Morley's Categoricity Theorem

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November 16, 2012

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This is a sketch of a proof of Morley's Categoricity Theorem. Readers who think some parts are too sketchy (or even wrong) will kindly inform me. My main reference is Marker [2], where the ingredients of the proof are scattered about.

Some material that might be developed during the course of the proof, but is not strictly needed for the proof, is in an appendix. (I typed up some of this because I was also looking at Chang and Keiser's somewhat different account [1].)

This document is typeset for a5 paper (so two of its pages fit on one side of the usual a4 paper).

Contents

1. The spectrum function

Given a theory T and an infinite cardinal κ, we let $\mathrm{I}(T,\kappa)$ denote the number of non-isomorphic models of T of cardinality κ. Some examples are as in Table 1, where κ is now an arbitrary *uncountable* cardinal. If

	T is the theory of:	$\mathrm{I}(T,\omega)$	$\mathrm{I}(T,\kappa)$
1)	bare sets	1	1
2)	vector spaces over $\mathbb{F}_q$	1	1
3)	vector spaces over $\mathbb{Q}$	ω	1
4)	algebraically closed fields of characteristic p	ω	1
5)	an infinite, co-infinite singulary relation	1	κ
6)	an infinite, co-infinite singulary relation with ∞ distinguished elements	ω	κ
7)	$(\omega, x \mapsto x+1)$	ω	1
8)	an equivalence with ∞ classes, all size ∞	1	2^κ
9)	an equivalence with ∞ classes, all size ∞, each having ∞ distinguished elements	2^ω	2^κ

Table 1: Numbers of nonisomorphic models

$\mathrm{I}(T,\kappa)=1$ for some infinite κ, then T is **κ-categorical.** If T is so for some κ that is not less than $|T|$, and T has no finite models, then T is complete, by the Łoś–Vaught Test.

Here $|T|$ is the number of inequivalent formulas (in countably many variables) with respect to T. For example, if the signature of T consists of uncountably many constants, but according to T they are all equal, then T is countable.

Henceforth T will range over the complete countable theories with infinite models. Our aim is to characterize those T that are κ-categorical for some uncountable κ and to show that they must be κ-categorical for *all* uncountable κ.

A *type* is a complete set of formulas in a given tuple of variables. We can understand types through the following formal development. Let $\mathrm{L}_n(T)$ be the n-th **Lindenbaum algebra** of T, namely the Boolean algebra of equivalence classes with respect to T of formulas in the tuple $(x_i \colon i < n)$ of variables. (Then by definition $|T| = \sup_{n\in\omega}|\mathrm{L}_n(T)|$.) An **$n$-type** of T is an ultrafilter of $\mathrm{L}_n(T)$. The set of all of these is $\mathrm{S}_n(T)$. This is the *Stone space* of $\mathrm{L}_n(T)$, a compact (and in particular Hausdorff) topological space that has, as basic open and closed sets, the sets $[\varphi]$, where $\varphi \in \mathrm{L}_n(T)$ and

$$[\varphi] = \{p \in \mathrm{S}_n(T) \colon \varphi \in p\}.$$

If $\mathfrak{M} \models T$ and $A \subseteq M$, when we may denote $\mathrm{S}_n(\mathrm{Th}(\mathfrak{M}_A))$ simply by $\mathrm{S}_n(A)$. (Thus $\mathrm{S}_n(T)$ is also $\mathrm{S}_n(\varnothing)$.) Then T is **ω-stable** just in case for all n in ω

$$|A| \leqslant \omega \implies |\mathrm{S}_n(A)| \leqslant \omega.$$

2. Necessity of ω-stability

If Ω is some set and, for some possibly finite cardinality κ, the set of κ-element subsets of Ω is partitioned, then a subset of Ω is **homogeneous** for the partition if all of its κ-element subsets belong to the same member of the partition.

