The Foundations of Arithmetic in Euclid

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Introduction

The geometry presented in the thirteen books of Euclid's *Elements* is founded on five postulates; but the arithmetic in Books VII, VIII, and IX makes no explicit use of postulates. Nonetheless, despite some concerns that there might be gaps in Euclid's reasoning, particularly in his proof of the lemma now often named for him, Euclid does prove this lemma rigorously. He even proves something that students of mathematics are rarely troubled with today: the commutativity of multiplication in any ordered ring (such as that of the integers) whose positive elements are well ordered. In fact Euclid's result is more general: again in modern terms, in every ordinal number that is closed under ordinal addition and multiplication and on which this addition is commutative, the multiplication is also commutative. Euclid's main tool for proving all of this is a theory of proportion founded on what we call the Euclidean Algorithm.

Such is the main mathematical burden of the present work. The burden is discharged in the last two chapters. Those chapters are so long, and they are preceded by two other chapters, because Euclid, if he is going to be read at all, deserves to be read with more care than we often read anything today. We may not need to read our contemporaries so carefully, because of some common understandings that we can take for granted. It is not so with Euclid.

The foregoing can be said briefly, as I have just done; *showing* it is what I try to do in the whole work. It is true that somebody who is in a hurry to see Euclid's mathematics interpreted in modern terms can turn directly to §4.2 (p. 118). All of the chapters and their sections might be summarized as follows.

Chapter 1. The study of Euclid is an instance of doing *history.* As such, the study can both benefit from, and illustrate, the philosophy of history developed by R. G. Collingwood in several of his books.

§1.1. Studying the *Elements* is like studying an ancient building, such as the Hagia Sophia in Istanbul: it is a kind of archeology. Archeology is in turn a kind of history. Historical inference resembles mathematical inference; but the way to understand this is by actually doing history and mathematics—which we do by studying Euclid.

§1.2. The present study of Euclid is motivated by such questions as: Can students today learn number theory from Euclid? Does Euclid correctly prove such results as the commutativity of multiplication and "Euclid's Lemma"? Why are the last three propositions of Book VII so strange? I am going to pursue my questions, making use of some scholarship that I know of, overlooking (of course) the scholarship that I do not know.

§1.3. We cannot decide whether a particular statement or argument of Euclid is correct or incorrect without first understanding what it means. The meaning is not always obvious.

§1.4. In studying Euclid, we have to re-enact his thoughts, as best we can.

§1.5. In reading Euclid (or any philosopher), we cannot just say that he is wrong (if we think he is), without also offering a correction that (in our best judgment) he (or she) can agree with.

Chapter 2. Mathematicians used to learn mathematics from Euclid. Since this no longer commonly happens, we may be *better* able now to understand Euclid properly. Nonetheless, some students do still learn mathematics from Euclid. Among those students are undergraduates in my own mathematics department in Istanbul. Reading with them brings out certain features of Euclid.

§2.1. Dedekind did *not* learn his construction of the real numbers from Euclid. Unlike some of his contemporaries, he understood Euclid well enough to see that what he was doing was different.

§2.2. Looking back at Euclid's Greek (for the sake of translating this into Turkish) brings out some misleading features of the standard English translation by Heath. Heath aids the reader by typographical means; but this may cause us to think wrongly that Euclid's propositions are like modern theorems. Euclid may have established a pattern for modern mathematical exposition; but this does not mean he is obliged to follow it.

§2.3. Contrary to modern mathematical practice, Euclid's *equality* is not sameness or identity. Thinking about what equality really means, one can see that Euclid's Proposition I.4, the "Side Angle Side" theorem of triangle congruence, is a real theorem.

§2.4. According to Hilbert, Euclid's Fourth Postulate, the equality of all right angles, is really a theorem. Examining Hilbert's proof of this theorem shows how different is his way of thinking of geometry from Euclid's.

Chapter 3. We turn to Book VII of Euclid's *Elements* and to the specific investigation of questions raised earlier.

§3.1. The definition of unity may well be a late addition to this book.

§3.2. By definition, if a number is of a second number the same *part* or *parts* that a third number is of a fourth, then the four numbers are *proportional.* Whatever else this means, a proportion is not an *equation* of ratios, but an *identification* of them.

§3.3. Euclid's numbers are finite sets of two (or possibly one) or more elements. Euclid distinguishes between *dividing* numbers into parts and *measuring* numbers by numbers. It so happens that for infinite sets, being divisible into two equipollent subsets is logically stronger than being measurable by a two-element set.

§3.4. If a number is *parts* of a number in Euclid's sense, this does not mean that the one number is a *fraction* of the other.

§3.5. When the numbers in Euclid's diagrams appear as line segments, one may think of these as lyre strings. The ancient musical treatise called *Sectio Canonis* suggests a way of thinking about numerical ratios that is useful for understanding the *Elements.*

§3.6. Lying behind the notion of a ratio is the alternating subtraction or *anthyphaeresis* used in the Euclidean Algorithm.

§3.7. Proposition VII.4, that the less number is either part or parts of the greater, should be understood as an explanation of what being the *same parts* means in the definition of proportion: it means that application of the Euclidean Algorithm requires the same pattern of alternating subtractions in either case.

§3.8. With this understanding, the proof of the commutativity of multiplication, Proposition VII.16, unfolds logically.

§3.9. So does the proof of "Euclid's Lemma," Proposition VII.30.

Chapter 4. We consider Euclid's mathematics in the modern symbolic fashion.

§4.1. Precisely *because* it must be understood in this fashion, The last proposition (numbered 39) of Book VII may be a late addition to the *Elements.*

§4.2. Euclid's proof of the commutativity of multiplication is summarized.

§4.3. Euclid works in a structure $(\mathbb{N}, 1, +, \times, <)$.

§4.4. There is a list of axioms that Euclid uses implicitly.

§4.5. There are some redundancies on this list, though there is no reason to think Euclid recognized them.

§4.6. Commutativity of multiplication is derived from the axioms, as closely to Euclid as possible.

1 Philosophy of History

1.1 Archeology

We are going to investigate foundational aspects of Euclid's arithmetic, as presented in Books VII, VIII, and IX of the thirteen books of the *Elements.* Our investigation might be called archeology, though it will require no actual digging. We want to know something about the *Elements,* and in particular its theory of numbers; however, we have no first-hand *testimony* about what Euclid was doing, or trying to do. We just have the *Elements* itself. As for the earlier tradition that Euclid came out of, we have only traces of it. We have other works possibly *by* Euclid, such as the *Sectio Canonis,* which we shall find to be of interest. We have later works about Euclid, especially parts of Pappus's *Collection,* as well Proclus's *Commentary on the First Book of Euclid's* Elements. We shall make some use of these later works, but they were written centuries after the *Elements.*

Euclid himself does not provide testimony about what he is doing; he just does it. If proper history requires such testimony,[1] then we cannot make an historical study of Euclid's arithmetic. We can still make an *archeological* study. We can read the *Elements* itself, for evidence of what Euclid was trying to do there. In the same way, one might read the Church of St Sophia in Constantinople, for evidence of the aims of its master builders, Anthemius of Tralles

[1] A *Wikipedia* article requires testimony in a stricter sense. Because of the rule of No Original Research, a *Wikipedia* article about a book should be based on the testimony of published sources *other than the book itself.*

and Isidore of Miletus: for indeed, the basilica itself can still be visited in Istanbul.

Concerning this basilica, called the Hagia Sophia or Ayasofya, we do also have some written testimony. Procopius saw the church constructed in the sixth century, at the command of Emperor Justinian, after an earlier church had been destroyed in the so-called Nika Insurrection. Justinian spared no expense to build a new church, says Procopius [85, I.i.22, p. 11],

> so finely shaped, that if anyone had enquired of the Christians before the burning if it would be their wish that the church should be destroyed and one like this should take its place, shewing them some sort of model of the building we now see, it seems to me they would have prayed that they might see their church destroyed forthwith, in order that the building might be converted to its present form.

We might infer that Anthemius and Isidore did actually show Justinian a model of the basilica that they planned to build. Like this model, or like the old church that the new Hagia Sophia would replace, some kind of mathematics came before the *Elements* as we have it now; but we have only hints (as in Plato's *Meno*) of what this was like. About the construction of the Hagia Sophia itself, from Procopius we know [85, I.i.68–74, pp. 29–31],

> it was not with money alone that the Emperor built it, but also with labour of the mind and with the other powers of the soul, as I shall straightway show. One of the arches which I just now mentioned (*lôri* the master-builders call them), the one which stands towards the east, had already been built up from either side, but it had not yet been wholly completed in the middle, and was still waiting. And the piers (*pessoi*), above which the structure was being built, unable to carry the mass which bore down upon them, somehow or other suddenly began to crack, and they seemed on the point of collapsing. So Anthemius and Isidorus, terrified at what had happened, carried the matter to the Emperor, having come to have no hope in their technical skill. And straightway the Emperor, impelled by I know not what, but I suppose by God (for he is not himself a master-builder), commanded them to carry the curve of this arch to its final completion. "For when it rests upon itself," he said, "it will no longer need the props (*pessoi*) beneath

> it." And if this story were without witness, I am well aware that it would have seemed a piece of flattery and altogether incredible; but since there were available many witnesses of what then took place, we need not hesitate to proceed to the remainder of the story. So the artisans carried out his instructions, and the whole arch then hung secure, sealing by experiment the truth of his idea. Thus, then, was this arch completed.

We have no such contemporary account of the *Elements.* Nobody who knew Euclid can tell us, for example, of Euclid's discovery of the Fifth or Parallel Postulate, as being needed to prove such results as the equality of an exterior angle of a triangle to the two opposite interior angles. Nobody can testify that, since the Parallel Postulate is phrased in terms of right angles, Euclid figured he needed another postulate, according to which all right angles are equal to one another, and thus arose the Fourth Postulate. We have no *testimony* that Euclid's thoughts proceeded in this way; we can only infer it (or something like it) from the edifice of the *Elements* as it has come down to us.

By some accounts then, there can be no properly *historical* study of the *Elements,* but only a "prehistorical" or archeological study. However, I am going to agree with the philosopher and historian R. G. Collingwood (1889–1943) that archeology *is* history. It is hard to say that we know more about the construction of the Hagia Sophia than of the *Elements,* simply because we have Procopius's fanciful story about how the Emperor saved a partially erected arch from collapse. That story might after all be true; but it can hardly be credited without independent reason for thinking it plausible.

I shall make use of Collingwood's ideas about history, because I know them and find them relevant. The Hagia Sophia is relevant as being, like the *Elements,* one of the great structures of the ancient world.[2] After the Turkish conquest of Constantinople, nine centuries

[2] One might classify the Hagia Sophia as medieval rather than ancient. The edifice at least *symbolizes* the ancient mathematical world, in that the master builder Isidore also compiled texts of Archimedes with the commentaries

after the construction of the Hagia Sophia, this edifice and especially its great dome became a model for Ottoman Imperial mosques [42]. Euclid's *Elements* became a model for mathematics, even as it is done today.

As for Collingwood, though some of his books have remained in print or been brought back into print, he is little known today. After speculating on why this is, in a review of a biography, Simon Blackburn concludes [13],

> A lucky life, then, rather than an unlucky one, is the explanation for Collingwood's unattractive features—unless, as Aristotle thought, we cannot even call men lucky when they die, since we can be harmed after our death. I do not know whether Aristotle was right, but if he was wrong, then in most respects the neglect of Collingwood's thought may be our tragedy rather than his.

In Blackburn's judgment, "Collingwood was the greatest British philosopher of history of the twentieth century" [12]. In her own assessment of Collingwood, Mary Beard argues [10],

> it is surely crucial that he was a product of the old Oxford 'Greats' (that is, classics) course, which focused the last two and a half years of a student's work on the parallel study of ancient history on the one hand, and ancient and modern philosophy on the other. Most students were much better at one side than the other . . . In the context of Greats, Collingwood was not a maverick with two incompatible interests. Given the educational aims of the course, he was a rare success, even if something of a quirky overachiever; his combination of interests was exactly what Greats was designed to promote.

Collingwood's combination of interests does not seem to have included mathematics especially. Nonetheless, he was aware of it and talked about it, as will be seen below. His sense of what it means to do history will help us as we bring our own interest in mathematics to the reading of Euclid.

In directing actual archeological excavations, Collingwood found himself "experimenting in a laboratory of knowledge," as he reports

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of Eutocius, and he is mentioned in the extant manuscripts of these texts [2, p. 368, n. 750].

in *An Autobiography* of 1939 [22, p. 24]. The study of Euclid can likewise serve as a laboratory of ideas about history. Collingwood likens history to mathematics in *The Idea of History,* posthumously edited and first published in 1946 [23, pp. 262–3]:

> One hears it said that history is 'not an exact science'. The meaning of this I take to be that no historical argument ever proves its conclusion with that compulsive force which is characteristic of exact science. Historical inference, the saying seems to mean, is never compulsive, it is at best permissive; or, as people sometimes rather ambiguously say, it never leads to certainty, only to probability. Many historians of the present writer's generation, brought up at a time when this proverb was accepted by the general opinion of intelligent persons (I say nothing of the few who were a generation ahead of their time), must be able to recollect their excitement on first discovering that it was wholly untrue, and that they were actually holding in their hands an historical argument which left nothing to caprice, and admitted of no alternative conclusion, but proved its point as conclusively as a demonstration in mathematics.

These words were originally intended for *The Principles of History* [25, p. 18], whose extant manuscripts were rediscovered in 1995, having been thought discarded after being (severely) edited to form part of *The Idea of History.* I do not know whether historians generally agree with Collingwood on the subject of historical inference. According to one archeologist's explanation of his field [7, p. 7],

> Since nobody *knows* what happened in the past (even in the recent historical past), there will never be an end to archaeological research. Theories will come and go, and new evidence or discoveries will alter the accepted fiction that constitutes the orthodox view of the past and which becomes established through general repetition and widespread acceptance. As Max Planck wrote, 'A scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents die and a new generation grows up that is familiar with it.'
>
> Archaeology is a perpetual search, never really a finding; it is an eternal journey, with no true arrival. Everything is tentative, nothing is final.

We shall return later (in §1.4, p. 32) to Paul Bahn's skepticism here. His reference to "accepted fiction" calls to my mind Gore Vidal's 1993 Introduction to his 1964 historical novel *Julian* [94]. Vidal quotes one historian describing another as being "The best in the field. Of course he makes most of it up, like the rest of us." Speaking for himself, Vidal says,

> Why write historical fiction instead of history? Because, when dealing with periods so long ago, one is going to make a lot of it up anyway, as Finley blithely admitted. Also, without the historical imagination even conventional history is worthless. Finally, there is the excitement when a pattern starts to emerge.

An emerging pattern in history is the opportunity for the flights of fancy that Vidal takes in novels like *Julian* and *Creation* [95]. Vidal does however admit a difference between fiction and history, as when he says,

> as every dullard knows, the historical novel is neither history nor a novel . . . I personally don't care for historical novels as such (I'm obliged to read history in order to write stories set in the past).

In any case, our present topic is not history as such, but Euclid. The editors of *The Principles of History* lament that, when declaring the compulsive force of historical inference, Collingwood does not give "at this point an example of historians actually reasoning to certain conclusions" [25, p. xxix]. But one should be able to supply one's own examples. Euclid will be *our* example, and in a footnote, the editors supply theirs. They report that, in a 1927 article, Collingwood said, "it is more certain than ever that the forts [on Hadrian's Wall] were built before the stone Wall." This conclusion is now held to be incorrect.

