: it distributes over addition from either side. Now suppose $\cdot$ is a multiplication on R. Then for every x in R, there is an element λ_x of $\mathrm{End}(R)$ given by

$$\lambda_x(y) = x \cdot y.$$

Moreover, the function $x \mapsto \lambda_x$ is a homomorphism of abelian groups. Let us say that $(R, \cdot\,)$ is a (p,q)**-ring** if $x \mapsto \lambda_x$ is a ring-homomorphism from $(R, \cdot\,)$ to $(\mathrm{End}(R), p\circ - q\circ')$. For example,

- an associative ring is a $(1,0)$-ring;
- a Lie ring is a $(1,1)$-ring.

In particular, if $(p,q) \in \{(1,0),(1,1),(0,0)\}$, then $(\mathrm{End}(R), p\circ - q\circ')$ is itself a (p,q)-ring. The converse is also true; but I have found no sign that the result has been published or observed.

Now go back to the Lie–Rinehart pairs $(V,K,*,D)$ discussed above. The theory of these structures is not inductive; but the theory of such structures in which $\dim_K(V) \leqslant m$ is inductive, and it has a model companion, which can be derived from the model companion of DF^m discussed in §3. Moreover, we can define in V an isomorphic copy of K as a field, using as a parameter just a single element t of K with a nonzero derivative. This t is in particular an endomorphism of the group-structure of V. Thus we can obtain a model-complete theory of Lie rings in a signature with a symbol for a singulary function. (I have written only the draft of an article showing this.)

Again, although it appears that the theory of Lie rings as such has no model companion, this does not rule out the possibility that there is a model-complete theory of Lie rings in the usual signature. This question should be settled.

I return to the work of Özcan Kasal. He studies Lie–Rinehart pairs, in characteristic 0, but in a signature *without* a symbol for the bracket operation. Let T be the theory of such structures. Kasal characterizes the existentially closed models of T and shows that the class of these models is not **elementary:** it is not the class of models of a particular theory. Therefore T has no model companion.

However, Kasal observes that there is a certain relation of dependence among the scalars: A scalar x depends on a set Y of scalars if $\delta x = 0$ for every derivation δ in V such that $\delta y = 0$ for every y in Y. (Here δx can also be written as δDx.) Kasal then enlarges the signature to contain, for each positive integer n, a symbol for the mutual dependence of n scalars. Then he shows that the theory of *expansions* to this signature of models of T has a model companion.

6 Recursion and induction

The remaining sections of this document concern ideas that come out of teaching. I feel about teaching the way Richard Feynman [22, pp. 166 f.] does:

> In any thinking process there are moments when everything is going good and you've got wonderful ideas. Teaching is an interruption, and so it's the greatest pain in the world. And then there are the *longer* periods of time when not much is coming to you. You're not getting any ideas, and if you're doing nothing at all, it drives you nuts! You can't even say 'I'm teaching my class.'
>
> If you're teaching a class, you can think about the elementary things that you know very well. These things are kind of fun and delightful. It doesn't do you any harm to think them over again. Is there a better way to present them? Are there any new problems associated with them? Are there any new thoughts you can make about them? The elementary things are *easy* to think about; if you can't think of a new thought, no harm done; what you thought about it before is good enough for the class. If you *do* think of something new, you're rather pleased that you have a new way of looking at it.
>
> The questions of the students are often the source of new research...

In teaching a course called Fundamentals of Mathematics, I observed that Peano [46] had apparently been confused about something that Dedekind [13] had got right: the Induction Axiom for $(\mathbb{N}, 1, x \mapsto x + 1)$ (mentioned in §1 on page 6) does not by itself justify recursive definitions of operations like addition, multiplication, and exponentiation. In general, one needs the other two of the so-called Peano Axioms:

1) $x \mapsto x + 1$ is not surjective;
2) $x \mapsto x + 1$ is injective.