Lemma 1 (Ramsey's Theorem)**.** *For every positive n in ω, if the set of n-element subsets of an infinite set Ω is partitioned into finitely many classes, then some infinite subset of Ω is homogeneous for the partition.*

Proof. We denote the set of n-element subsets of Ω by $[\Omega]^n$. The claim is easy when $n = 1$: it is an infinitary pigeonhole principle. Suppose the claim is true when $n = \ell$, but now $f\colon [\Omega]^{\ell+1} \to m$ for some m in ω, so that f induces a partition of $[\Omega]^{\ell+1}$. We define a sequence $((a_i, X_i)\colon i \in \omega)$ recursively so that $X_0 = \Omega$ and $a_0 \in X_0$, and if X_k is an infinite subset of ω with element a_k, then X_{k+1} is infinite and homogeneous for the partition of $[X_k \smallsetminus \{a_k\}]^\ell$ given by $X \mapsto f(X \cup \{a_k\})$, and $a_{k+1} \in X_{k+1}$. In particular $a_k \notin X_{k+1}$. By the same pigeonhole principle as before, there is some j in m such that the set of k in ω for which

$$\{f(X \cup \{a_k\})\colon X \in [X_{k+1}]^\ell\} = \{j\} \tag{$*$}$$

is infinite. Then the set of a_{k+1} such that $(*)$ holds is infinite and homogeneous for the partition induced by f. □

If $\mathfrak{M}$ is some structure, $A \subseteq M$, and $<$ is a linear ordering of A, then $(A, <)$ is **indiscernible** (or A is *order-indiscernible* with respect to $<$) if for all n in ω, all tuples $(a_0, \dots, a_{n-1})$ of elements of A such that

$$a_0 < \dots < a_{n-1}$$

have the same type in $\mathfrak{M}$.

Lemma 2. *For every linear order, T has a model in which the linear order embeds as an order-indiscernible set.*

Proof. By Compactness, we need only show that, for every n in ω, for every finite subset Γ of $\mathrm{L}_n(T)$, for every r in ω, there is a model of T in which is satisfied the conjunction of the formulas

$$\varphi(x_{f(0)}, \dots, x_{f(n-1)}) \to \varphi(x_{g(0)}, \dots, x_{g(n-1)}),$$

where $\varphi \in \Gamma$ and f and g are strictly increasing functions from n to r.

Let $\mathfrak{M}$ be a model of T, and let $<$ be a linear ordering of M. We can now identify $[M]^n$ with the set of $\vec{a}$ in M^n such that $a_0 < \dots < a_{n-1}$. Define h on $[M]^n$ by

$$h(\vec{a}) = \{\varphi \in \Gamma\colon \mathfrak{M} \models \varphi(\vec{a})\}.$$

Then h induces a finite partition of $[M]^n$. By Ramsey's Theorem, M has an infinite subset that is homogeneous for the partition; and then there is an increasing sequence $(a_i\colon i < r)$ of elements of this subset. This sequence satisfies in $\mathfrak{M}$ the desired conjunction of formulas. □

Theorem 1. *If T is κ-categorical for some uncountable κ, then T is ω-stable.*

Proof. There is a countable theory T^*, in an expansion of the signature of T, such that $T \subseteq T^*$, every model of T expands to a model of T^*, and T^* has *Skolem functions,* that is, for every formula $\varphi(\vec{x}, y)$ of the expanded signature, there is a term $t(\vec{x})$ such that

$$T^* \vdash \forall \vec{x}\ (\exists y\ \varphi(\vec{x}, y) \to \varphi(\vec{x}, t(\vec{x}))).$$

Suppose κ is an infinite cardinal. By the last lemma there is a model $\mathfrak{M}$ of T^* of size κ that includes an order-indiscernible set I of size κ. Because of the Skolem functions, we may assume that $\mathfrak{M}$ is *generated* by the set of entries in I.

Say A is a countable subset of M, and let X be a minimal subset of I such that every element of A is $t(\vec{b})$ for some term t and some tuple $\vec{b}$ of elements of X. Then X is countable. Let two tuples $\vec{c}$ and $\vec{d}$ of elements of I be called X-equivalent if they have the same order type and moreover

$$c_i < x \iff d_i < x, \qquad\qquad c_i = x \iff d_i = x,$$

for all x in X. In this case, for all n-tuples $(t_i : i < n)$ of terms (in the appropriate variables), by the order-indiscernibility of I, the tuples $(t_i(\vec{c}) : i < n)$ and $(t_i(\vec{d}) : i < n)$ realize the same types in $\mathrm{S}_n(A)$. But also the number of X-equivalance classes of tuples from I is countable. Therefore $\mathfrak{M}$ realizes only countably many types over A.