Collingwood's supposed historical mistake does not disprove his ideas of historical inference, any more than mathematical mistakes dispel our belief that mathematical correctness is possible. I suppose every mathematician has had the pleasure and excitement of discovering a theorem, only to find later that it was not a theorem after all. We may then give up mathematics, or we may simply return to it with more care.

Again, if Collingwood does not give examples of historical inference in *The Principles of History,* it is because he expects readers to come up with their own. He has already said this implicitly [23, p. 256] [25, pp. 11–2]:

> Like every science, history is autonomous. The historian has the right, and is under an obligation, to make up his own mind by the methods proper to his own science as to the correct solution of every problem that arises for him in the pursuit of that science. He can never be under any obligation, or have any right, to let someone else make up his mind for him. If anyone else, no matter who, even a very learned historian, or an eyewitness, or a person in the confidence of the man who did the thing he is inquiring into, or even the man who did it himself, hands him on a plate a ready-made answer to his question, all he can do is to reject it: not because he thinks his informant is trying to deceive him, or is himself deceived, but because if he accepts it he is giving up his autonomy as an historian and allowing someone else to do for him what, if he is a scientific thinker, he can only do for himself. There is no need for me to offer the reader any proof of this statement. If he knows anything of historical work, he already knows of his own experience that it is true. If he does not already know that it is true, he does not know enough about history to read this essay with any profit, and the best thing he can do is to stop here and now.

Actually, the best thing the reader could do would be to do some history—which is just what we are going to do with Euclid.

Meanwhile, the mathematician knows what Collingwood is talking about, at least as far as mathematics itself is concerned. In mathematics, the answers come from ourselves. It is pointless to teach calculus unless the student will learn that $\int_1^{\mathrm{e}} \log x \; \mathrm{d}x = 1$, for example, not because the teacher says so, or because that is the answer in the back of the book, but because the student herself—having accepted the conventional meaning of the symbols—can prove to herself that the equation is true.

Simon Blackburn was quoted above as saying, "I do not know whether Aristotle was right"—right that, " we cannot even call men lucky when they die, since we can be harmed after our death."

Nobody can just *tell* us that we can be harmed after death; it has to make sense to us, if we are going to accept it.

As Herodotus tells the story [49, I.30–2, 43, 86], when King Croesus of Lydia is at the height of his power, he invites the visiting Solon of Athens to tell him who in the world is the most blessed (or fortunate, or happy: ὄλβιος α ον). It is not he. The next most blessed? Again, not he. According to Solon, one cannot be judged blessed till after death. Croesus will see the wisdom of these words, first when his favorite son is killed in a hunting accident, and then when he finds himself being burnt at the stake by Cyrus the Great of Persia.

Meawhile, what can Solon even *mean* by his words? Aristotle takes up the question in chapters 10 and 11 of Book I of the *Ethics* [5, pp. 16–9]. For him, it is a question of what we call happiness (ἡ εὐδαιμονία [4, $1100^{\mathrm{a}}11$]). The ultimate conclusion seems to be that what happens after we die does matter to us, though only in the sense of making us more or less happy; it will not make a happy person unhappy, or an unhappy person happy.

The question is not one that will be answered for us when we die. It is a nice dream that all answers will be given to us then; but I read Aristotle as follows. Evidently we do care now what will happen to those we leave behind after we die. It makes sense to judge the happiness of others only if they may serve as models for ourselves. If our happiness depends on what we think will happen after death, we can only assess the happiness of others in the same terms.

1.2 Questions

In his own experiments in the historical or archeological "laboratory of knowledge," Collingwood was

> at first asking myself a quite vague question, such as: 'was there a Flavian occupation on this site?' then dividing that question into various heads and putting the first in some such form as this: 'are

> these Flavian sherds and coins mere strays, or were they deposited in the period to which they belong?' and then considering all the possible ways in which light could be thrown on this new question, and putting them into practice one by one, until at last I could say, 'There was a Flavian occupation; an earth and timber fort of such and such plan was built here in the year $a \pm b$ and abandoned for such and such reasons in the year $x \pm y$.' Experience soon taught me that under these laboratory conditions one found out nothing at all except in answer to a question; and not a vague question either, but a definite one. [22, p. 24]

Concerning Euclid's number-theory, my own vague initial question is, "Can Books VII–IX of the *Elements* serve as a text for undergraduates today, as Book I can?" Book I is indeed a text for students in my own mathematics department, as will be discussed in §2.2 (p. 46). In the number-theory books, Euclid proves several famous results that students should also learn:

- the efficacy of the "Euclidean Algorithm" for finding greatest common divisors (VII.1, 2);
- "Euclid's Lemma," that a prime divisor of the product of two numbers divides one of the numbers (VII.30);[3]
- that the set of prime numbers is unbounded (IX.20);
- that if the sum $\sum_{k=0}^{n} 2^k$ of consecutive powers of two (starting from one) is a prime p, then its product $p \cdot 2^n$ with the last term in the sum is a perfect number (IX.36).

Towards formulating a more precise question than whether Euclid is suitable as a textbook, I note a certain *unsuitability* in the textbooks of today. There is a common foundational error of assuming that recursive definitions of number-theoretic functions are justified by induction alone. Thus if we say

$$1! = 1, \qquad\qquad (k+1)! = k! \cdot (k+1),$$

[3] The result is called Euclid's Lemma by Burton [16, p. 24] and by *Wikipedia*; Hardy & Wright [45, p. 3, Thm 3] call it Euclid's First Theorem (the Second being, "The number of primes is infinite").

it may be thought that (1) we have defined $n!$ when $n = 1$, and (2) if we have defined $n!$ when $n = k$, then we have defined it when $n = k + 1$; therefore, (3) "by induction," we have defined $n!$ for all natural numbers n. Actually cannot always define functions by induction *alone.* Our definition of addition is *recursive.* It requires that $k + 1$ uniquely determine k, and that 1 be not of the form $k + 1$. This is more than a proof by induction requires. Clarification of this point leads to insight, as I have argued elsewhere [79]. It so happens that induction *is* enough to justify the recursive definitions of addition and multiplication, as Landau shows tacitly in *The Foundations of Analysis* [58]. Therefore congruence of integers with respect to a particular modulus respects addition and multiplication, since induction is valid on integers considered with respect to a modulus.[4] One can then use induction to prove associativity and commutativity of addition and multiplication, as well as distributivity of the latter over the former. The proofs of these results can be exercises for students today.

Euclid himself establishes commutativity of multiplication in Proposition VII.16 of the *Elements.* A more definite question for the present investigation is then, "Does Euclid give a valid proof of the commutativity of multiplication?"

Another question is whether Euclid gives a valid proof of "Euclid's Lemma." The question is investigated by Pengelley and Richman [77] and then by Mazur [63]; the former also review earlier, albeit still modern, investigations. Such investigations can be inhibited by the modern notion of a fraction. In the *Elements,* there are two definitions of proportion, to be quoted in full in §3.2 (p. 71): there is a clear definition of proportion of *magnitudes* at the head of Book V, and an unclear definition of proportion of *numbers* at the head of Book VII. By the latter definition, is A to B as C to

[4]The phrase "with respect to the modulus m" is, in Gauss's Latin, *secundum modulum m* [39, §2]. For reasons unknown to me, the English translation [40] sticks to Latin for this phrase, but drops the preposition *secundum* and puts *modulus* in the dative or ablative case, as in "modulo m."

D, just in case the fractions A/B and C/D are equal? To answer this, we should ask what question *Euclid* was trying to answer in his writing.

I shall propose that Euclid's question is, "Can we simplify the *anthyphaeretic* definition of proportion, that is, the definition based on the method of finding greatest common divisors [the Euclidean Algorithm]?" I think this proposal is at least hinted at by David Fowler in *The Mathematics of Plato's Academy* [37]. This book is all about anthyphaeresis, although it does not say a great deal about Euclid's arithmetic as such.

There is other writing on Euclid and his theory of numbers. I have been able to consult only some of it. As an apology for just going ahead anyway, and saying what I have to say, I cite Robert Pirsig, who coins a useful word in his philosophical novel *Lila* [82, ch. 26, pp. 370–2]. The word is defined by an analogy:

> Philosophology is to philosophy as musicology is to music, or as art history and art appreciation are to art, or as literary criticism is to creative writing.

One might add two more terms to the analogy, namely history (or philosophy) of mathematics and mathematics itself. And yet, as has already been suggested, you cannot properly study the history of mathematics without already being something of a mathematician. The point for now is that, according to Pirsig, philosophologists put

> a philosophological cart before the philosophical horse. Philosophologists not only start by putting the cart first; they usually forget the horse entirely. They say first you should read what all the great philosophers of history have said and *then* you should decide what *you* want to say. The catch here is that by the time you've read what all the great philosophers of history have said you'll be at least two hundred years old.

Some questions about Euclid have arisen in the course of my own mathematical life, and here I take them up. In the next chapter, I shall consider the question of how Richard Dedekind and David Hilbert (and my own students) read Euclid. One could raise the same question about every earlier mathematician, since just about all of them must have read Euclid. But one has only so much time.

One should also have something one is looking for: this was said by Collingwood above, and it is said by Pirsig, or more precisely by his persona, called Phaedrus:

> Phaedrus, in contrast, sometimes forgot the cart but was fascinated by the horse. He thought the best way to examine the contents of various philosophological carts is first to figure out what *you* believe and then to see what great philosophers agree with you. There will always be a few somewhere. These will be much more interesting to read since you can cheer what they say and boo their enemies, and when you see how their enemies attack them you can kibitz a little and take a real interest in whether they were right or wrong.

It is unfortunate that Pirsig thinks of intellectual life as an arena for attack and defense; but apparently he has reasons for this.

1.3 Evidence

We can read Euclid as if he were a contemporary mathematician. Then we may think that we understand him if we have translated his mathematics into our own terms. This is the approach that E. C. Zeeman takes and defends in an article (originally a talk) on Euclid [99, p. 16]:

> In our discussions we found ourselves following the traditional opposing roles of historian and mathematician. The historian thinks extrinsically in terms of the written evidence and adheres strictly to that data, whereas the mathematician thinks intrinsically in terms of the mathematics itself, which he freely rewrites in his own notation in order to better understand it and to speculate on what might have been passing through the mind of the ancient mathematician, without bothering to check the rest of the data.

The person given the role of historian here is the late David Fowler, mentioned above; but he was a mathematician as well. Zeeman seems to suggest that the mathematician has an advantage over the historian, because an understanding of mathematics can take a researcher places where a lack of evidence prevents the historian from going.

Though we have seen (or at least seen it argued by Collingwood) that mathematics is similar to history, the two disciplines are still different. Collingwood suggests this himself in *The Principles of History* [25, p. 5]. Mathematics and history are sciences; but

> anything that is a science at all must be more than merely a science, it must be a science of some special kind. A body of knowledge is never merely organized, it is always organized in some particular way.

There are sciences of observation, like meteorology; there are sciences of experiment, like chemistry. There is mathematics, not named as such by Collingwood here, but described as being

> organized not by observing events at all, but by making certain assumptions and proceeding with the utmost exactitude to argue out their consequences.

History is not like this. In meteorology and chemistry, the aim is "to detect the constant or recurring features in all events of a certain kind." In being different from this, history does *resemble* mathematics, but only to a certain point:

> It is true that in history, as in exact science, the normal process of thought is inferential; that is to say, it begins by asserting this or that, and goes on to ask what it proves. But the starting-points are of very different kinds. In exact science they are assumptions, and the traditional way of expressing them is in sentences beginning with a word of command prescribing that a certain assumption be made: 'Let ABC be a triangle, and let AB = AC.' In history they are not assumptions, they are facts, and facts coming under the historian's observation, such as, that on the page open before him there is printed what purports to be a charter by which a certain king grants certain lands to a certain monastery. The conclusions, too, are of different kinds. In exact science, they are conclusions about things which have no special habitation in space or time: if they are anywhere, they are everywhere, and if they are at any time they are at all times. In history, they are conclusions about events, each having a place and date of its own.

Collingwood appears to have the view of mathematics described by Timothy Gowers in *Mathematics: A Very Short Introduction* [41, pp. 39–40]. Gowers recognises it as a *twentieth-century* view. It is a view that has its place in history:

> [T]he steps of a mathematical argument can be broken down into smaller and therefore more clearly valid substeps. These steps can then be broken down into subsubsteps, and so on. A fact of fundamental importance to mathematics is that this process *eventually comes to an end* . . .
>
> What I have just said in the last paragraph is far from obvious: in fact it was one of the great discoveries of the early 20th century, largely due to Frege, Russell, and Whitehead. This discovery has had a profound impact on mathematics, because it means that *any dispute about the validity of a mathematical proof can always be resolved.* In the 19th century, by contrast, there were genuine disagreements about matters of mathematical substance. For example, Georg Cantor, the father of modern set theory, invented arguments that relied on the idea that one infinite set can be 'bigger' than another. These arguments are accepted now, but caused great suspicion at the time.

Cantor's arguments are accepted now, and one might say this is because they can be worked out in a formal system based on the Zermelo–Fraenkel theory of sets. Does Gowers mean to suggest that any dispute about the validity of a proof *in Euclid* can now be resolved? I do not know; but I myself would say that today's mathematics does not *automatically* give us a good criterion for assessing Euclid. Though Euclid may have inspired the formal systems alluded to by Gowers, Euclid himself is not using a formal system, or at least he is not *obviously* using one, even though it may well be possible to retrofit the *Elements* with a formal system, as is done by Avigad, Dean, and Mumma [6].

Mathematics can be held up as an example of the peaceable resolution of disputes. Gowers finds it to be unique in this way [41, p. 40]:

> There is no mathematical equivalent of astronomers who still believe in the steady-state theory of the universe, or of biologists who hold, with great conviction, very different views about how much is explained by natural selection, or of philosophers who disagree fundamentally about the relationship between consciousness and the physical world, or of economists who follow opposing schools

> of thought such as monetarism and neo-Keynesianism.

And yet, if all mathematical disputes are resoluble in principle, this is only because we accept the principle. Collingwood finds the same principle at work in history. "History," he says [25, p. 7],

> has this in common with every other science: that the historian is not allowed to claim any single piece of knowledge, except where he can justify his claim by exhibiting to himself in the first place, and secondly to anyone else who is both able and willing to follow his demonstration, the grounds upon which it is based. This is what was meant, above, by describing history as inferential.

Mathematics is inferential in the same way. Now, can the grounds of an inference be accepted by one person, yet rejected by another? Practically speaking, it is less likely in mathematics than elsewhere. Nonetheless, it can happen anywhere, because one can always be faced with a *skeptic,* whom Collingwood goes on to distinguish from a proper *critic*:

> a critic is a person able and willing to go over somebody else's thoughts for himself to see if they have been well done; whereas a sceptic is a person who will not do this; and because you cannot make a man think, any more than you can make a horse drink, there is no way of proving to a sceptic that a certain piece of thinking is sound, and no reason for taking his denials to heart. It is only by his peers that any claimant to knowledge is judged.

Mathematical disputes are resoluble in principle. In practice, they may not be resoluble; but in this case, we may say of the parties to the dispute that one is a skeptic in Collingwood's sense. This is possible in *any* dispute, be it in mathematics or history or anywhere else.