I first discussed these matters publicly at Logicum Colloquium 2008 in Bern in a contributed talk, and Alexandre Borovik has referred to this talk in his own work. One *can* prove the existence of addition and multiplication by induction alone, and Landau [39] does this, though without dwelling on the logical implications. However, this fact makes modular arithmetic possible. The set $\mathbb{Z}/n\mathbb{Z}$ of congruence-classes of integers with respect to a modulus n satisfies the Induction Axiom, and from this alone, it follows that $\mathbb{Z}/n\mathbb{Z}$ is a ring in the usual sense. Euler's first proof of Fermat's Theorem (reported by Gauss [23, ¶50, p. 32]) can be understood as a proof by induction in $\mathbb{Z}/p\mathbb{Z}$: With respect to the modulus p, we have $1^p \equiv 1$, and if $a^p \equiv a \pmod{p}$, then since $(a+1)^p \equiv a^p + 1$, we conclude $(a+1)^p \equiv a + 1$.

The different models of the Induction Axiom, and the operations that can be defined in them, were investigated by Henkin [31] (whose proof [30] of the *Completeness Theorem*—see §7 below—is the one used today). However, since Henkin's natural numbers began with 0 instead of 1, he missed the following observation: that the recursive definition

$$x^1 = x, \qquad\qquad x^{y+1} = x^y \cdot x$$

of multiplication is valid for $\mathbb{Z}/n\mathbb{Z}$ if and only if n is one of the numbers 1, 2, 6, 42, and 1806. (It turns out that these numbers were found by Dyer-Bennet [18] as the only moduli with respect to which the congruence of ${a_0}^{b_0}$ and ${a_1}^{b_1}$ can be inferred from that of the a_i and of the b_i.)

I wrote an unpublished article [52] on the relation between induction and recursion, and on confusions about it; John Baldwin referred to some of this in a talk [6]. We should distinguish two kinds of recursive definition. First, one can consider the set of natural numbers to be defined recursively by the rules that 1 is a natural number, and if n is, then so is $n+1$. The set of formulas of a logic is recursively defined in this way. Recursive definitions of sets justify proofs by induction on those sets. However, they do not alone justify recursive definitions of *functions on* those sets. This is understood by some writers, such as Enderton [20], but not others.

In the spirit of Feynman, we might make the presentation of basic logic more interesting in the following way. Starting with 'atomic' formulas, we build up other formulas recursively. Each formula can then be seen as the root of a tree whose leaves are the atomic formulas that appear in it. But then we must prove that the tree is uniquely determined by its root: this is needed to justify recursive definitions of functions on the set of formulas. An example of such a function is the one that assigns to each formula the relation that it defines on a given structure. We proceed to develop the notion of **formal proof:** from a given set of sentences to be considered as axioms, we define recursively the set of sentences that are to be considered as theorems provable from those axioms. Here then a theorem is the root of a tree whose leaves are axioms. However, this time the tree is not uniquely determined by its root. We have no obvious procedure for finding the proof of a theorem, other than to enumerate all of the possibilities. But then we have no obvious procedure for determining whether an arbitrary sentence is a theorem, unless the set of theorems is a complete theory. Indeed, there may be *no* such procedure, and *Gödel's Incompleteness Theorem* (see page 8 in §1 above and the next section) can be understood to establish this fact for certain systems of axioms for a theory of $(\mathbb{N}, +, \cdot)$.

The foregoing paragraph is just an example of how attention paid to tedious foundational details can give insight into deep results. The paper [52] ends with some apparently new results, although in preparing for a talk in the mathematics department of Istanbul Bilgi University, October 11, 2011, I found that revision is needed.

The Peano Axioms determine the natural numbers as composing a structure in a signature $\{1, \mathrm{S}\}$. This structure turns out to have a binary relation by which it is well-ordered. This observation leads naturally to the von Neumann definition of the natural numbers [67], whereby the least of them is $\varnothing$ (now called 0), and $n+1$ is $n \cup \{n\}$; by this definition, the natural numbers compose the set called ω. Then the natural numbers are just the finite examples of **ordinal numbers,** in von Neumann's definition.

This development of the ordinal numbers has a natural generalization. The structure determined by the Peano Axioms is a *free object* in the category whose objects are structures in $\{1, \mathrm{S}\}$ and whose arrows are embeddings. But for any signature $\mathscr{S}$ with no relation symbols, there is a free object in the category of structures in $\mathscr{S}$. There is then a set-theoretic definition of this free object, resembling von Neumann's definition of the natural numbers; and then there is a weakening of the definition that gives us a larger class, corresponding to the class of ordinals, in which the free object embeds.