Suppose now T is not ω-stable. Since κ is uncountable, T has a model of size κ realizing uncountably many types over some countable set. Such a model cannot be isomorphic to the reduction of $\mathfrak{M}$ to the signature of T. Thus T is not κ-categorical. □

Note that T can be ω-stable without being κ-categorical for any infinite κ. Indeed, all theories on the table above are ω-stable.

3. Necessity of having no Vaughtian pairs

A **Vaughtian pair** of models of T is a pair $(\mathfrak{N}, \mathfrak{M})$ of (distinct) models such that $\mathfrak{M} \prec \mathfrak{N}$, but some formula defines the same infinite relation in each model. For example, the theory of an infinite, co-infinite singulary relation has Vaughtian pairs.

Suppose φ defines the same infinite relation in each member of a Vaughtian pair of models of T. If we expand the signature to contain a new singulary relation-symbol P, then there is a theory T^* in the expanded signature such that $(\mathfrak{N}, M) \models T^*$ if and only if $\mathfrak{N} \models T$ and M is the universe of a proper elementary substructure $\mathfrak{M}$ of $\mathfrak{N}$ such that $\varphi^{\mathfrak{M}} = \varphi^{\mathfrak{N}}$ and this is infinite. Hence for example if T does have a Vaughtian pair, then it has a Vaughtian pair of countable models.

Lemma 3. *If T has a Vaughtian pair, then it has a Vaughtian pair of countable homogeneous models that realize the same types and are therefore (by the back-and-forth method) isomorphic to one another.*

Proof. Suppose $(\mathfrak{N}, \mathfrak{M})$ is a Vaughtian pair of countable models of T. We can write N as $\{b_n : n \in \omega\}$. Then we can create an elementary chain of countable Vaughtian pairs $(\mathfrak{N}_k, \mathfrak{M}_k)$ of models of T such that the type of $(b_i : i < k)$ is realized in $\mathfrak{M}_k$ for each k in ω. The union of this chain is a Vaughtian pair $(\mathfrak{N}_\omega, \mathfrak{M}_\omega)$ of models of T such that $\mathfrak{M}_\omega$ and therefore also $\mathfrak{N}_\omega$ realize every type that $\mathfrak{N}$ does. Now we can make another elementary chain whose union is a Vaughtian pair of countable models of T realizing the same types as one another.

Similarly, if $(\vec{a}, b)$ and $\vec{c}$ from N are such that $\vec{a}$ and $\vec{c}$ realize the same type in $\mathfrak{N}$, then $(\mathfrak{N}, \mathfrak{M})$ has an elementary extension $(\mathfrak{N}^*, \mathfrak{M}^*)$ such that $(\vec{a}, b)$ and $(\vec{c}, d)$ realize the same type in $\mathfrak{N}^*$ for some d in N^*. We can do the same for $\mathfrak{M}$. By repeating, we obtain an elementary chain whose union is a Vaughtian pair of homogeneous models of T.

By interweaving chains, we achieve what is desired. □

Theorem 2. *If T is κ-categorical for some uncountable κ, then it has no Vaughtian pairs.*

Proof. Suppose T has a Vaughtian pair. There is an elementary chain $(\mathfrak{M}_\alpha \colon \alpha \leqslant \kappa)$ of countable homogeneous models of T that realize the same types, with some formula defines the same infinite relation in each one. Indeed, $(\mathfrak{M}_1, \mathfrak{M}_0)$ is a pair as given by the last lemma. If $(\mathfrak{M}_{\alpha+1}, \mathfrak{M}_\alpha)$ has been defined as desired, we obtain $(\mathfrak{M}_{\alpha+2}, \mathfrak{M}_{\alpha+1})$ as a Vaughtian pair such that $\mathfrak{M}_{\alpha+1}$ and $\mathfrak{M}_{\alpha+2}$ are isomorphic under a map that restricts to an isomorphism from $\mathfrak{M}_\alpha$ to $\mathfrak{M}_{\alpha+1}$. If β is a limit, then we let

$$\mathfrak{M}_\beta = \bigcup_{\alpha<\beta} \mathfrak{M}_\alpha;$$

the $\mathfrak{M}_\alpha$ being homogeneous and realizing the same types, $\mathfrak{M}_\beta$ too is homogeneous and realizes the same types that the $\mathfrak{M}_\alpha$ realize. Then $\mathfrak{M}_\kappa$ is a model of T of size κ with a countably infinite definable subset. But T also has a model of size κ in which every infinite definable subset has size κ. Thus T is not κ-categorical for any infinite κ. □

4. Minimal formulas

We have the following general method for proving that T is *not* ω-stable.