According to Zeeman as quoted above, "The historian thinks extrinsically in terms of the written evidence and adheres strictly to that data," although this is admittedly a "traditional" view. It sounds like the obsolescent view of history that distinguishes history from archeology and that is described by Collingwood as "scissors and paste" [25, p. 30]:

> It is characteristic of scissors-and-paste history, from its least critical to its most critical form, that it has to do with ready-made statements, and that the historian's problem about any one of

> these statements is whether he shall accept it or not: where accepting it means reasserting it as a part of his own historical knowledge.

We can take an example from Euclid. Among the "definitions" at the head of Book VII of the *Elements,* there are the following two statements, numbered 3 and 4 in the Greek text established by Heiberg [33], and hence so numbered in translations like Heath's [34]:[5]

> **Μέρος** ἐστὶν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος,
> ὅταν καταμετρῇ τὸν μείζονα.
> **Μέρη** δέ, ὅταν μὴ καταμετρῇ.
>
> A number is **part** of a number, the less[6] of the greater,
> when it measures the greater.
> But **parts,** when it does not measure.

In the body of Book VII, Proposition 4 has the following enunciation.

> Ἅπας ἀριθμὸς παντὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος
> ἤτοι μέρος ἐστὶν ἢ μέρη.
>
> Every number is of every number, the less of the greater,
> either a part or parts.

If the first two statements are really definitions, then the last statement follows immediately from them; but Euclid gives a nontrivial proof anyway. Suppose we think of geometry as a body of knowledge consisting of, definitions, axioms, and the theorems that they entail; and suppose we are, as it were, historians of this geometry. If we are scissors-and-paste historians, then we are going to reject at least one of the three ready-made statements of Euclid, because they cannot all be respectively definitions and theorem.

In fact we are going to reject none of the statements out of hand; nor shall we just accept them as definitions and theorem respec-

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[5]Heiberg's bracketing of the tenth Greek definition leads to different numbering conventions for the later definitions: see note 8, page 71.

[6]Etymologically, "less" is a comparative form, although we seem not to have retained a positive form of its root, but we take "little" for the positive form [88]. The second S in "less" can be considered to stand for the R of the usual comparative suffix "-er." The word "lesser" is thus a double comparative, as "greaterer" would be, if there were such a word.

tively. We shall do something like what Collingwood goes on to describe:

> Confronted with a ready-made statement about the subject he is studying, the scientific historian never asks himself: 'Is this statement true or false?', in other words 'Shall I incorporate it in my history of that subject or not?' The question he asks himself is: 'What does this statement mean?' And this is not equivalent to the question 'What did the person who made it mean by it?', although that is doubtless a question that the historian must ask, and must be able to answer. It is equivalent, rather, to the question 'What light is thrown on the subject in which I am interested by the fact that this person made this statement, meaning by it what he did mean?'

Our ultimate interest is in our *own* statements, as mathematicians and historians of mathematics. Collingwood describes these statements on his next page:

> A statement to which an historian listens, or one which he reads, is to him a ready-made statement. But the statement that such a statement is being made is not a ready-made statement. If he says to himself 'I am now reading or hearing a statement to such and such effect', he is himself making a statement; but it is not a second-hand statement, it is autonomous. He makes it on his own authority. And it is this autonomous statement that is the scientific historian's starting-point . . .
>
> If the scientific historian gets his conclusions not from the statement that he finds ready made, but from his own autonomous statement of the fact that such statements are made, he can get conclusions even when no statements are made to him.

It is a nice distinction, the one between the statement that we hear or read and the statement *that* we are hearing or reading it. But the distinction is all-important for showing us our own role in the process of understanding.

Thus the mathematician's activity described by Zeeman, the rewriting of Euclid's mathematics in one's own notation, the better to understand it—this would seem to be an activity of the scientific historian, in Collingwood's terms.[7]"In scientific history," says Collingwood [25, p. 36],

> anything is evidence which is used as evidence, and no one can know what is going to be useful as evidence until he has had occasion to use it.

Thus the mathematics that *we* work out can serve as evidence of what Euclid was doing.

And yet this evidence must be used with care, since its use will be based on the presupposition of a kind of unity of mathematics—the presupposition about mathematical truths that, as Collingwood said above, "if they are anywhere, they are everywhere, and if they are at any time they are at all times." In the study of Euclid at least, it is desirable to question this presupposition. In his article [70] on the International Congresses of Mathematics, David Mumford's theme is that the unity of mathematics (as reflected in the very existence of the Congresses) cannot be taken for granted, but must be worked for, and will not be achieved by forcing all of mathematics into the same mold. This is especially true when the mathematics under consideration is spread out over more than two thousand years.

1.4 Re-enactment

In a quotation above from *The Principles of History,* Collingwood distinguishes between what a statement means and what the person who made the statement means. We want to understand the mathematics thought about by Euclid, and we want to understand what Euclid thought about the mathematics; but these are different. Collingwood does not seem so concerned with the distinction in *An Autobiography*; but he does establish there that all history

[7]Since we shall look briefly at Euclid's use of the definite article in §2.2 (p. 46), let us note that the word "the" in "the better" here is not the usual definite article descended from the Old English *þe*. It is rather a descendent of this pronoun's instrumental case, spelled as *þy* and *þon* [51].

is the history of thought. He distinguishes history from pseudo-history, the latter meaning

> the narratives of geology, palaeontology, astronomy, and other natural sciences which in the late eighteenth and the nineteenth centuries had assumed a semblance at least of historicity . . .
>
> History and pseudo-history alike consisted of narratives: but in history these were narratives of purposive activity, and the evidence for them consisted of relics they had left behind (books or potsherds, the principle was the same) which became evidence precisely to the extent to which the historian conceived them in terms of purpose, that is, understood what they were for. [22, pp. 107–9]

Thus we shall ask of Euclid's "definitions" of *part* and *parts* of a number, and of his propositions about the same: what are they for?

Collingwood continues [22, p. 110]:

> I expressed this new conception of history in the phrase: 'all history is the history of thought.' You are thinking historically, I meant, when you say about anything, 'I see what the person who made this (wrote this, used this, designed this, &c.) was thinking.' Until you can say that, you may be trying to think historically but you are not succeeding. And there is nothing else except thought that can be the object of historical knowledge. Political history is the history of political thought: not 'political theory', but the thought which occupies the mind of a man engaged in political work: the formation of a policy, the planning of means to execute it, the attempt to carry it into effect, the discovery that others are hostile to it, the devising of ways to overcome their hostility, and so forth.

As historians of mathematics, and in particular of Euclid's mathematics, we study what Euclid was thinking while composing the *Elements.*

We should acknowledge at some point that the author of the *Elements,* whom we call Euclid, was not necessarily one person. If different parts of the *Elements* do not seem to fit together, it may be because different hands wrote them. However, we can decide the question of the value of the *Elements* for us, without knowing whether Euclid was one person or many. This is a point made by Collingwood in *The Principles of Art* of 1938 [20, pp. 318–9]:

> Individualism would have it that the work of a genuine artist is altogether 'original', that is to say, purely his own work and not in any way that of other artists . . . All artists have modelled their style upon that of others, used subjects that others have used, and treated them as others have treated them already . . .
>
> The individualistic theory of authorship would lead to the most absurd conclusions. If we regard the *Iliad* as a fine poem, the question whether it was written by one man or by many is automatically, for us, settled. If we regard Chartres cathedral as a work of art, we must contradict the architects who tell us that one spire was built in the twelfth century and the other in the sixteenth, and convince ourselves that it was all built at once.

If the *Elements* is a fine work of mathematics, its authorship is not necessarily singular; if not, not necessarily multiple.

For Collingwood in *An Autobiography,* our study of Euclid is not only history, but typical or exemplary history. We want to understand the thoughts expressed in the *Elements,* and to do this, since they are mathematical thoughts, we have to be mathematicians:

> the historian must be able to think over again for himself the thought whose expression he is trying to interpret. If for any reason he is such a kind of man that he cannot do this, he had better leave that problem alone. The important point here is that the historian of a certain thought must think for himself that very same thought, not another like it. If some one, hereinafter called the mathematician, has written that twice two is four, and if some one else, hereinafter called the historian, wants to know what he was thinking when he made those marks on paper, the historian will never be able to answer this question unless he is mathematician enough to think exactly what the mathematician thought, and expressed by writing that twice two are four. When he interprets the marks on paper, and says, 'by these marks the mathematician meant that twice two are four', he is thinking simultaneously: (*a*) that twice two are four, (*b*) that the mathematician thought this, too; and (*c*) that he expressed this thought by making these marks on paper. I will not offer to help a reader who replies, 'ah, you are

> making it easy for yourself by taking an example where history really is the history of thought; you couldn't explain the history of a battle or a political campaign in that way.' I could, and so could you, Reader, if you tried.
>
> This gave me a second proposition: 'historical knowledge is the re-enactment in the historian's mind of the thought whose history he is studying.' [22, pp. 111–2]

Re-enactment is not discussed in *The Principles of History,* as we have the text; but re-enactment is discussed in some preliminary notes that Collingwood made for the book [25, pp. 239–40]:

> all genuine historians interest themselves in the past just so far as they find in it what they, as practical men, regard as living issues. Not merely issues *resembling* these: but *the same* issues . . . And this must be so, if history is the re-enactment of the past in the present: for a past so re-enacted is not a past that has finished happening, it is happening over again.
>
> People have been quite right to say that the historian's business is not to narrate the past in its entirety . . . but to narrate such of the past as has historical importance . . . historical importance means importance *for us.* And to call a thing important for us means that *we* are interested in it, i.e. that it is a past which we desire to re-enact in our present.

We are studying Euclid for the sake of doing mathematics today. Re-enacting Euclid means doing his mathematics. There are things called re-enactment that are not what Collingwood has in mind. It does not mean dressing in period costumes and aping archaic manners. Thus, the passage from *An Autobiography* mentioning political history continues [22, p. 110]:

> Consider how the historian describes a famous speech. He does not concern himself with any sensuous elements in it such as the pitch of the statesman's voice, the hardness of the benches, the deafness of the old gentleman in the third row: he concentrates his attention on what the man was trying to say (the thought, that is, expressed in his words) and how his audience received it (the thoughts in their minds, and how these conditioned the impact upon them of the statesman's thought).

Treating the *Elements* historically does not mean just reading it, or reproducing its propositions in lectures delivered at a blackboard.

It does involve *understanding* those propositions somehow. Historical study of Euclid does not even require learning Ancient Greek, though this does seem to be of value, and I shall usually quote Euclid in Greek as well as English. Euclid's Greek is easy, being formulaic and having a small lexicon, many of whose words can be seen in the mathematical language of today.

Proper re-enactment of the past is nonetheless a grand project. Many years before writing the books quoted so far, Collingwood engaged in another grand project, in *Speculum Mentis* of 1924 [19, p. 9]:

> This book is the outcome of a long-growing conviction that the only philosophy that can be of real use to anybody . . . is a critical review of the chief forms of human experience . . . We find people practicing art, religion, science, and so forth, seldom quite happy in the life they have chosen, but generally anxious to persuade others to follow their example. Why are they doing it, and what do they get for their pains? This question seems, to me, crucial for the whole of modern life . . .

After using the bulk of the book to review the "forms of experience" called art, religion, science, history, and philosophy, Collingwood sums things up [19, pp. 306–9]:

> We set out to construct a map of knowledge on which every legitimate form of human experience should be laid down, its boundaries determined, and its relations with its neighbors set forth . . .
>
> Such a map of knowledge is impossible . . .
>
> Beginning, then, with our assumption of the separateness and autonomy of the various forms of experience, we have found that this separateness is an illusion . . .
>
> The various countries on our initial map, then, turn out to be variously-distorted versions of one and the same country . . . What, then, is this one country? It is the world of historical fact, seen as the mind's knowledge of itself.

I would read the emphasis on historical fact as follows. To know a theorem, such as Fermat's "Little" Theorem, means knowing that it is true. This means knowing, as historical fact, that one has worked through a proof of the theorem and verified its correctness. One may have established a proof by induction, having noted that

1) $1^p \equiv 1 \pmod p$, and

2) if $a^p \equiv a \pmod p$, then $(a+1)^p \equiv a^p + 1 \equiv a + 1 \pmod p$.

One can then file the proof away. Asserting the theorem as true does not require reopening the file; but it requires summoning up the historical fact of the proof's having been placed in the file.

One may find this "historical fact" inadequate; for, in the words of Paul Bahn quoted earlier (§1.1, p. 14), "nobody *knows* what happened in the past (even in the recent historical past)." If these words are to be taken seriously, then the memory of having proved a theorem does not suffice to provide knowledge that the theorem is true. Maybe we made a mistake when we proved the theorem yesterday. Maybe as students we accepted a teacher's proof, but had not yet acquired sufficient mathematical skepticism to find the gaps in the proof.

We may then try to ensure that the stating of a theorem includes within itself an actual proof. I think this is Mazur's "self-proving theorem principle" [63, p. 229], namely:

> if you can restate a theorem, without complicating it, so that its proof, or the essence of its proof, is already contained in the statement of the theorem, then you invariably have
>
> - a more comprehensible theorem,
> - a stronger theorem, and
> - a shorter and more comprehensible proof!

For Mazur, the self-proving formulation of the Euclidean Algorithm is,

> Suppose you are given a pair of numbers A and B with A greater than B. Any common divisor of A and B is a common divisor of B and $A - B$; and conversely, any common divisor of B and $A - B$ is a common divisor of A and B.

This formulation gives us a key part of the theorem; but it omits that the process of continually replacing the greater of two numbers with the difference of the two numbers must eventually come to an end. That the natural numbers are well ordered must still be summoned up somehow.

Collingwood seems to have anticipated our difficulty. As we were reading him in *Speculum Mentis,* he was speaking of a country, namely "the world of historical fact":

> Can we, then, sketch this country's features in outline?
>
> We cannot. To explore that country is the endless task of the mind; and it only exists in its being explored. Of such a country there is no map, for it is itself its own map . . .
>
> There is and can be no map of knowledge, for a map means an abstract of the main features of a country, laid before the traveller in advance of his experience of the country itself. Now no one can describe life to a person who stands on the threshold of life. The maxims given by age to youth are valueless not because age means nothing by them but because what it means is just its own past life. To youth they are empty words. The life of the spirit cannot be described except by repeating it: an account of it would just be itself.

The bare statement of a theorem, without a reiteration of its proof, is thus a valueless maxim given by age to youth. This would appear to be so, even if "age" here is our former, younger self. As Wordsworth wrote in "My Heart Leaps Up" [98, p. 85],

> The child is the father of the man;
> And I could wish my days to be
> Bound each to each by natural piety.

The child, or younger person, passes along a theorem to the man, or older person; but how can the older person properly receive this legacy? How can one day be remembered in the next, without being simply relived? To give a proper account of Euclid would seem to require us to repeat him; not to parrot him, but, as it were, to *be* him.

The same goes for Collingwood too, by the way. If I parrot him here, in the sense of just quoting him, it is because I think his words need little translation, but usually make good-enough sense as they are. Further sense will come, if it comes at all, from applying them to one's own experience, such as the experience of reading Euclid.

What Serge Lang says of mathematics, in the Foreward of his *Algebra* [59, p. v], is true for any of Collingwood's "forms of human experience":

> Unfortunately, a book must be projected in a totally ordered way on the page axis, but that's not the way mathematics "is," so readers have to make choices how to reset certain topics in parallel for themselves, rather than in succession.