We can see the elements of this larger class as a new kind of set. Each of these new sets has a **type,** and each of its elements falls into one or more **grades.** Specifically, there is a type for each symbol of $\mathscr{S}$, and then there is one more type for *limits.* If a symbol is n-ary, then the sets of its type have elements of n different grades. The constant symbols are 0-ary; sets of their type are empty.

Thus there is an analogy between sets in the usual sense, and natural numbers (in the usual sense):

1. The unique least natural number corresponds to the unique empty set $\varnothing$.
2. There is *one* way of getting new numbers, namely by adding 1 to old numbers; correspondingly, sets are determined by their elements.
3. A new number is obtained by adding 1 to a *single* natural number; correspondingly, an element of a set is an element in only one way.

7 The Russell Paradox

This section gives my idea of set theory, as developed while teaching it. There are some leads that may be pursued further.

Every student of mathematics should be familiar with the Russell Paradox [59]. I present it as follows. Our language gives us the notion of a **collective noun,** which is a singular noun that refers to many things at once. Suppose we attempt to declare that one particular collective noun, such as *set,* is going to be the most general. Then the sets that do not contain themselves must compose a set. This set contains itself if and only if it does not; and that is absurd. Therefore there is no most general collective noun.

We can nonetheless choose a collective noun that will be most general for our purposes. A similar move is made in model theory, in the study of a particular theory with infinite models. There is no largest model of this theory, but there is a *'monster model',* which is larger than all of the other models that one wants to study; these models can be assumed to be elementary substructures of the monster model.

For the collective noun that is most general for our purposes, the obvious choice is *collection.* Such an understanding seems to be behind accounts of set theory that can be found in standard textbooks of other areas of mathematics. Indeed, in two such books are found the following statements.

> A collection of objects viewed as a single entity will be called a *set.*
> [2, p. 32]

> Intuitively we consider a class to be a collection A of objects (elements) such that given any object x it is possible to determine whether or not x is a member (or element) of A. [36, p. 2]

However, such statements suggest questions such as,

- Is there a collection of things that are not objects?
- Is there a collection that is not viewed as a single entity?
- Is there a collection in which membership is impossible to determine?

If the answers are no, then the quoted statements can be reduced to the following, respectively:

A collection will be called a *set.*

Intuitively we consider a class to be a collection.

These are not very useful. But if any of the questions above are answered yes, then examples should be given.

I approach set theory as the study of a particular kind of collection, to be called a **set.** We do not say what a set is, beyond its being an element of some model of the theory that we develop. The models of set theory have one sort, and their signature has just one symbol, which the binary relation symbol $\in$ for membership. Thus all elements of sets must be sets themselves.

We may imagine that set theory has an *intended model,* to be called $\mathbf{V}$. This is a collection of sets, but not necessarily a set itself. A singulary formula in the signature of set theory defines a *collection* of elements of $\mathbf{V}$; but there is no reason to assume that this collection is actually *in* $\mathbf{V}$; that is, there is no reason to assume that it is a set. We refer to such a collection as a **class.** Then there is a class of sets that do not contain themselves: this is the class defined by the formula $x \notin x$. Call this class $\boldsymbol{R}$. The Russell Paradox now becomes the theorem that $\boldsymbol{R}$ is not a set. This theorem can be expressed formally as a sentence of set theory, namely

$$\neg\exists x\ \forall y\ (y \in x \leftrightarrow y \notin y).$$

The Russell Paradox has an echo in the proof of Gödel's Incompleteness Theorem mentioned above in §1 and the last section. Gödel assigns to each symbol S in the logic of $(\mathbb{N}, +, \cdot)$ a distinct element $\ulcorner S \urcorner$ of $\mathbb{N}$. Then to a formula $S_1 \cdots S_n$ can be assigned the number

$$2^{\ulcorner S_1 \urcorner} \cdot 3^{\ulcorner S_2 \urcorner} \cdot 5^{\ulcorner S_3 \urcorner} \cdots,$$

that is, $\prod_{i=1}^{n} {p_i}^{\ulcorner S_i \urcorner}$, where $(p_k : k \in \mathbb{N})$ is the sequence of primes. If the formula is φ, then the number thus assigned to it can itself be denoted by $\ulcorner \varphi \urcorner$.

