From Euclid to Descartes

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First-year mathematics students at Mimar Sinan take a course in Book I of Euclid's *Elements* in the fall semester, and then take an analytic geometry course in the spring. These notes are intended to be useful to teachers of both courses.

Euclid distinguishes between *equality* and *sameness.* For example, in Book I of the *Elements,* Proposition 35 is that parallelograms on the *same* base and in the same parallels are equal to one another; this is used to prove Proposition 36, that parallelograms on *equal* bases and in the same parallels are equal to one another.

An implicit definition of equality can be found in what we call Common Notion 4: things that coincide with one another are equal to one another. More precisely, things that can be *made* or *caused* to coincide are equal. Hence the radii of a circle are equal to one another, by the very construction of the circle: As one compass point swings around the fixed other point, the gap between the two points is made to coincide with each radius in turn.

In Proposition 4, the Side-Angle-Side Theorem, the hypothesis comprises three equalities: of two sides of a triangle to two sides (respectively) of another triangle, and of the included angles to one another. The conclusion comprises four equalities: of the remaining sides, of the angles subtended by the original equal sides, and of the triangles themselves. Indeed, the equalities in the hypothesis mean that the two sides and included angle of the one triangle can be superimposed on, or *applied* to, those of the other triangle. When this is done, then the whole triangles coincide, and therefore the remaining sides and angles are equal, and the triangles themselves are equal.

Today we are inclined to think that Proposition 4 is 'really' another postulate. Its proof does not rely on any earlier propositions or postulates. But I think the proposition really does have a proof, which makes use of the implicit definition of equality.

Equality is not identity, but is what we call an equivalence-relation. Given a (bounded) straight line—what we call today a line segment—, we may consider the class of straight lines that are equal to the given straight line. This class itself can be understood as the *length* of that straight line. Today we think of this length as a *number,* namely a positive element of a certain ordered field. But it takes work to justify this way of thinking. In fact this way of thinking took centuries to appear.

I think it is implicit in Euclid that lengths are the elements of an *ordered abelian semigroup:* by this I mean an abelian semigroup with a (linear) ordering such that always

$$a < a + b.$$

This rule is suggested by Common Notion 5: the whole is greater than the part.

In the same way, the *area* of a figure can be understood as the class of all figures that are equal to it. Equality of figures is implied by *congruence.* However, incongruent figures can also be equal, as in Propositions 35, quoted above. There, two incongruent parallelograms are shown to be equal by cutting them up (in different ways) into congruent pieces.

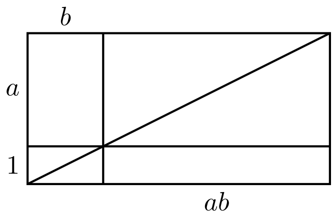
The ordering of lengths is *dense,* by Proposition 10 (to bisect a given bounded straight line).

Proposition 44 is to construct, in a given angle, and on a given straight line as base, a parallelogram equal to a given triangle. Using Propositions 42 and 45, we may replace the triangle in Proposition 44 with an arbitrary rectilineal figure. If we consider the given straight line in that proposition as a unit, and if we let the given angle be right, then we obtain a function from areas of rectilineal figures to lengths. Any such figure determines the height of a rectangle that is equal to it and sits on a base of unit length.

Having fixed a unit length, we can multiply lengths: the product of two lengths is the height of the rectangle of unit width that is equal to a rectangle whose base and height are the two given lengths respectively.

Obviously then this multiplication is commutative. It is fairly obvious that multiplication distributes over addition. It is not obvious from Book I of Euclid that multiplication is associative; but we can still prove it, as follows.

Multiplication can be effected as in the figure below. The two small



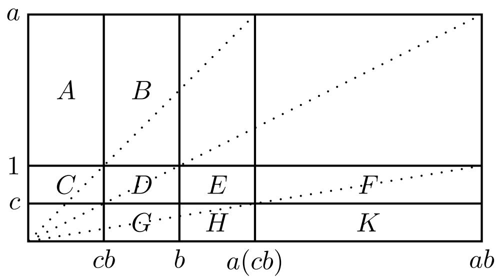
rectangles on either side of the diagonal of the large rectangle are equal to one another by Proposition 43. The converse is also true, in the following sense: given that those two small rectangles have a common vertex as in the figure, then the two straight lines from that vertex to the remaining vertices of the large rectangle must lie on the same straight line.