Thoughts projected on the axis of time become totally ordered; but sometimes a different ordering is needed for understanding them. My present anthology of quotations of Collingwood (and others) is intended to offer such an ordering. In any case, our main purpose is to understand *Euclid*; Collingwood is here to serve that purpose. On the other hand, if that purpose *is* served, this in turn will illuminate Collingwood.

There is a certain pessimism or futility in *Speculum Mentis*: the old—Euclid—can give nothing to the young—us—, but we must go over everything thoroughly for ourselves. And yet progress is possible [19, p. 312]:

> In the toil of art, the agony of religion, and the relentless labour of science, actual truth is being won and the mind is coming to its own true stature.

Progress *of a sort* is possible [19, p. 316]:

> This process of the creation and destruction of external worlds might appear, to superficial criticism, a mere futile weaving and unweaving of Penelope's web, a declaration of the mind's inability to produce solid assets, and thus the bankruptcy of philosophy. And this it would be if knowledge were the same thing as information, something stored in encyclopaedias and laid on like so much gas and water in schools and universities. But education does not mean stuffing a mind with information; it means helping a mind to create itself, to grow into an active and vigorous contributor to the life of the world. The information given in such a process is meant to be absorbed into the life of the mind itself, and a boy leaving school with a memory full of facts is thereby no more educated than one who leaves table with his hands full of food is thereby fed. At the completion of its education, if that event ever happened, a mind would step forth as naked as a new-born babe,

> knowing nothing, but having acquired the mastery over its own weaknesses, its own desires, its own ignorance, and able therefore to face any danger unarmed.

How does the mind "create itself," with the help of education? Collingwood continued to work on this question. The pessimism of *Speculum Mentis* did not lead to despair. Collingwood continued to think, as the mathematician continues to think after finding an error in a supposed proof.

The answer given in *An Autobiography* is summarized in three "propositions," two of which we have seen: history is history of thought, and historical knowledge is re-enactment of thought. Then there is [22, p. 114]

> my third proposition: 'Historical knowledge is the re-enactment of a past thought incapsulated in a context of present thoughts which, by contradicting it, confine it to a plane different from theirs.'
>
> How is one to know which of these planes is 'real' life, and which mere 'history'? By watching the way in which historical problems arise. Every historical problem ultimately arises out of 'real' life. The scissors-and-paste men think differently: they think that first of all people get into the habit of reading books, and then the books put questions into their heads. But am not talking about scissors-and-paste history. In the kind of history that I am thinking of, the kind I have been practising all my life, historical problems arise out of practical problems. We study history in order to see more clearly into the situation in which we are called upon to act. Hence the plane on which, ultimately, all problems arise is the plane of 'real' life: that to which they are referred for their solution is history.

To the question how we know which plane of life is real, and which history, I am not sure whether Collingwood's answer is any clearer than saying, "We just *do* know, at least if we are paying attention." Collingwood works through the example of Admiral Nelson, on the deck of the *Victory,* wondering whether to remove his decorations so as to become a less conspicuous target to snipers on enemy ships. To understand Nelson's answer, as apparently Collingwood tried as a boy, we have to understand what Nelson thinks about the question; and yet we still know that the question does not actually arise in our own lives. Nonetheless, young Collingwood may have had a personal interest in the question, knowing that Nelson had a friend and colleague bearing the name of Collingwood, who took command at the Battle of Trafalgar after Nelson's death [54].

We are interested in Euclid now, and difficulties in reading him arise precisely when the questions that underlie what he tells us are not our own questions. We must *make* them our own questions, and this will be re-enacting them. But the very difficulty of doing this will tell us that we are doing history. This is how I understand Collingwood's "third proposition."

1.5 Science

Before continuing with Euclid, I want to note again that, for Collingwood in *The Principles of History,* history is a science. This classification agrees with the general account of science given in *An Essay on Metaphysics* of 1940 [24, p. 4]:

> The word 'science', in its historical sense, which is still its proper sense not in the English language alone but in the international language of European civilization, means a body of systematic or orderly thinking about a determinate subject-matter. This is the sense and the only sense in which I shall use it.

And yet earlier, in *An Essay on Philosophical Method* of 1933 [26, p. 26], Collingwood distinguished history from science:

> Historical thought concerns itself with something individual, scientific thought with something universal; and in this respect philosophy is more like science than history, for it likewise is concerned with something universal: truth as such, not this or that truth; art as such, not this or that work of art. In the same way exact science considers the circle as such, not this or that individual instance of it; and empirical science considers man as such, not, like history, this man as distinct from that.

In *An Autobiography* [22, p. 117], Collingwood described *An Essay on Philosophical Method* as

> my best book in matter; in style, I may call it my only book, for it is the only one I ever had the time to finish as well as I knew how, instead of leaving it in a more or less rough state.

Did Collingwood nonetheless change his mind about science later? The passage from *An Essay on Metaphysics* continues:

> There is also a slang sense of the word ['science'], unobjectionable (like all slang) on its lawful occasions, parallel to the slang use of the word 'hall' for a music-hall or the word 'drink' for an alcoholic drink, in which it stands for natural science.

In the earlier *Essay on Philosophical Method,* was Collingwood using the word "science" in its slang sense? Does the term "natural science" in the later *Essay* encompass both empirical and exact science? These questions are not of great importance for us, but provide an opportunity to note the grand theme of the earlier *Essay* [26, p. 31], which is the "overlap of classes":

> The specific classes of a philosophical genus do not exclude one another, they overlap one another. This overlap is not exceptional, it is normal; and it is not negligible in extent, it may reach formidable dimensions.

Thus in *The Principles of Art* of 1938 [20, p. 21], Collingwood will be at pains to distinguish art from craft, although, for example,

> The distinction between planning and executing certainly exists in some works of art, namely those which are also works of craft or artifacts; for there is, of course, an overlap between these two things, as may be seen by the example of a building or a jar, which is made to order for the satisfaction of a specific demand, to serve a useful purpose, but may none the less be a work of art. But suppose a poet were making up verses as he walked . . .

The poem may be unplanned, and so not be craft, but nonetheless be a work of art; yet other things are both craft and art.

The overlap of classes has a practical result [26, p. 105]. First:

> On a matter of empirical fact it is possible, when asked for example 'where did I leave my purse?' to answer 'not in the taxi, I am sure', without having the least idea where the purse was actually left . . .

Likewise, when asked, "Is this proof correct?" we may answer "No," without having the least idea of a correct proof, if indeed there is a correct proof. Nonetheless,

> In philosophy this is not so. The normal and natural way of

> replying to a philosophical statement from which we dissent is by saying, not simply 'this view seems to me wrong', but 'the truth, I would suggest, is something more like this', and then we should attempt to state a view of our own . . . This is not a mere opinion. It is a corollary of the Socratic principle (itself a necessary consequence of the principle of overlapping classes) that there is in philosophy no such thing as a transition from sheer ignorance to sheer knowledge, but only a progress in which we come to know better what in some sense we know already.

Euclid's *Elements* is not philosophy; and yet perhaps the best reason for reading it is philosophical: to deepen our understanding of mathematics as we already know it. In this case, if we detect what we think are errors in Euclid, we ought to be prepared to correct them, and correct them *in a way that Euclid himself would agree with.*

Mathematics is not philosophy; but there is an overlap. After writing *An Essay on Philosophical Method,* with its doctrine of the overlap of classes, Collingwood applied the method to *nature*; the resulting lectures were posthumously published in 1945 as *The Idea of Nature.* What Collingwood says at the beginning about natural science applies as well to mathematics [21, p. 2]:

> The detailed study of natural fact is commonly called natural science, or for short simply science; the reflection on principles, whether those of natural science or of any other department of thought or action, is commonly called philosophy. Talking in these terms, and restricting philosophy for the moment to reflection on the principles of natural science, what I have just said may be put by saying that natural science must come first in order that philosophy may have something to reflect on; but that the two things are so closely related that natural science cannot go on for long without philosophy beginning; and that philosophy reacts on the science out of which it has grown by giving it in future a new firmness and consistency arising out of the scientist's new consciousness of the principles on which he has been working.
>
> For this reason it cannot be well that natural science should be assigned exclusively to one class of persons called scientists and philosophy to another class called philosophers.

Collingwood saw the need for a bridge between science and philosophy, and thought himself not alone in this [21, p. 3]:

> In the nineteenth century a fashion grew up of separating natural scientists and philosophers into two professional bodies, each knowing little about the other's work and having little sympathy with it. It is a bad fashion that has done harm to both sides, and on both sides there is an earnest desire to see the last of it and to bridge the gulf of misunderstanding it has created. The bridge must be begun from both ends; and I, as a member of the philosophical profession, can best begin at my end by philosophizing about what experience I have of natural science. Not being a professional scientist, I know that I am likely to make a fool of myself; but the work of bridge-building must go on.

Why build bridges? Collingwood had seen the great destructive folly of the First World War, albeit from a desk in London at the Admiralty Intelligence Division [22, p. 29]. The War had been a disaster in every way, except [22, pp. 90–1]:

> The War was an unprecedented triumph for natural science. Bacon had promised that knowledge would be power, and power it was: power to destroy the bodies and souls of men more rapidly than had ever been done by human agency before. This triumph paved the way to other triumphs: improvements in transport, in sanitation, in surgery, medicine, and psychiatry, in commerce and industry, and, above all, in preparations for the next war.
>
> But in one way the War was an unprecedented disgrace to the human intellect . . . nobody has ever supposed that any except at most the tiniest fraction of the combatants wanted it . . . I seemed to see the reign of natural science, within no very long time, converting Europe into a wilderness of Yahoos.

Collingwood saw salvation in history, pursued scientifically in the sense described earlier, and not by scissors and paste.

It was suggested earlier that a kind of salvation from conflict lay in mathematics, where all disputes could be resolved amicably. And yet, strictly speaking, what can be so resolved are disputes about mathematics that can be expressed in formal systems. Mathematics, and number theory in particular for its relevance to encryption, can no longer be distinguished from other sciences as Hardy distinguished it in 1940 by saying [44, §21, p. 121],

> science works for evil as well as for good (and particularly, of course, in time of war); and both Gauss and less mathematicians may be justified in rejoicing that there is one science at any rate, and that their own, whose very remoteness from ordinary human activities should keep it gentle and clean.

The *Elements* is an old book, and certain old books have been, and continue to be, the nominal causes of bloody disputes. Euclid is different. There have been no wars in his name. There can still be academic disputes about him. If history has the potential that Collingwood saw in it, Euclid may be as good a place as any for a mathematician, at least, to try out the possibilities.

2 Euclid in History

2.1 Dedekind and Hilbert

Today we may be better able to learn Euclid's mathematics, precisely because the *Elements* is *not* commonly used as a textbook. We can approach Euclid more easily now, without assuming that he is doing just what we are doing when we do mathematics.

In the preface of his 1908 translation of the *Elements* [34, v. I, p. vii], Thomas Heath said,

> no mathematician worthy of the name can afford not to know Euclid, the real Euclid as distinct from any revised or rewritten versions which will serve for schoolboys or engineers.

I do not know why engineers and schoolchildren, boys *and* girls, do not also deserve to know the real Euclid; but in any case, I suppose mathematicians of Heath's day *did* know Euclid. Whether they knew him more directly than through the "revised or rewritten versions" that Heath refers to, I do not know. However, a few years before Heath, David Hilbert introduced his *Foundations of Geometry* [50, p. 1] by saying,

> Geometry, like arithmetic, requires for its logical development only a small number of simple, fundamental principles. These fundamental principles are called the axioms of geometry. The choice of the axioms and the investigation of their relations to one another is a problem which, since the time of Euclid, has been discussed in numerous excellent memoirs to be found in the mathematical literature. This problem is tantamount to the logical analysis of our intuition of space.

Presumably Hilbert had actually read Euclid, at least in some form. Thus he could disagree explicitly with Euclid on whether the congruence of all right angles is an axiom or a theorem [50, p. 13].

On the other hand, this disagreement may represent a somewhat careless reading of Euclid; I shall discuss this point in §2.4 (p. 63).

A few years before Hilbert, Richard Dedekind [28, pp. 39–40] complained about critics who thought that his theory of irrational numbers could be found already in the *Traité d'Arithmétique* of Joseph Bertrand. According to Dedekind, the theory in Bertrand's work had already been present in *Euclid's* work; but his own theory was different. Dedekind traced his own definition of irrational numbers to the idea that

> an irrational number is defined by the specification of all rational numbers that are less and all those that are greater than the number to be defined . . . That an irrational number is to be considered as fully defined by the specification just described, this conviction certainly long before the time of Bertrand was the common property of all mathematicians who concerned themselves with the irrational . . . [I]f, as Bertrand does exclusively in his book (the eighth edition, of the year 1885, lies before me,) one regards the irrational number as the ratio of two measurable quantities, then is this manner of determining it already set forth in the clearest possible way in the celebrated definition which Euclid gives of the equality of two ratios (*Elements*, V., 5).

In the 1849 edition of his *Traité* [11] (the one that I was able to find), Bertrand indeed defines irrational square roots and other irrational numbers as measures of quantities with respect to a predetermined unit. Thus he writes:

> **267.** Lorsqu'un nombre N, n'est le carré d'aucun nombre entier ou fractionnaire, la définition de sa racine carrée exige quelques développements.
>
> On dit qu'un nombre est plus grand ou plus petit que $\sqrt{\mathrm{N}}$ suivant que son carré est plus grand ou plus petit que N; d'après cela, pour définir *les grandeurs dont* $\sqrt{\mathrm{N}}$ *est la mesure*, supposons, par exemple, qu'après avoir adopté une certaine unité de longueur on regarde tous les nombres comme exprimant des longueurs portées sur une mème ligne droite à partir d'une origine donnée. Une portion de cette ligne recevra les extrémités des longueurs dont la mesure est moindre que $\sqrt{\mathrm{N}}$, et une autre portion celles des lignes dont la mesure est plus grande que $\sqrt{\mathrm{N}}$; entre ces deux régions, il ne pourra évidemment exister aucun intervalle, mais, seulement,

> un *point de démarcation.* La distance à laquelle se trouve ce point, est, par définition, mesurée par $\sqrt{\mathrm{N}}$.

That is, if N is not a square, we define $\sqrt{N}$ by saying first $x > \sqrt{N}$ if $x^2 > N$, and $x < \sqrt{N}$ if $x^2 < N$, where implicitly the x are rational. If such x are taken to measure lengths along a straight line from a given origin, a unit length having been chosen, then the extremities of the lengths measuring less than $\sqrt{N}$ will be separated from the extremities of the lengths measuring more than $\sqrt{N}$ by a single point, whose distance from the origin is $\sqrt{N}$.

Dedekind seems right in saying that Euclid had "already set forth in the clearest possible way" the approach that Bertrand would take. Thus, suppose A and B are magnitudes that *have a ratio to one another* in the sense of Definition V.4 of the *Elements.* This means some multiple of either magnitude exceeds the other. Though Euclid has no such notation, we can understand a multiple of A as

$$\underbrace{A + \cdots + A}_{n},$$

where n belongs to the set $\mathbb{N}$ of positive integers; and then we may write the multiple more simply as

$$A \cdot n.$$

I am writing the multiplier n on the right, in accordance with the modern convention for ordinal arithmetic; but one could just as well write nA. In either case, the meaning is "A, n times," or "n times A." Here n is not a thing in itself. If A and B are magnitudes which are *numbers of units,* say m and n units respectively, then we can define $A \times B$ as $A \cdot n$ and $B \times A$ as $B \cdot m$, and these are not obviously equal. See p. 79 and especially p. 98.