Given a recursive collection Δ of axioms for $(\mathbb{N}, +, \cdot)$, Gödel has a system of *formal proof* (see the last section) of deriving their logical consequences. Then there is a binary formula $\varphi(x, y)$ such that, for any singulary formula $\psi(x)$ and any n in $\mathbb{N}$, there is a formal proof of $\psi(n)$ from Δ if and only $\varphi(\ulcorner \psi \urcorner, n)$ is true in $(\mathbb{N}, +, \cdot)$. Now let $\theta(x)$ be $\neg\varphi(x, x)$. Then

$$\psi(\ulcorner \psi \urcorner) \text{ is provable if and only if } \neg\theta(\ulcorner \psi \urcorner) \text{ is true.}$$

Here the resemblance to the Russell Paradox comes out. Replace ψ with θ. Writing σ for $\theta(\ulcorner \theta \urcorner)$, we have that σ is provable if and only if $\neg\sigma$ is true, that is, σ is false. But every provable sentence is true. Therefore σ is not provable; but it is true.

There now two possibilities.

1. One possibility is that our proof system is *incomplete.* In particular, although there is no formal proof of σ from Δ, maybe σ is still a **logical consequence** of Δ: that is, maybe σ is true in every model of Δ. In this case, we have not ruled out the possibility that every sentence in the complete theory of $(\mathbb{N}, +, \cdot)$ is a logical consequence of Δ.

Indeed, such a possibility is realized in second-order logic. The second-order Peano Axioms for $(\mathbb{N}, 1, \mathrm{S})$ have only one model, up to isomorphism; therefore every sentence of the theory of this model is a logical consequence of the Peano Axioms. Thus there can be no complete proof-system for the second-order logic of $(\mathbb{N}, 1, \mathrm{S})$, and therefore of $(\mathbb{N}, +, \cdot)$.

However, Gödel had already proved a **Completeness Theorem** for his proof system for first-order logic [24]. If there is no first-order proof of σ

from Δ, then σ is not a logical consequence of Δ. Therefore the second possibility must be realized:

2. Not all sentences in the first-order theory of $(\mathbb{N}, +, \cdot\,)$ are logical consequences of Δ. The theory axiomatized by Δ is not complete.

We have not gone through the details of what a recursive collection is, nor the derivation of $\varphi(x, y)$ above. However, it is easier to carry out related arguments in set theory.

As an example, in set theory we establish the **Undefinability of Truth.** This is a theorem proved by Tarski [63, p. 247], who notes his debt to Gödel. We can encode each formula φ now as a set $\ulcorner\varphi\urcorner$ in $\mathbf{V}$. Suppose there were a singulary formula defining the collection of codes of all true sentences. Then there would be a singulary formula φ defining the collection of codes of all singulary formulas ψ such that $\psi(\ulcorner\psi\urcorner)$ is *false.* In short,

$$\varphi(\ulcorner\psi\urcorner) \iff \neg\psi(\ulcorner\psi\urcorner).$$

Now we really do have the Russell Paradox. Putting φ for ψ gives the contradiction.

8 Apollonius

Apollonius of Perga wrote eight books on conic sections. The last four were lost in the original Greek, perhaps because they were too difficult; any case, books V–VII do survive in Arabic translation.

I myself have spent much time only with Book I [1]. Here Apollonius assigns the names *parabola, ellipse,* and *hyperbola* to the conic sections: these are names that in Greek allude to the properties of the curves *as* sections of a cone. Students today are commonly *told* in textbooks that these curves can be obtained by cutting a cone; but a proof is rarely seen. A lovely geometrical account of the sections, with visual proofs, is given by Hilbert and Cohn-Vossen [33]; but this account uses *right* cones, like most other descriptions of conic sections as such. However, Apollonius shows that oblique cones can be used. He in effect derives our modern equations for the cones; these equations work in rectangular or oblique coordinate systems. Then, given a curve satisfying one of the equations, he shows how to obtain a cone from which that curve can be cut.