We can adjust the figure so that the height of the large rectangle is not the sum of a and the unit, but is the larger of the two. Now we shall not label straight lines with their lengths, but we shall label points with their distances from the lower left vertex. Then associativity of multiplication can be shown by means of the next diagram. Indeed, by the converse of Proposition 43, we shall have $c(ab) = a(cb)$, provided

$$C + D + E = K.$$

By Proposition 43 itself, we have

$$A + B = E + F + H + K,$$
$$B = E + F,$$



and therefore

$$A = H + K.$$

But again by Proposition 43, we have

$$A = D + E + G + H,$$

and therefore

$$D + E + G = K.$$

We finish by noting

$$C = G.$$

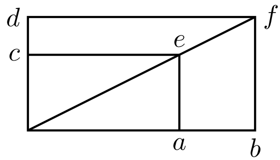
Therefore $c(ab) = a(cb)$. We have assumed $c < 1 < a$ and $b < a(cb)$. Strictly we should consider four more cases:

(1) $c < 1 < a$, but $a(cb) = b$;
(2) $c < 1 < a$, but $a(cb) < b$;
(3) $c < a < 1$; and
(4) $1 < c < a$.

With lengths, equality becomes sameness, identity. By writing $c(ab) = a(cb)$, we mean that two equivalence-classes are the *same,* and in particular, every element of one of the classes is *equal* to every element of the other class.

Again, the product of lengths a and b is the height of a unit-width rectangle that is equal to a rectangle whose width is a and height is b. Then the *quotient* of a by b is the height of a rectangle of width b that is equal to a unit-width rectangle of height a. So quotients of lengths always exist. Thus lengths are the positive elements of an ordered field.

By Proposition 47, the so-called Pythagorean Theorem, all lengths have square roots. By the same theorem, if again points in the next diagram are labelled with their distances from the lower left vertex, then

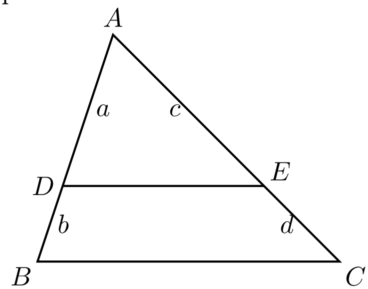


$$ad = bc, \qquad \frac{a}{b} = \frac{c}{d},$$

and therefore

$$af = a\sqrt{b^2 + d^2} = \sqrt{a^2b^2 + a^2d^2} = \sqrt{a^2b^2 + b^2c^2} = b\sqrt{a^2 + c^2} = be,$$
$$\frac{a}{b} = \frac{e}{f}.$$

Proposition 2 of Book VI of the *Elements* is that, in the following diagram, where DE is parallel to the base BC of the triangle, this straight



line DE cuts the other two sides of the triangle *proportionally:* that is, AD is to DB as AE is to EC. We may write this as

$$AD : DB :: AE : EC.$$

Here the expression $AD : DB$ can be understood to stand for a certain equivalence-class; and then the sign :: stands for identity of equivalence-classes. But these equivalence-classes are such that the same conclusion is expressed by

$$\frac{a}{b} = \frac{c}{d}.$$

We can prove this by dropping an altitude from A.

But Euclid's proof is not like that. For him, the meaning of $A : B :: C : D$ is that, for all natural numbers k and m,

$$kA > mB \iff kC > mD.$$

This is Definition 5 of Book V. Euclid does not define the ratio $A : B$ as such. We can understand it as the equivalence class of all pairs (C, D) such that $A : B :: C : D$; or we can understand it as the class of all pairs (k, m) such that $kA > mB$. In the latter case, a ratio is just a *Dedekind cut.* It would be foolish then to be impressed that Euclid somehow 'anticipated' Dedekind; the point is that Dedekind learned from Euclid.