Implicitly, if the arbitrary magnitudes A and B have a ratio to one another, but are unequal, then the greater exceeds the less by some magnitude, and this magnitude has a ratio to A and B. Thus, in our terms, A and B are positive elements of an archimedean ordered group.[1] We derive from these magnitudes the set

$$\{m/n\colon B\cdot m < A\cdot n\} \tag{2.1}$$

consisting of positive rational numbers; here m and n range over $\mathbb{N}$. The set (2.1) determines the ratio of A to B, in the sense that, if C and D are also magnitudes having a ratio to one another, then they have the *same ratio* that A and B have, provided

$$\{m/n\colon B\cdot m < A\cdot n\} = \{m/n\colon D\cdot m < C\cdot n\}.$$

Such is Euclid's Definition v.5, in modern form. I suggested in §1.2 (p. 19) that fractions were problematic; but in the present context, it makes no difference if we replace m/n with the ordered pair (m, n).

Dedekind's insight was that a set of the form $\{m/n\colon B\cdot m < A\cdot n\}$ had properties that could be specified without reference to magnitudes like A and B; and then the sets of rational numbers with those properties could be used to define the irrational numbers. In the *Geometry* [29], Descartes had justified algebraic manipulations geometrically, by showing how the product of two line-segments could be understood as another segment, once a unit length was chosen. Essential to Descartes's work was a geometric theory of proportion, and specifically Proposition vii.2 of Euclid's *Elements,* that a straight line parallel to the base of a triangle cuts the sides proportionally.[2] Bertrand, for one, seems to have continued the tradition of finding the ultimate foundation of mathematics in geometry. Dedekind saw that arithmetic also could serve as a foundation of

[1] If $A < B$, then Euclid requires the difference $B - A$ to have a ratio to A and B, in order to prove that A and B cannot have the same ratio to another magnitude. In the paper [99] discussed earlier (p. 21), Zeeman shows that, for the proof, $B - A$ need not be defined, but it is enough to assume $A \cdot (n + 1) < B \cdot n$ for some positive integer n.

[2] My article [78] has some model-theoretic developments from Descartes's idea.

mathematics, even a better foundation. This was a significant advance, not always appreciated by Dedekind's contemporaries, who were perhaps too strongly attached to ideas traceable to Euclid.

Or did Dedekind perhaps understand Euclid more clearly than others did? Dedekind showed that we need not *assume* that the number line is continuous; we can *make* it continuous. He observed [28, p. 38] that if all of the points M of a plane have algebraic coordinates (according to some coordinatization),

> then is the space made up of the points M, as is easy to see, everywhere discontinuous; but in spite of this discontinuity, and despite the existence of gaps in this space, all constructions that occur in Euclid's *Elements*, can, so far as I can see, be just as accurately effected as in perfectly continuous space; the discontinuity of this space would not be noticed in Euclid's science, would not be felt at all. If any one should say that we cannot conceive of space as anything else than continuous, I should venture to doubt it and to call attention to the fact that a far advanced, refined scientific training is demanded in order to perceive clearly the essence of continuity and to comprehend that besides rational quantitative relations, also irrational, and besides algebraic, also transcendental quantitative relations are conceivable.

Mathematicians had drifted away from the rigor of Euclid without realizing it. They worked with kinds of numbers never contemplated by Euclid, while still founding their ideas of such numbers in geometric intuition. For Dedekind, this would not do. There is more to the story; but the point for now is that when there has been a continuous tradition of building up mathematics from Euclid, the tradition may lose sight of what Euclid actually did. Now that we no longer have this tradition, we may be better able to do what Dedekind could do, and understand what Euclid really did do.

2.2 What Translation Can Miss

Some students do still learn mathematics from Euclid. My own mathematics department in Turkey now has a first-semester undergraduate course based on the first book of the *Elements.* Students go to the board and demonstrate propositions, more or less on the pattern of my own *alma mater,* St. John's College in the United States [80]. However, our students in Istanbul use a Turkish translation of Euclid prepared in collaboration with my colleague Özer Öztürk [36]. The translation is from the original Greek, as established by Heiberg [33].

One could just translate Heath's English [34]; but there are ways that Heath is inaccurate. In translating the first proposition of the first book of the *Elements,* Heath begins,

> *On a given finite straight line to constuct an equilateral triangle.*
>
> Let AB be the given finite straight line.
>
> Thus it is required to construct an equilateral triangle on the straight line AB.
>
> With centre A and distance AB let the circle BCD be described.

But what Euclid says (in Heiberg's transcription and in my literal translation) is,

> Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συςτήσασθαι. Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ. Δεῖ δὴ ἐπὶ τῆς ΑΒ εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι. Κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΓΔ. // [1] On the given finite straight [line] to construct an equilateral triangle. [2] Let the given finite straight [line] be AB.[3] [3] Thus it is required on the AB straight [line] to construct an equilateral triangle. [4] With center A and distance AB let a circle have been drawn, ΒΓΔ.

The four sentences here are, respectively, the four parts of a proposition that Proclus [84, p. 159] calls (1) **enunciation,** (2) **exposition** (or "setting out"), (3) **specification,** and (4) **construction.** (The fourth sentence is only *part* of the construction.) The remaining

[3] Alternatively, Let there be a given finite straight [line, namely] AB.

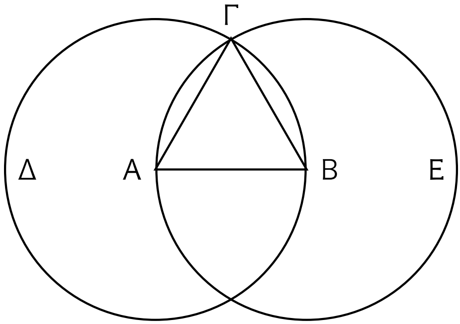

Figure 2.1. Proposition I.1

two parts of a proposition are the (5) **demonstration** and (6) **conclusion.** We shall find it useful to analyze Propositions VII.37 and 38 into these parts in §4.1 (p. 112).

In Proposition I.1, the construction continues with the drawing of a second circle as in Figure 2.1, and with the connection of one of the points of intersection of the two circles with the endpoints of the original finite straight line. Then follows the (easy) demonstration that the resulting triangle is indeed equilateral, and the conclusion that what was to be done has been done. The literal translation of the Greek above shows six differences from Heath, as follows.

2.2.1 Lines that need not be straight

For Euclid, a line is what we call a curve; our lines are Euclid's *straight* lines. Heath is faithful to Euclid by writing "straight line" when this is what is meant. However, Euclid usually abbreviates "straight line" (ἡ εὐθεῖα γραμμή) to "straight" (ἡ εὐθεῖα). Heath cannot do this if, as he does, he wants to maintain good English style. One can do it in Turkish though: here a line in Euclid's sense is *çizgi,* while a straight line is *doğru çizgi* or just *doğru.*

2.2.2 The generic article

In the enunciation of the proposition, Euclid refers not to *a* straight line, but *the* straight line. The definite article "the" here can be understood as being generic, as in Wordsworth's verse, "The child is the father of the man," quoted already on p. 33. I discuss the mathematical use of the Greek article in more detail elsewhere [81]. The straight line of Euclid's enunciation can also be understood as the straight line in Euclid's diagram, which exists before we start to read the proposition.

2.2.3 Diagrams already there

Today, if we say that we are going to construct an equilateral triangle on *a* straight line, we mean an arbitrary straight line; but then we proceed to draw our own straight line, assigning to its endpoints, as Heath does, the letters A and B. This is not what Euclid does, as Reviel Netz explains in *The Shaping of Deduction in Greek Mathematics* [72, pp. 24–5]:

> Nowhere in Greek mathematics do we find a moment of specification *per se*, a moment whose purpose is to make sure that the attribution of letters in the text is fixed. Such moments are very common in modern mathematics, at least since Descartes. But specifications in Greek mathematics are done, literally, *ambulando.* The essence of the 'imperative' element in Greek mathematics—'let a line be drawn . . . ', etc.—is to do some job upon the geometric space, to get things moving there . . . [4]
>
> What we see, in short, is that while the text is being worked through, the diagram is assumed to exist. The text takes the diagram for granted.

Thus, at the start of Euclid's proposition, we have already been given a straight line, its endpoints labelled A and B. The exposition or "setting out" of the proposition tells us to understand the straight line of the enunciation as the straight line AB. We are *not* told to

[4] The first ellipsis here is Netz's; the second, mine.

let AB be some straight line that we proceed to create for ourselves. (See p. 115 for further illustration of this point.)

2.2.4 Letters as adjectives

Today it is standard to use expressions like "the straight line *AB*" or "the circle *BCD*"; but sometimes this phrasing does not quite fit Euclid. Netz proposes that the letters in the diagrams of Greek mathematics are not *symbols,* but *indices* [72, p. 47]:

> the letter *alpha* signifies the point next to which it stands, not by virtue of its being a symbol for it, but simply because it stands next to it. The letters in the diagram are useful signposts. They do not stand *for* objects, they stand *on* them.

In his ensuing pages, Netz gives five arguments for his idea. One argument is based on what we have just observed: in the proposition under consideration, the letters A and B have a meaning before we start reading, since they are already in the diagram. Thus the letters are not symbols, whose meaning would be established by convention; they are indices.

A related argument, though one that Netz seems not quite to make, is that, strictly speaking, the letters used by Euclid to indicate points or lines are *adjectives.* They are like the *colors* that Oliver Burne uses in his remarkable edition of the first six books of the *Elements* [15]: here the triangle to be constructed in Proposition I.1 has black, red, and yellow sides respectively, and these sides are referred to in the text merely by colored pictures of themselves; but if we were reading out loud, we should speak of the *yellow line,* and not the "line yellow."[5]

---

[5]Burne himself uses expressions like "the yellow line" and "the red angle" when describing his system of geometry in his Introduction. Meanwhile, Netz has referred to Peirce for a general theory of signs, according to which these may be indices, *icons,* or symbols. Icon are signs "signifying by virtue of a similarity with their object." Burne's colored lines in text would seem to be icons in this sense, for they are similar to the colored lines in the diagrams. Netz calls Peirce's distinction "well-known" and gives no further reference. (Three pages later he quotes Peirce, with a precise reference, to

Letters and colors may indeed be nouns, as in the song verse, “Black is the color of my true love’s hair,” or in Revelation 1:8, Ἐγώ εἰμι τὸ Ἄλφα καὶ τὸ Ὦ, λέγει κύριος ὁ θεός [1], “I am *the* Alpha and *the* Omega, says *the* Lord God.” In the Biblical passage, the Greek articles here are neuter, neuter, and masculine respectively; thus the letters take their native neuter gender [89, p. 46, 199d], rather than the masculine gender of “God.”

Like other adjectives though, letters and colors may also be used elliptically, in place of a noun phrase. Thus “the A” may mean the A Train, the subway train that can take you from downtown Manhattan to Harlem; and to be “in the black” apparently means to be in the black *ink,* that is, to have your worth written on the credit side of the account. Used elliptically in this way, the Greek adjective takes the gender of the missing noun, as when the adjective εὐθύς -εῖα -ύ “straight” takes the feminine form εὐθεῖα when standing for εὐθεῖα γραμμή, because the noun γραμμή is feminine. In the exposition of the Proposition I.1, the phrase ἡ ΑΒ, “the AB,” uses the feminine article ἡ, because the phrase stands for ἡ ΑΒ εὐθεῖα γραμμή, “the AB straight line.”

In mathematics in English today, we use letters as nouns, often in apposition to other nouns, as in “the straight line AB” or “the circle BCD.” Euclid does something similar, as in the fourth sentence of the quotation from Proposition I.1, where for example the noun phrase τῷ ΑΒ, in the dative case, is in apposition to διαστήματι, which is the dative case of the noun τὸ διάστημα, “distance.” As this noun is neuter, so is the article in τῷ ΑΒ. Actually the dative masculine article is also spelled τῷ, but the feminine is τῇ; thus τῷ ΑΒ stands for τῷ ΑΒ διαστήματι, “the AB distance,” and not τῇ ΑΒ εὐθείᾳ “the AB straight [line].”

---

the effect that letters used in mathematics are indeed indices.) Peirce gives the following examples in [75, p. 104]: The streak of a lead pencil is an icon for a geometrical line; a bullet-hole is an index for a shot; an utterance of speech is a symbol.

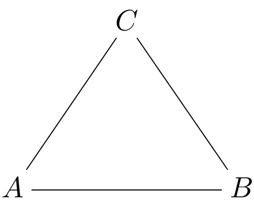

Figure 2.2. Letters as points

We see such a noun phrase spelled out in the third sentence as τῆς ΑΒ εὐθείας, "the AB straight [line]," which is in the genitive case as being the object of the preposition ἐπί "on." Here AB is not in apposition to "straight line," but is an adjective modifying it.

Again, the straight line in question is that straight line in the diagram whose endpoints are labelled A and B. If the letters were *symbols* for those endpoints, then our diagram might be as in Figure 2.2—which is indeed how we make commutative diagrams and Hasse diagrams today. Of course there are practical reasons not to draw geometrical diagrams that way: we draw the lines and points first, and then add the letters. This itself is a reason why those letters will be conceived not as *being* points, but only as describing them.

We shall see later (§4.1, p. 109) an instance in the *Elements* where letters *are* used symbolically: where they are not strictly labels on objects in a diagram, because what they symbolize is not an object, but a ratio of objects, if not simply a verbal expression. This will give us reason to question the authenticity of the proposition in question (which is VII.39).

2.2.5 The perfect imperative

Meanwhile, in Proposition I.1, Heath tells us to let a circle be drawn. Using chalk or a pen, we can then draw a circle. But apparently Euclid had nothing like our blackboards or whiteboards, with which a diagram could be constructed during a lecture. Again, his diagrams would already have been drawn, perhaps on a wax tablet. Referring to the passage quoted earlier, about how the text takes the diagram for granted, Netz says,

> This, in fact, is the simple explanation for the use of *perfect* imperatives in the references to the setting out—'let the point A have been taken'. It reflects nothing more than the fact that, by the time one comes to discuss the diagram, it has already been drawn.

Though it is seen in the command "Have done with it," the English imperative in the perfect aspect is awkward, and Heath avoids it. English does not even have a *third-person* imperative in any aspect, except in some formulas like "God bless you"; otherwise we achieve the same effect periphrastically,[6] by means of the second-person imperative "let," as in "Let it be done" or "Let there be light." Turkish, like Greek, does have third-person imperatives, which are commonly used:

let [somebody] draw	γραφέτω	*çizsin*
let [it] be drawn	γραφέσθω	*çizilsin*
let [it] have been drawn	γέγραφθω	*çizilmiş olsun*

This similarity between Turkish and Greek was a particular reason to translate directly from Euclid's Greek into Turkish for our students in Istanbul.

2.2.6 Continuous text

Heath *italicizes* the enunciations of Euclid's propositions; but Euclid had no such means of emphasizing text. He did not even have

[6]"Periphrasis & civilisation are by many held to be inseparable; these good people feel that there is an almost indecent nakedness, a reversion to barbarism, in saying *No news is good news* instead of *The absence of intelligence is an indication of satisfactory developments.* Nevertheless, *The year's penultimate month* is not in truth a good way of saying *November.*" H. W. Fowler, *A Dictionary of Modern English Usage* [38, Periphrasis, p. 430].

the medieval distinction between minuscule and capital letters.