In *Rules for the Direction of the Mind* [15], Descartes suggests that the ancient mathematicians must have had some kind of algebra for discovering their theorems, but that they felt obliged to conceal it. I last worked through Book I of Apollonius for a course at the Nesin Mathematics Village in the summer of 2008; then I got the feeling that Descartes was probably not correct, but that Apollonius's geometric arguments really did show how Apollonius thought of things.

However, this is a point worth further investigation. Some historians read Apollonius, but probably few mathematicians: the reading is difficult, and our Cartesian methods of analysis seem to work more efficiently. And yet a good understanding of Apollonius might help prevent inaccurate generalizations about mathematics. I said in §1 that most mathematics involves structures. This may be true today; but it makes little sense for ancient mathematics. And yet ancient mathematics is unquestionably mathematics, sometimes at the highest level.

9 Euclid and Archimedes

Euclid is more commonly read than Apollonius, by mathematicians and others. The same may be true of Archimedes. In the last three summers at the Nesin Mathematics Village, I have taught a course that I called Non-standard Analysis. The title comes from the book of Abraham Robinson [57] that makes rigorous Leibniz's notion of infinitesimals in calculus. But I have found it worthwhile to go back further in history, to study Archimedes's use of infinitesimal methods, in the quadrature of the parabola for example [3, 64, 29], or in showing that the surface of a sphere is equal to a circle whose radius is the diameter of the sphere [4].

I have also worked through the construction of the real numbers from the natural numbers. I have aimed to make the construction as transparent as possible. Treatments of the construction that I know of, as in Landau [39] or Spivak [61], have the aim of satisfying the reader (and the writer) that

$\mathbb{R}$ really exists. But in the spirit of Feynman (§6 above), one can get more out of the construction.

For example, one can rediscover Hölder's theorem [35] that there is a unique complete densely ordered abelian group, namely $(\mathbb{R}, +, <)$; and every archimedean ordered abelian group embeds in this. Hölder derives his results from considering *magnitudes* as used Book V of Euclid's *Elements.*

All of this has led to ongoing collaboration with Alexandre Borovik on constructions of $\mathbb{R}$ and their possible generalizations. One observation is the following. There seem to be two standard constructions of $\mathbb{R}$.

1. One is Dedekind's [13], whereby a real number is a set of rational numbers with an upper bound, but no maximum element, and containing every rational that is less than one of its elements. This construction uses only the ordering of $\mathbb{Q}$, and so, first of all, $(\mathbb{R}, \subseteq)$ is obtained as the **completion** of $(\mathbb{Q}, <)$. But $\mathbb{Q}$ is also an abelian group, and this structure extends to $\mathbb{R}$; likewise, the completion of every ordered abelian group has the structure of an abelian group, *if and only if* the ordered group is archimedean (this is part of Hölder's theorem).

2. The other standard construction of $\mathbb{R}$ obtains it as the quotient of the ring of Cauchy sequences of $\mathbb{Q}$ by the maximal ideal of sequences convergent to 0. But this construction can be applied to any ordered field, even a non-archimedean one. In this case, the Cauchy sequences should have length equal to the cofinality of the field. (All Cauchy sequences shorter than this are eventually constant.) So this construction should be distinguished from Dedekind's.

References

[1] Apollonius of Perga, *Conics. Books I–III*, revised ed., Green Lion Press, Santa Fe, NM, 1998, Translated and with a note and an appendix by R. Catesby Taliaferro, With a preface by Dana Densmore and William H. Donahue, an introduction by Harvey Flaumenhaft, and diagrams by Donahue, Edited by Densmore. MR MR1660991 (2000d:01005)

[2] Tom M. Apostol, *Mathematical analysis*, second ed., Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1974. MR 49 #9123

[3] Archimedes, *The works of Archimedes*, Dover Publications Inc., Mineola, NY, 2002, Reprint of the 1897 edition and the 1912 supplement, Edited by T. L. Heath. MR MR2000800 (2005a:01003)

[4] ———, *The works of Archimedes. Vol. I*, Cambridge University Press, Cambridge, 2004, The two books on the sphere and the cylinder, Translated into English, together with Eutocius' commentaries, with commentary, and critical edition of the diagrams by Reviel Netz. MR MR2093668 (2005g:01006)

[5] James Ax, *The elementary theory of finite fields*, Ann. of Math. (2) **88** (1968), 239–271. MR MR0229613 (37 #5187)

[6] John Baldwin, *The impact of modern model theory on mathematics and philosophy*, `http://lumiere.ens.fr/~dbonnay/files/talks/baldwin.pdf`, June 2010, slides of a talk at Philosophy and Model Theory, Université Paris Ouest Nanterre La Défense.