Lemma 4. *Suppose that, for some property of basic open subsets of type spaces, for some A, for some n, every nonempty subset $[\varphi]$ of $\mathrm{S}_n(A)$ with the property has disjoint nonempty subsets $[\varphi_0]$ and $[\varphi_1]$ with the property. Then T is not ω-stable.*

Proof. Under the given conditions, by recursion we obtain for each k in ω, for each α in 2^k, a certain formula φ_α such that $[\varphi_\alpha]$ is nonempty and has the given property, and

$$[\varphi_{\alpha 0}] \cup [\varphi_{\alpha 1}] \subseteq [\varphi_\alpha], \qquad [\varphi_{\alpha 0}] \cap [\varphi_{\alpha 1}] = \varnothing.$$

By compactness of $\mathrm{S}_n(A)$, for each σ in 2^ω, the descending chain

$$[\varphi_{\sigma\restriction 0}] \supset [\varphi_{\sigma\restriction 1}] \supset [\varphi_{\sigma\restriction 2}] \supset \cdots$$

has nonempty intersection, with an element p_σ. The function $\sigma \mapsto p_\sigma$ on 2^ω is injective. There are countably many formulas φ_α, so we can go back and assume A is countable. Thus T is not ω-stable. □

If $\mathfrak{M} \models T$, a formula φ (with parameters from M) is **minimal** for $\mathfrak{M}$ if $\varphi^{\mathfrak{M}}$ is infinite, but for every formula ψ in the same variables, either $\varphi \wedge \psi$ or $\varphi \wedge \neg\psi$ defines a finite set. Then an immediate application of the Lemma 4 is the following.

Lemma 5. *If T is ω-stable, then every model of T has a minimal formula.*

The relation defined by a minimal formula is a **minimal set.** Suppose D is a minimal set. If $A \subseteq D$, then an element of D is **algebraic** over A if it belongs to a finite subset of D defined by a formula with parameters from A and from the formula defining D in the first place. The set of elements of D that are algebraic over A can be denoted by

$$\operatorname{acl}(A).$$

Lemma 6. *In a minimal set, if $a \in \operatorname{acl}(\{b\} \cup C)$, then either $a \in \operatorname{acl}(C)$ or $b \in \operatorname{acl}(\{a\} \cup C)$.*

Proof. Suppose a satisfies $\exists^{\leqslant n}x\ \varphi(x, b)$. If the formula

$$\varphi(a, y) \wedge \exists^{\leqslant n}x\ \varphi(x, y)$$

has only finitely many solutions, then since b is one of them, it is in $\operatorname{acl}(\{a\} \cup C)$ as desired. So suppose there are infinitely many solutions. Then for some m

$$\exists^{\leqslant m}y\ \neg(\varphi(a, y) \wedge \exists^{\leqslant n}x\ \varphi(x, y)).$$

Suppose the formula $\exists^{\leqslant m}y\ \neg(\varphi(x, y) \wedge \exists^{\leqslant n}x\ \varphi(x, y))$ has solutions a_i, where $i \leqslant n$. Then the negation of the formula

$$\bigwedge_{i \leqslant n} \varphi(a_i, y) \wedge \exists^{\leqslant n}x\ \varphi(x, y)$$

has at most $m \cdot (n + 1)$ solutions. In particular, the formula itself has at least one solution. But then the a_i must not all be distinct. This shows $a \in \operatorname{acl}(C)$. □

Now the proofs that work for fields and vector spaces work here: a minimal set has a **basis,** namely a maximal algebraically independent subset or a minimal subset over which the whole set is algebraic; also all bases have the same size.