Like Heath, Heiberg emphasizes Euclid's enunciations: not however by a change of font, b u t b y s p a c i n g o u t t h e l e t t e r s l i k e t h i s (the effect on Greek letters is seen on page 69). I assume Euclid did not consider spacing out letters for emphasis, since he would not even have separated *words* with spaces, BUTINSTEADHEWROTECONTINUOUSLYLIKETHIS.[7] Although there might have been the possibility of underlining for emphasis, presumably Euclid did not use this either.[8]

In *The Mathematics of Plato's Academy* [37, §6.2], David Fowler discusses what we know about ancient manuscript style. He also [37, §10.4] looks at what he calls the *protasis*-style of Euclid. Πρότασις is the term of Proclus that we translate as "enunciation." Before Euclid, we have no evidence of any mathematics written in the Euclidean style, with an enunciation followed by justification. Aristotle's *Prior Analytics* might be an exception, except that it is not really mathematics, but logic.[9] Today the *protasis*-style is ubiquitous in mathematics; and yet we signal our protases with rubrics like "Theorem N," and our justifications with the word "Proof" (and a box □ at the end). Strictly speaking, this is not Euclid's style.

It is often not students' style either. In performing the task of demonstrating something, students will write down various state-

[7]Observing this in a footnote [72, n. 102, p. 50], Netz mentions a papyrus from 350–325 BCE in which, "the continuous text is, as usual, unspaced. Letters referring to the diagram are spaced from the rest of the text." This is another of Netz's reasons why letters referring to diagrams should be understood as indices, not symbols.

[8]In the same footnote mentioned in the previous note, Netz mentions later papyri in which mathematical letters are marked by superscribed lines.

[9]At the beginning of the *Prior Analytics,* Aristotle defines πρότασις as "an affirmative or negative statement of something about some subject" [3, 24ᵃ10, p. 199]. We then translate the word as premiss or just proposition. Πρότασις is used in Greek today, as "proposition" is used in English, for an *entire* proposition of Euclid [35]; but I am not aware of any ancient basis for this usage.

ments, without being clear about the logical relations between them. Even professional mathematicians will do this at the board, expecting the attentive listener to know what is meant (and expecting all listeners to be attentive).

At least Euclid establishes a set pattern for his propositions. Read a bit, and you see that the *Elements* is a sequence of assertions, each followed by a justification, the justification itself being laid out (more or less) according to the outline given by Proclus. Actually, some of Euclid's enunciations are not assertions, but tasks, as in the very Proposition I.1 that we have been looking at. The distinction between an assertion and a task can be indicated by the labels *theorem* (θεόρημα) and *problem* (πρόβλημα). However, according to Pappus, "among the ancients some described them all as problems, some as theorems" [93, p. 567]. Euclid himself did not use such labels at all.

Heath helps the reader typographically, by italicizing enunciations, by breaking the text into short paragraphs, and by centering some phrases. This typography may be misleading, if it causes us to think of Euclid's propositions just as if they were modern theorems with proofs. In some cases at least, Euclid's propositions are not like that. An example mentioned above in §1.3 (p. 25) is Proposition VII.4, and we shall consider this further in §§3.4 & 3.7 (pp. 80 & 90).

2.3 Equality

2.3.1 Side Angle Side

Another proposition of Euclid that is not like a modern theorem is Proposition I.4. This is where Euclid establishes the principle of triangle congruence that we call "Side Angle Side." For Hilbert, this principle is an axiom, the one that he numbers **IV, 6.**[10] Euclid

[10] More precisely, Hilbert's axiom is that if two sides and the included angle of a triangle are respectively congruent to two sides and the included angle of

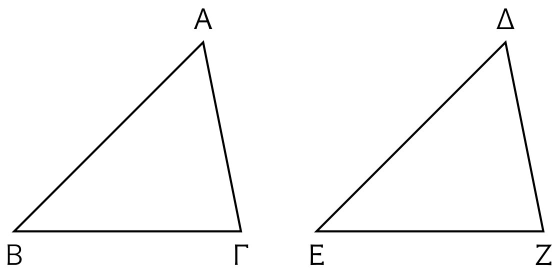

Figure 2.3. Proposition I.4

proves the principle, but without using any postulate or any previous proposition. Can Euclid's proof then be a "real" proof? Given triangles ABΓ and ΔEZ as in Figure 2.3, let us suppose AB = ΔE and AΓ = ΔZ and ∠ BAΓ = ∠ EΔZ. If we apply triangle to triangle so that A falls on Δ, and AB falls along ΔE, then B will fall on E, and AΓ will fall along ΔZ, so that Γ will fall on Z; and then BΓ will coincide with EZ (see §2.4, page 62 below). So argues Euclid. Do we accept it?

Students are sometimes disturbed by an expression like AB = ΔE; they want to make it |AB| = |ΔE|. Apparently they think that the sign of equality in fact denotes identity. Obviously the straight lines AB and ΔE are not identical; but they may have identical *lengths,* which can then be denoted indifferently by |AB| or |ΔE|.

Euclid does not use a symbol for equality; he just says AB is *equal* (ἴσος -α -ον) to ΔE. Along with definitions and postulates, the preamble of the *Elements* contains so-called common notions; and of the five of these that Heiberg and Heath accept as genuine, the first four establish what equality means.[11] According to the assigned numbering, these Common Notions are that (4) things

another triangle, then the remaining angles are respectively congruent to the remaining angles. That the remaining side is congruent to the remaining side is Hilbert's Theorem 10.

[11] The fifth common notion is quoted on p. 61.

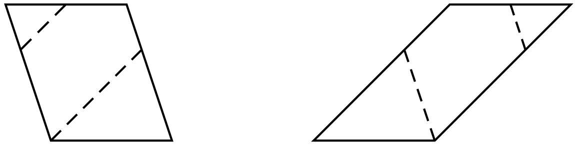

Figure 2.4. Proposition I.36

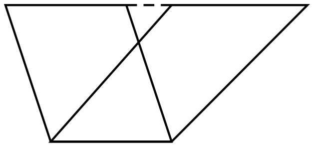

Figure 2.5. Proposition I.35

congruent to one another are equal to one another, and (1) things equal to the same thing are equal to one another; but moreover, if equals be (2) added to or (3) subtracted from equals, the results are equal. In expressing Common Notion 4, I use the word congruent for Euclid's participle ἐφαρμόζων, to emphasize that the notion of congruence found in Hilbert originates in Euclid's notion of equality; but Heath says "things which coincide with" instead of things congruent to. For Euclid, figures are equal if congruent parts can be added or subtracted so as to obtain congruent figures. Thus parallelograms on equal bases and in the same parallels can be shown to be equal to one another by cutting out and rearranging the parts shown in Figure 2.4. The equality of these parallelograms is Proposition I.36.[12] Actually Euclid derives this equality from Proposition I.35, where the bases of the parallelograms are not equal, but *the same* (ὁ αὐτός), as in Figure 2.5. *Here* the proof is by cutting and pasting; then I.36 is proved by means of a third parallelogram, which shares a base with either of the first two parallelograms, as

[12]Equality is thus for Euclid an equivalence relation, with respect to which the class of a figure is its "motivic measure" as in the account of Hales [43].

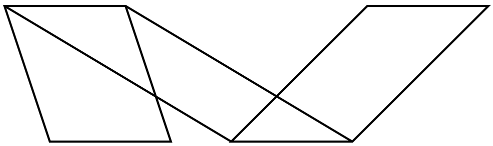

Figure 2.6. Proposition I.36 as Euclid does it

in Figure 2.6.

In the Declaration of Independence of the United States of America, when Thomas Jefferson wrote the self-evident truth, "that all men are created equal" [47, p. 15], he did not mean that all persons were the same person. However, in mathematics today, we confuse equality and identity. Thus when we write the equation $(k+1)! = k! \cdot (k+1)$ as above (p. 18), we mean that $(k+1)!$ and $k! \cdot (k+1)$ are to be considered as the same element of the set $\mathbb{N}$, although we read the sign $=$ as "equals."

In the article called "When is one thing equal to some other thing?" [64, p. 222], the notion of equality that Barry Mazur contemplates is not distinct from the notion of sameness. Thus the article begins,

> One can't do mathematics for ten minutes without grappling, in some way or other, with the slippery notion of *equality.* Slippery, because the way in which objects are presented to us hardly ever, perhaps never, immediately tells us—without further commentary—when two of them are to be considered equal. We even see this, for example, if we try to define real numbers as decimals, and then have to mention aliases like $20 = 19.999\ldots$, a fact not unknown to the merchants who price their items \$19.99.
>
> The heart and soul of much mathematics consists of the fact that the "same" object can be presented to us in different ways. Even if we are faced with the simple-seeming task of "giving" a large number, there is no way of doing this without also, at the same time, giving a hefty amount of extra structure that comes as a result of the way we pin down—or the way we present—our large number. If we write our number as 1729 we are, sotto voce, offering a preferred way of "computing it" (add one thousand to

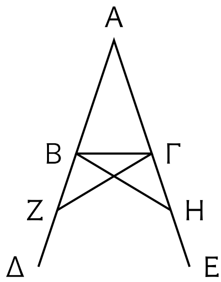

Figure 2.7. Proposition I.5

> seven hundreds to two tens to nine). If we present it as $1+12^3$ we are recommending another mode of computation, and if we pin it down—as [Ramanujan] did—as the first number expressible as a sum of two cubes in two different ways, we are being less specific about how to compute our number, but we have underscored a characterizing property of it within a subtle diophantine arena.

When we are presented with the angles ΒΑΓ and ΕΔΖ in the triangles in Figure 2.3, there is nothing about this presentation itself that tells us that the two angles are equal, just as there is nothing about the two expressions 20 and 19.999... that tells us they stand for equal numbers. However, even when the expressions are understood, the equal angles ΒΑΓ and ΕΔΖ are not *interchangeable* in the way that 20 and 19.999... are. We say that the numbers 20 and 19.999... are the *same* number; but ΒΑΓ and ΕΔΖ are different angles of different triangles.

In fact they *could* be the same angle, as they are for example in Euclid's Proposition I.5, whose diagram is in Figure 2.7. Here the sides ΑΒ and ΑΓ of the triangle ΑΒΓ are given as being equal to one another. By construction, $\mathsf{AZ} = \mathsf{AH}$. Since also the angle ΖΑΗ is common to the triangles ΑΖΓ and ΑΗΒ, these two triangles are congruent to one another, by Proposition I.4—but strictly speaking, this conclusion requires us to recognize that the common angle of the two triangles is equal to *itself.* We must also recognize that this angle can be expressed indifferently as ΖΑΓ or ΗΑΒ. Euclid does not say this explicitly.

Mazur's concern is more with the question of whether your triangles are the same as my triangles, or your numbers are the same as my numbers. He says [64, p. 225],

> *Equivalence* (of structure) in the above 'compromise' is the primary issue, rather than *equality* of mathematical objects

—where again I think "equality" can be read as sameness. The "compromise" is between the treatment of the number 5 as a particular standard five-element set, and its treatment as the class of all five-element sets. The compromise is to let you use your five, and let me use mine, as long as what each of us does with it can be "translated" into what the other does with it. Mazur elaborates on how the language of category theory lets us talk about these things. However, I think there is no question that, in any instance of actually doing mathematics today, there is only one number 5. There may be different five-element sets, but there is only one *five,* be it a *particular* five-element set reserved as a standard, or be it the unique class consisting precisely of all five-element sets.

For Euclid, a number is simply a finite set, usually with more than one element. There might then be two numbers described by the same adjective "five"; thus the numbers themselves would be equal, though not the same. We shall consider an example in §3.3 (p. 74).

Meanwhile, I want to answer the question raised at the beginning of this section. Euclid's proof of Proposition I.4 *is* a real proof, because it is based on several applications of the principle that equal straight lines can be made to coincide, and likewise equal angles. This principle is not explicitly stated; but should it have been? *What else can equality of straight lines or angles mean?* (The question of why, in the proof, ΒΓ should coincide with ΕΖ, once their endpoints coincide, will be considered in the next section.)

We have noted Common Notion 4, that things congruent to one another are equal to one another. We have noted that the converse fails: sometimes things are equal, only because they have congruent *parts*. Sometimes not even this is so. Thus Proposition XII.7 is

> Πᾶν πρίσμα τρίγωνον ἔχον βάσιν διαιρεῖται εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους βάσεις ἐχούσας. // Every prism having a triangular base is divided into three pyramids, equal to one another, having triangular bases.

To prove this, rearranging congruent parts is not enough, by Dehn's solution of Hilbert's Third Problem.[13] Proposition XII.7 is a corollary of Proposition XII.5, which belongs to the theory of proportion, so it relies on the "Archimedean" axiom alluded to earlier (p. 44):

> Αἱ ὑπὸ τὸ αὐτὸ ὕψος οὖσαι πυραμίδες καὶ τριγώνους ἔχουσαι βάσεις πρὸς ἀλλήλας εἰσὶν ὡς αἱ βάσεις. // Pyramids under the same height, having triangular bases, are to one another as the bases.

Thus the scope of the notion of equality in the *Elements* becomes broader; but it originates in simple congruence. To say that two straight lines or two angles are equal is to say that they can be made to coincide. This gives us Proposition I.4 as a theorem.

2.3.2 Side Side Side

The status of Proposition I.8, "Side Side Side," is not so clear. Given again the triangles ΑΒΓ and ΔΕΖ as in Figure 2.3 (on p. 55), but now letting ΑΒ = ΑΓ and ΑΓ = ΔΖ and ΒΓ = ΕΖ, we can apply the base ΒΓ to the base ΕΖ, since they are equal. Euclid argues that Α must fall on Δ, because otherwise it falls on a point Η as in Figure 2.8, and then a contradiction to Proposition I.7 arises. If Euclid allows such an argument, as he does, perhaps he will allow Hilbert's proof (discussed in the next section) that all right angles are equal to one another.

Alternatively though, if angles ΑΓΒ and ΔΖΕ are right, then they are equal by the fourth postulate (considered in the next section),

[13] Secondary references include Struik [90, p. 192–4], who gives a complete list of Hilbert's 23 problems and mentions Dehn's (negative) solution of Problem 3, this problem being in effect to prove Euclid's VII.7 by cutting and pasting. Kline [55, p. 1016, n. 20] gives the precise reference to Dehn's paper: *Math. Ann.,* 55, 1902, 465–78.

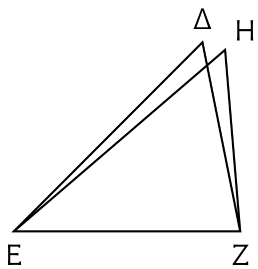

Figure 2.8. Proposition I.8

and so Proposition I.8 follows by I.4. If only ΑΓΒ is right, then a perpendendicular to ΕΖ can be erected at Ζ by Proposition I.11; and the perpendicular can be made equal to ΑΓ, by I.3. In this case, a contradiction to I.7 arises as before. As it is, Euclid's proof of I.11 relies on I.8; but it need not. If neither of the angles ΑΒΓ nor ΔΖΕ is right, then a perpendicular to ΒΓ can be dropped from Α, by I.12; again Euclid's proof of this relies on I.8, but need not.

There is no reason not to think that Euclid was aware of this alternative approach to I.8, but preferred to leave it as an exercise for the reader. Euclid does not *say* he is doing this; but then there is a lot that he leaves to the reader without spelling it out, as we shall discuss in §3.7 (p. 93).