[7] J. L. Bell and A. B. Slomson, *Models and ultraproducts: An introduction*, North-Holland Publishing Co., Amsterdam, 1969, reissued by Dover, 2006. MR MR0269486 (42 #4381)

[8] Lenore Blum, *Differentially closed fields: a model-theoretic tour*, Contributions to algebra (collection of papers dedicated to Ellis Kolchin), Academic Press, New York, 1977, pp. 37–61. MR MR0491149 (58 #10415)

[9] Alexandru Buium, *Arithmetic analogues of derivations*, J. Algebra **198** (1997), no. 1, 290–299. MR 98k:12004

[10] C. C. Chang and H. J. Keisler, *Model theory*, third ed., North-Holland Publishing Co., Amsterdam, 1990, first edition, 1973. MR 91c:03026

[11] Chen Chung Chang, *On unions of chains of models*, Proc. Amer. Math. Soc. **10** (1959), 120–127. MR MR0103812 (21 #2576)

[12] Zoé Chatzidakis and Ehud Hrushovski, *Model theory of difference fields*, Trans. Amer. Math. Soc. **351** (1999), no. 8, 2997–3071. MR 2000f:03109

[13] Richard Dedekind, *Essays on the theory of numbers. I: Continuity and irrational numbers. II: The nature and meaning of numbers*, authorized translation by Wooster Woodruff Beman, Dover Publications Inc., New York, 1963. MR MR0159773 (28 #2989)

[14] Descartes, *The geometry of René Descartes*, Dover Publications, Inc., New York, 1954, Translated from the French and Latin by David Eugene Smith and Marcia L. Latham, with a facsimile of the first edition of 1637.

[15] ———, *Règles pour la direction de l'esprit*, Classiques de la philosophie, Le Livre de Poche, 2002, Traduction et notes par Jacques Brunschwig. Préface, dossier et glossaire par Kim Sang Ong-Van-Cung.

[16] Jean-Louis Duret, *Sur la théorie élémentaire des corps de fonctions*, J. Symbolic Logic **51** (1986), no. 4, 948–956. MR MR865921 (88d:03066)

[17] ______, *Équivalence élémentaire et isomorphisme des corps de courbe sur un corps algébriquement clos*, J. Symbolic Logic **57** (1992), no. 3, 808–823. MR MR1187449 (94g:03073)

[18] John Dyer-Bennet, *A theorem on partitions of the set of positive integers*, Amer. Math. Monthly **47** (1940), 152–154. MR MR0001234 (1,201b)

[19] Paul Eklof and Gabriel Sabbagh, *Model-completions and modules*, Ann. Math. Logic **2** (1970/1971), no. 3, 251–295. MR MR0277372 (43 #3105)

[20] Herbert B. Enderton, *A mathematical introduction to logic*, Academic Press, New York, 1972. MR MR0337470 (49 #2239)

[21] Euclid, *Euclid's Elements*, Green Lion Press, Santa Fe, NM, 2002, All thirteen books complete in one volume, the Thomas L. Heath translation, edited by Dana Densmore. MR MR1932864 (2003j:01044)

[22] Richard P. Feynman, *'Surely you're joking, Mr. Feynman!'*, W.W. Norton & Company, New York and London, 1985, Adventures of a Curious Character; as told to Ralph Leighton; edited by Edward Hutchings.

[23] Carl Friedrich Gauss, *Disquisitiones arithmeticae*, Springer-Verlag, New York, 1986, Translated into English by Arthur A. Clarke, revised by William C. Waterhouse.

[24] Kurt Gödel, *The completeness of the axioms of the functional calculus of logic (1930a)*, From Frege to Gödel (Jean van Heijenoort, ed.), Harvard University Press, 1976, pp. 582–91.

[25] ______, *On formally undecidable propositions of* principia mathematica *and related systems I (1931)*, From Frege to Gödel (Jean van Heijenoort, ed.), Harvard University Press, 1976, pp. 596–616.