Lemma 7. *If T has no Vaughtian pairs, then every minimal formula for a model of T is minimal for every elementary extension of that model.*

Proof. Suppose $\varphi(x)$ is minimal for some model $\mathfrak{M}$ of T, but not for some elementary extension of $\mathfrak{M}$. Then there is a formula $\theta(x, y)$ such that, for each n in ω,

$$T \vdash \exists y\ (\exists^{\geqslant n} x\ (\varphi(x) \wedge \theta(x, y)) \wedge \exists^{\geqslant n} x\ (\varphi(x) \wedge \neg\theta(x, y))).$$

In particular, there is a formula $\theta(x, y)$ and, for each n in ω, a tuple b_n from M such that $\theta(x, b_n)$ has finitely many solutions, but at least n solutions. Thus if $\mathfrak{M} \prec \mathfrak{N}$, then $\theta(x, b_n)$ defines in each structure the same subset of size at least n. By Compactness, T has a Vaughtian pair. $\square$

A minimal formula that satisfies the conclusion of the lemma is **strongly minimal.** By the last two lemmas, if T is ω-stable with no Vaughtian pairs, then every model of T has a strongly minimal formula.

5. Sufficiency

Lemma 8. *If T is ω-stable, then the isolated points of each space $\mathrm{S}_n(A)$ are dense.*

Proof. Suppose the isolated points of some $\mathrm{S}_n(A)$ are not dense. Then some nonempty subset $[\varphi]$ contains no isolated points. Consequently $[\varphi]$ is the union of two disjoint nonempty subsets $[\varphi_0]$ and $[\varphi_1]$, neither of which (of course) contains isolated points. (Indeed, $[\varphi]$ has distinct elements p and q, and then $q \notin [\psi]$ for some ψ in p; now let φ_0 be $\varphi \wedge \psi$, and let φ_1 be $\varphi \wedge \neg\psi$.) By Lemma 4, T must not be ω-stable. $\square$

If $\mathfrak{M} \models T$ and $A \subseteq M$, then $\mathfrak{M}$ is **prime** over A if $\mathfrak{M}_A$ embeds elementarily in every other model of $\mathrm{Th}(\mathfrak{M}_A)$.

Lemma 9. *If T is ω-stable, $\mathfrak{M} \models T$, and $A \subseteq M$, then $\mathfrak{M}$ has an elementary substructure that is prime over A.*

Proof. For some maximal ordinal γ, for all β in γ, we can define elements b_β of M recursively so that b_β realizes an isolated type over $\{b_\alpha \colon \alpha < \beta\}$, but is not in $A \cup \{b_\alpha \colon \alpha < \beta\}$. Let $B = A \cup \{b_\alpha \colon \alpha < \gamma\}$. By Lemma 8 and the Tarski–Vaught Test, B is the universe of an elementary substructure $\mathfrak{B}$ of $\mathfrak{M}$. Now suppose $\mathfrak{M}_A \equiv \mathfrak{N}_A$. By recursion we can embed $\mathfrak{B}_A$ elementarily in $\mathfrak{N}_A$. □

Theorem 3. *If T is ω-stable with no Vaughtian pairs, then T is κ-categorical for all uncountable κ, and $\mathrm{I}(T, \omega) \leqslant \omega$.*

Proof. Suppose T is ω-stable with no Vaughtian pairs. By Lemmas 9, 5 and 7, T has a prime model $\mathfrak{M}_0$ with a strongly minimal formula φ. If $\mathfrak{M}$ is a model of T of size κ, then we may assume $\mathfrak{M}_0 \preccurlyeq \mathfrak{M}$, so φ defines in $\mathfrak{M}$ a minimal set. Since T has no Vaughtian pairs, $\mathfrak{M}$ must be prime over its interpretation of φ, by Lemma 9. This interpretation has a countable basis, if $\kappa = \omega$; but if κ is uncountable, then this is the size of a basis.

Suppose $\mathfrak{N}$ is a model of T such that the bases of $\varphi^{\mathfrak{N}}$ and $\varphi^{\mathfrak{M}}$ have the same size. By induction, a bijection between the bases is elementary and extends to an isomorphism of the minimal sets themselves, and then to an isomorphism of the structures. □

A. Appendix

A model of T is **saturated** if it realizes all types in a small set of parameters, where *small* means having cardinality less than the model itself. All saturated models of T having the same size are isomorphic to one another; indeed, an isomorphism can be constructed by the back-and-forth method.

One can say more generally that a model of T is *κ-saturated* if it realizes all types in fewer than κ parameters. So a model of size κ is saturated if and only if it is κ-saturated.