Meanwhile, as it is, Euclid's proof of I.8 assumes that we can compare two angles that have different vertices: we can bring them together, so to speak, so that, if they are unequal, then one will become a part of the other, in the sense of the fifth (and last) [14]common notion: "The whole is greater than the part." The possibility of bringing angles together in the sense of *adding* them (rather than subtracting the less from the greater) is required by the Fifth Postulate—to be considered in the next section.

[14]The first four being on p. 55.

2.4 Straight lines and right angles

We do not know what the preamble of the *Elements* consisted of when this collection of thirteen books was first compiled. In *The Forgotten Revolution* [87, pp. 323–4], Lucio Russo argues that Euclid's obscure definition of straight line is only a later addition to the *Elements*: in origin it is a truncated sentence from a student's crib-sheet. The definition in the *Elements* is

Εὐθεῖα γραμμή ἐστιν,
ἥτις ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς σημείοις κεῖται,

which is practically the same as the first part of

Εὐθεῖα μὲν οὖν γραμμή ἐστιν,
ἥτις ἐξ ἴσου τοῖς ἐπ᾽ αὐτῆς σημείοις κεῖται
ὀρθὴ οὖσα καὶ οἷον ἐπ᾽ ἄκρον τεταμένη ἐπὶ τὰ πέρατα.

Russo noticed the latter sentence among the *Definitions of Terms in Geometry* attributed to Hero [48, p. 16, ll. 22–4]; it means something like

A straight line is
one that equally with respect to all points on itself lies
right and maximally taut between its extremities.[15]

The italicized part is what is missing from the *Elements,* though it is where the meaning of the definition lies; the reference to "all points on itself" allows for the case of an unbounded straight line, where any two points must serve as extremities for the purpose of the definition.

Here I just want to assert the plausibility of Russo's argument. There is reasonable doubt about the authenticity of the definition of straight line in the *Elements. Some* understanding of straightness is needed for the conclusion in Proposition I.4 that, when points Β and Γ are made to coincide with points Ε and Ζ, then straight line ΒΓ will coincide with straight line ΕΖ. Whether Euclid meant this understanding to be part of Postulate 1, or part of the definition of straight line, or even just part of Proposition I.4, I do not know.

[15] Russo's translation, as rendered in English by Silvio Levy, is "A straight (εὐθεῖα) line is a line that equally with respect to all points on itself lies straight (ὀρθή) and maximally taught between its extremities."

The definition in the *Elements* of a right angle (ὀρθὴ γωνία) is given as,

> When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right,* and the straight line standing on the other is called a *perpendicular* to that on which it stands.

This is Heath's translation, with his italics. As the italics suggest, surely it is right angles that are being characterized, and not equality of angles. Equality of angles is implicitly understood; and indeed, what else can it mean but that one angle can be picked up and placed on another?

Postulate 4 is that all right angles are equal to one another. I take this to mean that we have a toolkit containing a carpenter's square. The square is not for *drawing* right angles: this is achieved by Propositions I.11 and 12, mentioned at the end of the last section. But the carpenter's square reminds us that indeed all right angles *are* equal to one another, because they can be made to coincide with this one standard angle. (I take an actual carpenter's square to my own Euclid classes, to serve as a reminder or symbol of Postulate 4.)

Euclid's first three postulates are that we can do things that can be done with a straightedge and compass, or just with a length of cord. We can (1) connect two points with a straight line, (2) extend a given straight line, and (3) draw a circle with given center, passing through another given point. Finally, Postulate 5 tells us something else that we can do, though it requires no new tool: we can find a point of intersection of two straight lines, if we extend them far enough, provided that a line falling across them makes the interior angles on the same side less than two right angles. Here the angles are *together* less than two right angles, though one of them might be greater than a right angle. Implicitly then, we can bring two angles together, for comparison with two right angles.

I noted in §2.1 (p. 41) that Hilbert thought Euclid's Postulate 4 was actually a theorem. In the style of Euclid, Hilbert's proof

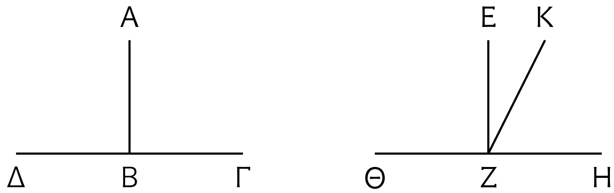

Figure 2.9. Postulate 4

would seem to be as follows. Suppose, as in Figure 2.9, straight line ΑΒ is perpendicular to ΓΔ, and ΕΖ is perpendicular to ΗΘ. If right angles ΑΒΓ and ΕΖΗ are not equal to another, then one is greater. Suppose the latter is greater. Then ΑΒΓ will fall inside it as ΚΖΗ. The supplement ΚΖΘ of ΚΖΗ must be equal to the supplement ΑΒΔ of ΑΒΓ, though this needs further discussion: it is Hilbert's Theorem 12. Absurdity results, since in short

$$\begin{aligned}\angle \mathsf{AB\Gamma} = \angle \mathsf{KZH} < \angle \mathsf{EZH} &= \angle \mathsf{EZ\Theta} \\ &< \angle \mathsf{KZ\Theta} = \angle \mathsf{AB\Delta} = \angle \mathsf{AB\Gamma}.\end{aligned}$$

How did we obtain angle ΚΖΗ? Hilbert can use his axiom **IV, 4,** which he summarizes as being

> that every angle in a given plane can be *laid off* upon a given side of a given half-ray in one and only one way.

For Euclid, this is a *theorem,* namely Proposition I.23. Hilbert's toolkit contains a protractor, or else triangles with angles of all posssible sizes; and these can be used to draw with. But again, the only triangles in Euclid's toolkit are right triangles, and they cannot be used to draw with (and their acute angles cannot be used in any way). In Euclid's Proposition I.4, we are able to "lay off" one given angle on another, because this possibility is implicit in the assumption that the two angles are equal to one another. In Figure 2.9, there is no assumed equality of anything on the left with anything on the right; so for Euclid, there can be no "laying off."

We have noted in the previous section that Euclid's proof of Proposition I.8 does require a laying off of one angle along the side of another, as in Hilbert's axiom; but there is an alternative proof, on a Euclidean basis, that does not require this laying off. We have noted that Euclid's fifth postulate also requires this laying off, for the sake of adding two angles that have different vertices; but Euclid will not use this postulate until I.29, when it has been shown (in I.23) how to lay off an angle.

Corresponding to Euclid's fifth postulate, Hilbert has his Axiom **III:**

> In a plane α there can be drawn through any point A, lying outside of a straight line a, one and only one straight line which does not intersect the line a. This straight line is called the parallel to a through the given point A.

This introduces a whole new tool: parallel rulers.

In his proof of the equality of all right angles, Hilbert could have assumed $\mathsf{AB} = \mathsf{EZ}$, and likewise for the other corresponding straight lines, because of his axiom **IV, 1,** summarized as being

> that every segment can be *laid off* upon a given side of a given point of a given straight line in one and only one way.

This is Euclid's Proposition I.3.

There must have been a great change in thinking about mathematics, if Hilbert was trying to build up geometry on a minimal foundation, yet could treat as axiomatic what for Euclid needed proof. Hilbert did achieve the logical economy of proving Euclid's fourth postulate, but at the expense of an abundance of tools whose practical use by the student of geometry is left unexplained.

Hilbert seems not to have been troubled by his change of approach to geometry. Perhaps this is because he was reading Euclid according to a tradition whose changes to the meaning of Euclid had gone unnoticed. Mathematics had evolved, but still Euclid was revered as the father of rigorous mathematics; this may have caused contemporary notions of mathematics to be read into Euclid. At any rate, the continuous tradition may instill the assumption that, when one is doing mathematics, one is still only doing the same kind of thing that Euclid did. It is an assumption that ought to be questioned, at least from time to time.

3 Euclid's Foundations of Arithmetic

3.1 Unity

Not only (as in the previous section) is the definition of straight line plausibly held to be a late addition to the *Elements,* but Russo says the same of the definition of unity at the head of Book VII. His reason is that, six centuries after Euclid, Iamblichus described the same definition as being due to "more recent authorities."

Heath discusses the same passage of Iamblichus, both in his edition of the *Elements* [34, v. II, p. 279] and in his *History of Greek Mathematics* [46, p. 69]. The discussion is more brief in the latter, where the very part of the passage that is relevant for us now is not considered. In his notes on the *Elements,* Heath gives no explicit indication that the authenticity of the definition of unity is called into question by the words of Iamblichus. The definition in Euclid is,

Μονάς ἐστιν, καθ᾽ ἣν ἕκαστον τῶν ὄντων ἓν λέγεται.

Unity is that according to which each entity is said to be one thing.

The translation is mine; Heath's is "A **unit** is that by virtue of which each of the things that exist is called one." I propose to try out unity instead of Heath's "unit," because the latter is a made-up word, albeit made up precisely to translate Euclid's μονάς. In his preface to Billingsley's English translation of the *Elements,* published in 1570, John Dee wrote,

> Number, we define, to be, a certayne Mathematicall Summe, of Vnits. And, an Vnit, is that thing Mathematicall, Indiuisible, by participation of some likenes of whose property, any thing, which is in deede, or is counted One, may resonably be called One.

As Billingsley's was the first English translation of the *Elements* [34, v. I, p. 109], so is the passage from Dee the first of the illustrative quotations in the UNIT article of the *Oxford English Dictionary* [71]. In the etymology section of that article, Dee's marginal note—apparently on the passage above—is quoted:

> Note the worde, Vnit, to expresse the Greke Monas, and not Vnitie: as we haue all, commonly, till now, vsed.

However, at UNITY, the *OED* gives Billingsley's translation of Euclid's definition:

> Vnitie is that, whereby euery thing that is, is sayd to be on.

Evidently Billingsley did not perceive a need for Dee's new word "unit." Billingsley and Dee could have used "monad" (from the stem μονάδ- of μονάς): this is now an English word, though its earliest quotation in the *OED* is from 1615. The French *monade* dates to 1547 [30, p. 484]. Meanwhile, French is content to use the old *unité* where English puts the newfangled "unit": in the quotation from Bertrand's *Traité d'Arithmétique* in §2.1 (p. 42), "unité de longueur" serves where we say "unit of length."[1]

By the way, it is not clear whether Euclid's μονάς and ἕν ("one") are etymologically related. The Greek cardinal numeral "one" is declined as an adjective and, as such, has three distinct genders: the masculine, feminine, and neuter are respectively εἷς, μία, ἕν. Despite appearances, these three forms *are* related to one another: each comes from σεμ, according to Smyth's old *Greek Grammar* [89, ¶349] and the more recent Chantraine [17, II, 326, **εἷς**]. The same root is seen also in the Latin sources of our "simple" and "single." But whether the μ of μία is related to the μ of μονάς is unclear. The latter word comes from μόνος, -η, -ον, meaning "only, alone, sole," as we may see from English words like "monotheism." Chantraine just traces μόνος to a conjectural *μονϜός and expresses doubt that it is related to μανός "thin, scanty."[2]

[1] The word *unité* is Anglo-French, being traced to the *Psautier d'Oxford* of 1120 [30].

[2] The adjective μόνος is attached to μονάς in the definition of prime number, quoted on page 75. By the way, here is a case that might seem to justify the

As Smyth [89, ¶354e] describes it, the ending -άς, -άδος creates "abstract and collective numbers," so that ἑνάς or μονάς is "the number one, unity, monad." I think we may usefully consider the suffix as indicating a *set* with the number of elements indicated by the stem to which it is grafted. Thus a δεκάς, a decade, is a set of ten years. As far as I can tell though, the only specific number that Euclid ever uses is δύας, in IX.32 and later: it is a pair or double or dyad.[3]

Meanwhile, our question was whether the definition of unity now in Euclid is authentic, and in particular whether some words of Iamblichus bear on this question. Those words are from *On Nicomachus's Introduction to Arithmetic* [52, p. 11, l. 1].[4] The relevant paragraph begins as follows; as there seems to be no full published English translation, I give my own attempt.

> Μονὰς δέ ἐστι ποσοῦ τὸ ἐλάχιστον ἢ ποσοῦ τὸ πρῶτον καὶ κοινὸν μέρος ἢ ἀρχὴ ποσοῦ· ὡς δὲ Θυμαρίδας π ε ρ α ί ν ο υ σ α π ο σ ό τ η ς , ἐπεὶ ἑκάστου καὶ ἀρχὴ καὶ τέλος πέρας καλεῖται,

---

tedium of using accents, which were introduced only around 200 BCE and not used regularly for 800 years [89, ¶161]: formed to agree with μονάς in gender, the feminine μόνη is different from the feminine noun μονή, a *nom d'action* derived from the verb μένω *rester* [17, III, 686].

[3] Similarly, Mueller observes, "in Euclid's arithmetic particular numbers play virtually no role and are never explicitly characterized." But then he goes on to say, "Euclid does not prove that $2+2=4$ or that a 2 and a 2 combined yield a 4, nor does he even have the apparatus for doing so. Such facts, insofar as they are used in the Elements, are used without proof" [69, p. 59]. The same can probably be said of just about any number theory text today. Landau's *Foundations of Analysis* [58] is a textbook not of mathematics proper, but of its foundations, as the title says; the book establishes the needed apparatus for proving specific facts, but the proofs themselves are considered a triviality: "The multiplication table is not to be found in this book, not even the theorem $2 \cdot 2 = 4$," which is left as an exercise for the student. So it seems hardly remarkable that Euclid does not prove such theorems. I referred on page 19 to the foundational issues, elaborated in [79], whose absence from most texts today is a more serious mathematical concern.

[4] The Greek text names the work as being of Iamblichus of Chalcis in the valley of Syria (ΙΑΜΒΛΙΧΟΥ ΧΑΛΚΙΔΕΩΣ ΤΗΣ ΚΟΙΛΗΣ ΣΥΡΙΑΣ).

ἔστι δὲ ὧν καὶ τὸ μέσον, ὥσπερ ἀμέλει κύκλου καὶ σφαίρας. ὁι δὲ νεώτεροι καθ' ἣν ἕκαστον τῶν ὄντων ἓν λέγεται· ἔλειπε δὲ τῷ ὅρῳ τούτῳ τὸ κἂν συστηματικὸν ῇ. συγκεχυμένως δὲ οἱ Χρυσίππειοι λέγοντες 'μονάς ἐστι πλῆθος ἕν'· μόνη γὰρ αὕτη ἀντιδιέσταλται τῷ πλήθει.[5] // Unity is the least of an amount or the first and common part of an amount or the beginning of an amount: thus Thymaridas [called it] "[the] limiting quantity," since the beginning and end of everything is called a limit. (But there are things of which there is a middle, such as, of course, the circle and sphere.)[6] More recent authorities [called unity] "that according to which each entity is said to be one thing"; but they left out the restriction, "even though it be collective."[7] The Chrysippians, confusedly, saying, "Unity is multitude one"; for one is opposed to multitude.

Thymarides was "an ancient Pythagorean, probably not later than Plato's time" [46, p. 69]. Since Euclid was later than Plato's time, it is not clear whether Iamblichus's "more recent authorities" (νεώτεροι) included Euclid or not. As suggested above, Russo [87, p. 320] thinks Euclid is *not* one of these authorities.

Heath [34, v. II, p. 279] apparently thinks he *is*. At any rate, Heath continues to refer to the definition of unity in the *Elements* as "Euclid's definition," even though he has taken note of Iamblichus's observation of what was missing from this definition. Apparently, according to Iamblichus, the full definition should be something like,

Unity is that according to which each entity is said to be one thing, even though it be collective.