[26] Bradd Hart, Ehud Hrushovski, and Michael C. Laskowski, *The uncountable spectra of countable theories*, Ann. of Math. (2) **152** (2000), no. 1, 207–257. MR 1792295 (2002a:03063)

[27] Bradd Hart and Michael C. Laskowski, *A survey of the uncountable spectra of countable theories*, Algebraic model theory (Toronto, ON, 1996), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 496, Kluwer Acad. Publ., Dordrecht, 1997, pp. 107–118. MR 1481440 (99c:03053)

[28] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR MR0463157 (57 #3116)

[29] Thomas L. Heath (ed.), *The* Method *of Archimedes recently discovered by Heiberg: A supplement to* The Works of Archimedes *1897*, University Press, Cambridge, UK, 1912.

[30] Leon Henkin, *The completeness of the first-order functional calculus*, J. Symbolic Logic **14** (1949), 159–166. MR MR0033781 (11,487d)

[31] ———, *On mathematical induction*, Amer. Math. Monthly **67** (1960), 323–338. MR MR0120156 (22 #10913)

[32] Jean-Claude Herz, *Pseudo-algèbres de Lie. I*, C. R. Acad. Sci. Paris **236** (1953), 1935–1937. MR MR0055323 (14,1057c)

[33] D. Hilbert and S. Cohn-Vossen, *Geometry and the imagination*, Chelsea Publishing Company, New York, N. Y., 1952, Translated by P. Neményi. MR MR0046650 (13,766c)

[34] Wilfrid Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993. MR 94e:03002

[35] Otto Hölder, *The axioms of quantity and the theory of measurement*, J. Math. Psych. **40** (1996), no. 3, 235–252, Translated from the 1901 German original and with notes by Joel Michell and Catherine Ernst, With an introduction by Michell. MR MR1423724 (98c:00001)

[36] Thomas W. Hungerford, *Algebra*, Graduate Texts in Mathematics, vol. 73, Springer-Verlag, New York, 1980, Reprint of the 1974 original. MR MR600654 (82a:00006)

[37] Özcan Kasal, *Model theory of derivation spaces*, Ph.D. thesis, Middle East Technical University, Ankara, February 2010.

[38] Piotr Kowalski, *Derivations of the Frobenius map*, J. Symbolic Logic **70** (2005), no. 1, 99–110. MR MR2119125 (2005m:03069)

[39] Edmund Landau, *Foundations of analysis. The arithmetic of whole, rational, irrational and complex numbers*, third ed., Chelsea Publishing Company, New York, N.Y., 1966, translated by F. Steinhardt; first edition 1951; first German publication, 1929. MR 12,397m

[40] Jerzy Łoś and Roman Suszko, *On the extending of models (IV): Infinite sums of models*, Fund. Math. **44** (1957), 52–60. MR MR0089813 (19,724c)

[41] Angus Macintyre, *On existentially complete Lie algebras*, Notices of the AMS **21** (1974), no. 3, A–379, Preliminary report.

[42] ———, *Generic automorphisms of fields*, Ann. Pure Appl. Logic **88** (1997), no. 2-3, 165–180, Joint AILA-KGS Model Theory Meeting (Florence, 1995). MR 99c:03046

[43] Tracey McGrail, *The model theory of differential fields with finitely many commuting derivations*, J. Symbolic Logic **65** (2000), no. 2, 885–913. MR 2001h:03066

[44] Michael Morley, *Categoricity in power*, Trans. Amer. Math. Soc. **114** (1965), 514–538. MR MR0175782 (31 #58)

[45] Richard S. Palais, *The cohomology of Lie rings*, Proc. Sympos. Pure Math., Vol. III, American Mathematical Society, Providence, R.I., 1961, pp. 130–137. MR MR0125867 (23 #A3164)

[46] Giuseppe Peano, *The principles of arithmetic, presented by a new method (1889)*, From Frege to Gödel (Jean van Heijenoort, ed.), Harvard University Press, 1976, pp. 83–97.