In the definition of proportion, it is understood that A, B, C, and D are *magnitudes,* such as (bounded) straight lines or figures; moreover A and B *have a ratio,* as do C and D. By Definition 4 of the same book, this means for example some multiple of A exceeds B, and vice versa. It is a tacit assumption that any two straight lines have a ratio to one another. In our terms, the ordered semigroup of lengths is assumed to be archimedean. However, we have already shown that this assumption is not required.

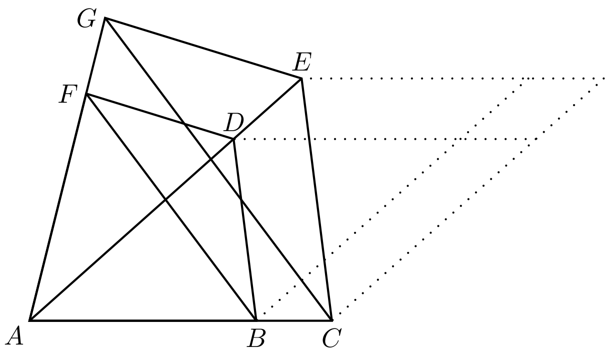
The use of capital letters to denote points, and minuscule letters to denote lengths, is a convention established by Rene Decartes at the beginning of his *Geometry* of 1637. Descartes gives a geometric definition of the product of two lengths; but he uses Euclid's theory of proportion for this. In particular, in the notation of our last diagram, Decartes defines

$$ad = bc,$$

so that if d is a unit, then a is the product of b and c. But this definition should be shown to be independent of angle BAC. It *is* independent, by Euclid's Proposition VI.2; but again, this proposition uses the archimedean assumption.

Descartes does not bother with the question of whether his multiplication is commutative or associative. A positive answer may be taken as an implicit consequence of Euclid's theory of proportion. As we have seen, that theory makes more assumptions than we need; but without the archimedean assumption, we have to do more work.

In the next figure, if $BD \parallel CE$ and $DF \parallel EG$, then by Euclid's theory



and in particular VI.2, we have

$$AB : BC :: AD : DE :: AF : AG,$$

and so $BF \parallel CG$. Without the archimedean assumption, we can make a similar argument, using quotients of lengths. But can we draw the same conclusion, without doing all of the work to show that lengths are the positive elements of an ordered field?

Using the idea of I.43, we know that the parallelogram BE is equal to the parallelogram DC. This is enough, if we can conclude also that the *rectangles* with the same sides as these parallelograms are also equal to one another. But it is not clear that this conclusion can be established without developing a full theory of proportion.

A theorem expressed in the language of proportion is Euclid's VI.30: to cut a given bounded straight line in so-called *extreme and mean ratio*. Given a length a, the problem is to find a shorter length x so that

$$\frac{a}{x} = \frac{x}{a - x}.$$

We can rewrite this equation as

$$a(a - x) = x^2.$$

This is the form of I.11: to cut a given straight line so that rectangle whose width is one segment and whose height is the whole straight line is equal to the square on the other segment. Euclid presents the solution to this problem *synthetically:* that is, he presents x as

$$\sqrt{a^2 + \frac{a^2}{4}} - \frac{a}{2},$$

then shows that this works. A Cartesian solution proceeds in the other direction, *analytically:* We assume that there *is* some x such that

$$a(a - x) = x^2,$$

and then we find

$$\begin{aligned} a^2 &= x^2 + ax, \\ a^2 + \frac{a^2}{4} &= x^2 + ax + \frac{a^2}{4} \\ &= \left(x + \frac{a}{2}\right)^2, \\ x &= \sqrt{a^2 + \frac{a^2}{4}} - \frac{a}{2}. \end{aligned}$$

However, *synthesis* and *analysis* are both Greek words, and the ancient Greek mathematicians were aware of the method of analysis, whereby one assumes that one has the x that one wants to find, in order to be able to figure out what x is.