By contrast, T itself is **κ-stable** if every model realizes only κ types in κ parameters. That is, T is κ-stable just in case for all n in ω

$$|A| \leqslant \kappa \implies |\mathrm{S}_n(A)| \leqslant \kappa.$$

Another application of Lemma 4 is the following.

Theorem 4. *If T is ω-stable, then T is κ-stable for all infinite κ.*

Proof. Suppose $[\varphi]$ is a subset of $\mathrm{S}_n(A)$ that is larger than κ. Let p be the subset of $\mathrm{L}_n(A)$ that contains ψ if $[\varphi \wedge \psi]$ is larger than κ; otherwise let p contain $\neg\psi$. Then $[\varphi]$ consists of every point in

$$\bigcup_{\psi \in p} [\varphi \wedge \neg\psi],$$

along with p if this is indeed a point of $\mathrm{S}_n(A)$. Thus for some ψ, both $[\varphi \wedge \psi]$ and $[\varphi \wedge \neg\psi]$ are larger than κ. □

A formula φ in $\mathrm{L}_n(T)$ is **complete** if $[\varphi]$ has only one element. This one element is then just an isolated point in $\mathrm{S}_n(T)$. The theory T is **atomic** if every formula is entailed (with respect to T) by a complete formula, that is, the isolated points of each $\mathrm{S}_n(T)$ are dense in the space.

More generally if $\mathfrak{M} \models T$ and $A \subseteq M$, then $\mathfrak{M}$ is **atomic** over A if the type over A of every tuple of elements of M is isolated.

Theorem 5. *The prime model over A guaranteed by Lemma 9 is also atomic over A.*

Proof. With the notation of the proof of Lemma 9, suppose $\beta < \gamma$, and for all n in ω, for all $(\alpha(i) \colon i < n)$ from β^n, the type of $(b_{\alpha(0)}, \dots, b_{\alpha(n-1)})$ is isolated. We have to show that the type of $(b_{\alpha(0)}, \dots, b_{\alpha(n-1)}, b_\beta)$ is isolated. Suppose

$$\mathfrak{B}_A \models \chi(b_{\alpha(0)}, \dots, b_{\alpha(n-1)}, b_\beta).$$

The type of b_β over $\{b_\alpha\colon \alpha < \beta\}$ is isolated by some formula

$$\varphi(b_{\alpha(n)}, \ldots, b_{\alpha(n+m-1)}, y),$$

so we have

$$\mathfrak{B}_A \models \forall y\ (\varphi(b_{\alpha(n)}, \ldots, b_{\alpha(n+m-1)}, y) \to \chi(b_{\alpha(0)}, \ldots, b_{\alpha(n-1)}, y)).$$

By inductive hypothesis, the type of $(b_{\alpha(0)}, \ldots, b_{\alpha(n+m-1)})$ is isolated by a formula $\theta(x_0, \ldots, x_{n+m-1})$, so we have

$$\begin{aligned}\mathfrak{B}_A \models \forall x_0\ \cdots \forall x_{n+m-1}\ (\theta(x_0, \ldots, x_{n+m-1}) \to \\ \forall y\ (\varphi(x_n, \ldots, x_{n+m-1}, y) \to \chi(x_0, \ldots, x_{n-1}, y))),\end{aligned}$$

equivalently

$$\begin{aligned}&\mathfrak{B}_A \models \forall x_0\ \cdots \forall x_{n-1}\ \forall y \\ &\quad (\exists x_n\ \cdots \exists x_{n+m-1}\ (\theta(x_0, \ldots, x_{n+m-1}) \land \varphi(x_n, \ldots, x_{n+m-1}, y)) \\ &\qquad\qquad \to \chi(x_0, \ldots, x_{n-1}, y)).\end{aligned}$$

Thus the formula

$$\exists x_n\ \cdots \exists x_{n+m-1}\ (\theta(x_0, \ldots, x_{n+m-1}) \land \varphi(x_n, \ldots, x_{n+m-1}, y))$$

isolates the type of $(b_{\alpha(0)}, \ldots, b_{\alpha(n-1)}, b_\beta)$. $\square$

References

[1] C. C. Chang and H. J. Keisler, *Model theory*, third ed., Studies in Logic and the Foundations of Mathematics, vol. 73, North-Holland Publishing Co., Amsterdam, 1990. MR 91c:03026

[2] David Marker, *Model theory: an introduction*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002. MR 1 924 282