[47] David Pierce, *Function fields and elementary equivalence*, Bull. London Math. Soc. **31** (1999), no. 4, 431–440. MR 2001a:03080

[48] ______, *Differential forms in the model theory of differential fields*, J. Symbolic Logic **68** (2003), no. 3, 923–945. MR 2 000 487

[49] ______, *Geometric characterizations of existentially closed fields with operators*, Illinois J. Math. **48** (2004), no. 4, 1321–1343. MR MR2114160

[50] ______, *Fields with several commuting derivations*, ArXiv e-prints (2007).

[51] ______, *Model-theory of vector-spaces over unspecified fields*, Arch. Math. Logic **48** (2009), no. 5, 421–436. MR MR2505433

[52] ______, *Numbers*, ArXiv e-prints (2011).

[53] David Pierce and Anand Pillay, *A note on the axioms for differentially closed fields of characteristic zero*, J. Algebra **204** (1998), no. 1, 108–115. MR 99g:12006

[54] George S. Rinehart, *Differential forms on general commutative algebras*, Trans. Amer. Math. Soc. **108** (1963), 195–222. MR MR0154906 (27 #4850)

[55] Abraham Robinson, *On the concept of a differentially closed field*, Bull. Res. Council Israel Sect. F **8F** (1959), 113–128 (1959). MR MR0125016 (23 #A2323)

[56] ______, *Complete theories*, second ed., North-Holland Publishing Co., Amsterdam, 1977, With a preface by H. J. Keisler, Studies in Logic and the Foundations of Mathematics, first published 1956. MR MR0472504 (57 #12202)

[57] ______, *Non-standard analysis*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1996, Reprint of the second (1974) edition, With a foreword by Wilhelmus A. J. Luxemburg. MR MR1373196 (96j:03090)

[58] Julia Robinson, *Definability and decision problems in arithmetic*, J. Symbolic Logic **14** (1949), 98–114. MR 0031446 (11,151f)

[59] Bertrand Russell, *Letter to Frege (1902)*, From Frege to Gödel (Jean van Heijenoort, ed.), Harvard University Press, 1976, pp. 124–5.

[60] Michael F. Singer, *Model theory of partial differential fields: from commuting to noncommuting derivations*, Proc. Amer. Math. Soc. **135** (2007), no. 6, 1929–1934 (electronic). MR MR2286106 (2007m:03077)

[61] Michael Spivak, *Calculus. 2nd ed.*, Berkeley, California: Publish or Perish, Inc. XIII, 647 pp., 1980 (English).

[62] Jim Stasheff, *Homological reduction of constrained Poisson algebras*, J. Differential Geom. **45** (1997), no. 1, 221–240. MR MR1443334 (98i:17026)

[63] Alfred Tarski, *Logic, semantics, metamathematics*, second ed., Hackett Publishing Co., Indianapolis, IN, 1983, Papers from 1923 to 1938, Translated by J. H. Woodger, Edited and with an introduction by John Corcoran. MR 736686 (85e:01065)

[64] Ivor Thomas (ed.), *Selections illustrating the history of Greek mathematics. Vol. II. From Aristarchus to Pappus*, Harvard University Press, Cambridge, Mass, 1951, With an English translation by the editor. MR 13,419b

[65] Lou van den Dries, *Classical model theory of fields*, Model theory, algebra, and geometry, Math. Sci. Res. Inst. Publ., vol. 39, Cambridge Univ. Press, Cambridge, 2000, pp. 37–52. MR 1773701 (2001i:12012)

[66] Xavier Vidaux, *Multiplication complexe et équivalence élémentaire dans le langage des corps*, J. Symbolic Logic **67** (2002), no. 2, 635–648. MR 1 905 159

[67] John von Neumann, *On the introduction of transfinite numbers (1923)*, From Frege to Gödel (Jean van Heijenoort, ed.), Harvard University Press, 1976, pp. 346–354.

[68] Carol Wood, *The model theory of differential fields of characteristic $p \neq 0$*, Proc. Amer. Math. Soc. **40** (1973), 577–584. MR 48 #8227

[69] ———, *Prime model extensions for differential fields of characteristic $p \neq 0$*, J. Symbolic Logic **39** (1974), 469–477. MR 50 #9577

[70] Yoav Yaffe, *Model completion of Lie differential fields*, Ann. Pure Appl. Logic **107** (2001), no. 1-3, 49–86. MR 1 807 840