I see no indication by Heath or Russo that such a definition can be found anywhere else than in Iamblichus's comment. However, it does bear some resemblance to a definition that Sextus Empiricus attributes to Plato in §11 of "Against the Arithmeticians" (which is Book IV of *Against the Professors,* here from the edition of Bekker

[5]Three passages are letterspaced in Pistelli's text, as discussed on page 53; but only the third of these passages is bounded by inverted commas.

[6]Pistelli, the editor, marks this passage, "abesse malim," which seems to mean he would rather it were not there.

[7]The translation "even though it be collective" is Heath's [34, v. II, p. 279].

[31, p. 724, l. 6], with the translation of Bury [32, pp. 310 f.]):

> Τὴν τοῦ ἑνὸς τοίνυν νόησιν διατυπῶν ἡμῖν πυθαγορικώτερον ὁ Πλάτων φησὶν 'ἕν ἐστιν οὗ μηδὲν χωρὶς λέγεται ἕν' ἢ 'οὗ μετοχῇ ἕκαστον ἕν τε καὶ πολλὰ λέγεται.' τὸ γὰρ φυτόν, εἰ τύχοι, καὶ τὸ ζῷον καὶ ὁ λίθος προσαγορεύεται μὲν ἕν, οὐκ ἔστι δὲ κατὰ τὸν ἴδιον λόγον ἕν, ἀλλ' ἐν μετοχῇ ἑνὸς νοεῖται, τούτου μηδενὸς τούτων καθεστῶτος. // Now Plato, in formulating in rather Pythagorean fashion the concept of the one, declares that "One is that without which nothing is termed one," or "by participation in which each thing is termed one or many." For the plant, let us say, or the animal, or the stone is called one, yet is not one according to its own proper description, but is conceived as one by participation in the One, none of them actually being the One.

This has little to do with mathematics, and neither does Sextus's entire essay. (See however page 99 and its note 24.) Sextus does not give me the impression of somebody who knows what Euclid is about. It is plausible that the definition of unity now in the *Elements* was a later addition.

3.2 Proportion

What we refer to as *proportion* could also be called *analogy.* The Greek for proportional is ἀνάλογον, and Euclid defines it for numbers in what is now listed as the twentieth "definition" at the head of Book VII:[8]

> Ἀριθμοὶ **ἀνάλογόν** εἰσιν, ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ἰσάκις ᾖ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ὦσιν. // Numbers are **proportional** when the first is of the second, and the third is of the fourth, equally multiple, or the same part, or the same parts.

We saw the third and fourth "definitions," of *part* and *parts,* in §1.3 (p. 25). These are followed by:

[8] It is reckoned as the 21st of 23 definitions in Heiberg's Greek text and hence in commentators like Mueller [69]. But Heiberg brackets the tenth of the Greek definitions and omits it from his Latin translation, renumbering the remaining definitions accordingly. Heath follows suit, as shall I.

Πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάσσονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάσσονος. // And the greater [number] is a **multiple** of the less when it is measured by the less.

The notions of multiple and part are thus correlative, and we have three ways to say the same thing:

1) B is a multiple of A;
2) A is a part of B;
3) A measures B.

Measurement is the basic undefined notion.

Proportionality of *magnitudes* is defined at the head of Book V:

Ἐν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρῶτον πρὸς δεύτερον καὶ τρίτον πρὸς τέταρτον, ὅταν τὰ τοῦ πρώτου καί τρίτου ἰσάκις πολλαπλάσια τῶν τοῦ δευτέρου καὶ τετάρτου ἰσάκις πολλαπλασίων καθ᾽ ὁποιονοῦν πολλαπλασιασμὸν ἑκάτερον ἑκατέρου ἢ ἅμα ὑπερέχῃ ἢ ἅμα ἴσα ᾖ ἢ ἅμα ἐλλείπῃ ληφθέντα κατάλληλα. Τὰ δὲ τὸν αὐτὸν ἔχοντα λόγον μεγέθη **ἀνάλογον** καλείσθω. // Magnitudes are said to be **in the same ratio,** the first to the second and the third to the fourth, when equimultiples of the first and third, by whatever multiplication, are respectively either alike in excess of, or alike equal to, or alike falling short of, equimultiples of the second and fourth. And magnitudes having the same ratio are called **proportional.**

In the definition of proportionality of numbers, Euclid does not mention ratios; but he does mention them later, as for example in Proposition 17 in Book VII:

Ἐὰν ἀριθμὸς δύο ἀριθμοὺς πολλαπλασιάσας ποιῇ τινας, οἱ γενόμενοι ἐξ αὐτῶν **τὸν αὐτὸν ἕξουσι λόγον** τοῖς πολλαπλασιασθεῖσιν. // If a number multiply two numbers, the numbers produced **will have the same ratio** as the multiplicands.

A third way of referring to numbers in proportion is seen in Proposition VII.11:

Ἐαν **ᾖ ὡς** ὅλος **πρὸς** ὅλον, **οὕτως** ἀφαιρεθεὶς **πρὸς** ἀφαιρεθέντα, καὶ ὁ λοιπὸς **πρὸς** τὸν λοιπὸν **ἔσται, ὡς** ὅλος **πρὸς** ὅλον. // If **as** whole **be to** whole, **so** subtrahend **to** subtrahend, also remainder to remainder **will be, as** whole **to** whole.[9]

Thus, whether we are working with arbitrary magnitudes or numbers, we have different ways of expressing what seems to be the same thing. We can say in words that

1) A, B, C, and D are proportional, or
2) A is to B as C is to D, or
3) A has the same ratio to B that C has to D.

I do not see an important distinction to make between these modes of expression as such. We might abbreviate any of them by writing

$$A : B :: C : D. \tag{3.1}$$

However, I think it is important not to write

$$A : B = C : D. \tag{3.2}$$

Neither definition of proportion describes an equation of two things. In a proportion, two ratios are not *equal,* but the *same.*[10] This is related to the fact that a ratio is not a thing that can be drawn in a diagram; it is a relation between two things in a diagram—two things that might have the same relation to one another that two other things have. (See §4.1, p. 109, for possible exceptions to the rule that a ratio cannot be depicted.)

The notion of equality does however appear in the definitions of proportion. I used the expression equally multiple in translating the Book-VII definition, and equimultiples in the Book-V definition; but these stand for the same Greek phrase ἰσάκις πολλαπλάσιος, which appears in singular form in Book VII, plural in Book V. Heath uses "same multiple" for my equally multiple; but I have followed Heath in using the word "equimultiples" in the other case.

---

[9] Heath uses "number subtracted" for my subtrahend; but I remember being taught the latter word in third grade, along with its correlate, "minuend" (the number to be diminished). According to the *OED*, the two words go back three centuries in English.

[10] Taisbak makes the same observation in explaining his own reluctance to use the sign of equality in a proportion [91, p. 55].

Like "unit," the word "equimultiple" seems to have been coined for translating Euclid: the earliest example in the *Oxford English Dictionary* is again from Billingsley's 1570 translation. In any case, Euclid's ἰσάκις is an adverb derived from the adjective ἰσός "equal"; it is not the adjective itself. The Book-V definition of proportionality does not describe things that are equal, though both definitions describe multiplying two things equally. In Greek as in English, it appears there is no adverb "samely." (The *OED* lists an *adjective* "samely," meaning "without variety; monotonous"; the earliest illustrative quotation is from 1799.)

Instead of (3.1), much less than (3.2) should we write

$$A/B = C/D, \tag{3.3}$$

since A/B suggests a fraction, and none is indicated by Euclid's definition. Still, as Collingwood says in the Introduction to his first book, *Religion and Philosophy* [18, p. xvii],

> I am afraid we cannot escape the difficulty by any method so simple as recourse to the dictionary. The question is not what words we use, but what we mean by them.[11]

What do we mean, what do we *want* to mean, by writing (3.1), (3.2), or (3.3) to express a proportion of numbers according to Euclid's definition? This is what we are going to investigate.

3.3 Numbers and sets

In addition to distinguishing equality from identity, we should distinguish measurement from division. Given twelve apples, we can describe the same operation in two ways: we can *measure* the twelve

[11] For the record, the difficulty that Collingwood happens to be addressing has nothing to do with mathematics, but lies with his assertion, "every one has to some degree that unified life of all the faculties which is a religion. He may be unconscious of it . . . But the thing, in some form, is necessarily and always there . . . " An imagined interlocutor objects, "But at least, that is not the way we use the word; and you can't alter the use of words to suit your own convenience."

apples by three apples, or we can *divide* them into four equal parts. Euclid refers to dividing, as distinct from measuring, in the sixth definition of Book VII:

Ἄρτιος ἀριθμός ἐστιν ὁ δίχα διαιρούμενος.
An **even number** is one that is divided in twain.

I use the archaic "twain" here because it is usually seen only in the phrase "in twain," and this translates the single Greek adverb δίχα.[12]

Euclid's numbers seem to be indistinguishable from our finite sets with at least two elements. After the definition of unit or unity in the *Elements* (quoted in §3.1, p. 67), there comes,

Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος.
And a **number** is a multitude of unities.

In addition to the expression multitude, which Heath uses, other possible translations of πλῆθος are "mass, throng, crowd" [68]. We use the word "number" in this way too, as when we say that a number of people are marching in the street.

In this sense, one is not a number. Thus we have, in the eleventh, twelfth, and thirteenth definitions at the head of Book VII:[13]

> **Πρῶτος ἀριθμός** ἐστιν ὁ μονάδι μόνῃ μετρούμενος. **Πρῶτοι πρὸς ἀλλήλους ἀριθμοί** εἰσιν οἱ μονάδι μόνῃ μετρούμενοι κοινῷ μέτρῳ. **Σύνθετος ἀριθμός** ἐστιν ὁ ἀριθμῷ τινι μετρούμενος. // A **prime number** is a number measured by unity alone. **Numbers prime to one another** are numbers measured only by unity as a common measure. A **composite number** is a number measured by some number.

Every number is measured by unity; if this were a number too, then by definition every number would be composite.

We did question the authenticity of the definition of unity; we may do the same for other definitions. But consider Proposition 16 of Book IX:

[12] In the *OED* under TWAIN, the phrase "in twain" merits its own definition: "into two parts or pieces, in two, asunder." Of the Old English numeral for two, "twain" represents the masculine form *tweġen*; "two," the feminine and neuter *twā* and *tū* [51].

[13] By the numbering of Heiberg's Latin, and of Heath, as in note 8, page 71.

Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, οὐκ ἔσται ὡς ὁ πρῶτος πρὸς τὸν δεύτερον, οὕτως ὁ δεύτερος πρὸς ἄλλον τινά. // If two numbers be prime to one another, it will not be the case that the first is to the second as the second is to some other number.

Unity is prime to every number, and unity will be to a number as that number is to its product with itself. Thus unity is not a number in the sense of Proposition IX.16.

On the other hand, Proposition VII.15 concerns a unit and three numbers, given as Α, ΒΓ, Δ, and ΕΖ, where Α is unity; but Δ is described as the *third* number; ΒΓ, the second; and ΕΖ, the fourth. (The Greek is on page 103.) Implicitly then, Α is the first of *four* numbers, although it is unity.[14]

As the status of unity is ambiguous, so is that of measurement. In the given definitions of prime and composite numbers, being measured means being measured by some *other* number, and thus being measured *a number of times.* Elsewhere, a number is allowed to measure itself. For example, Proposition VII.2 is the problem of finding the greatest common measure of two numbers that are not prime to one another. If one of the numbers measures the other, then it is the greatest common measure, since, as Euclid notes, it also measures itself. In this sense, a prime number is a number measured only by unity *and itself.*

Throughout the number-theoretic books of the *Elements,* numbers are diagrammed as bounded straight lines, or what are today called line segments. These are not obviously sets of units. We shall consider this feature of the *Elements* in §3.5 (p. 82); for now I note a curious move that Euclid makes in proving Proposition VII.8. The enunciation is,

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρη ᾖ, ἅπερ ἀφαιρεθεὶς ἀφαιρεθέντος, καὶ ὁ λοιπὸς τοῦ λοιποῦ τὰ αὐτὰ μέρη ἔσται, ἅπερ ὁ ὅλος τοῦ ὅλου. // If a number be the very parts of a number that a subtrahend is

[14]Mueller too [69, p. 58] observes that Euclid is not consistent in distinguishing numbers from units. Mueller gives the example that Proposition VII.12, stated in terms of numbers, is applied to units in the proof of Proposition 15.

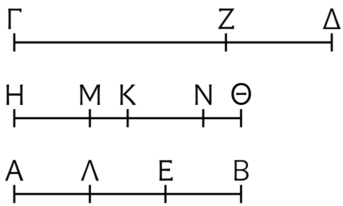

Figure 3.1. Proposition VII.8

> of a subtrahend, also the remainder will be the very same parts of the remainder that the whole is of the whole.[15]

In our symbols, if $A : B :: C : D$, at least according to the "parts" condition in Euclid's definition of a proportion of numbers, then

$$A - C : B - D :: A : B,$$

by the same condition. In Euclid's proof, the four numbers are the four line segments AB, ΓΔ, AE, and ΓZ, with AB assumed to be the same parts of ΓΔ that the subtrahend AE is of the subtrahend ΓZ. The segments are drawn as in Figure 3.1, with E lying on AB, and Z on ΓΔ. We want to break up AB into parts, each equal to the same part of ΓΔ; and we want to break up AE into the equally numerous parts of ΓZ. Doing this directly makes the diagram complicated; so Euclid takes HΘ equal to AB and divides *it* instead, along with AE. Specifically, HK and KΘ are the parts of ΓΔ, and AΛ and ΛE, equally numerous, are each that same part (τὸ αὐτὸ μέρος) of ΓZ; and HM and KN are made equal to AΛ and ΛE respectively. Thus each of HM and KN is the same part of ΓZ that each of HK and KΘ is of ΓΔ; hence, by Proposition VII.7, each of the remainders MK and NΘ is that same part of the remainder ZΔ. Hence MK and NΘ together are the same *parts* of ZΔ that HK and KΘ together are of ΓΔ; but these sums are equal to the remainder EB and to AB, respectively. The point for now is that HΘ is different from AB as

[15] On subtrahend see note 9, page 73.

a set of units; but the two sets are equipollent, or as Euclid says, equal.[16]

The above definition of evenness of a number is meaningful for an arbitrary set, possibly infinite. A set B is even, just in case it can be divided in two, in the sense of having a subset A for which there is a bijection f from A to $B \smallsetminus A$. Then the collection $\big\{\{x, f(x)\} \colon x \in A\big\}$ is a partition of B into two-element subsets: this means B is *measured* by a two-element set, a dyad.

Conversely, being measurable by a dyad implies being even, but only by the principle whereby one sock from each pair in a collection of pairs of socks can be selected, as in Bertrand Russell's illustration of the Axiom of Choice [86, pp. 92–3].[17] In fact the needed principle is weaker than the full Axiom of Choice, but stronger than bare Zermelo–Fraenkel set theory [53, 5.4 & 7.4]. Thus there is mathematical reason to distinguish between measuring by two and dividing in two, as Euclid's language does.

[16] Mueller too observes that in Euclid, equipollent numbers are not always the same: "there is no unique 2 or 3; any pair of units is a 2, for example" [69, p. 59]. If I understand him, Mueller finds this problematic for the foundations of Euclid's mathematics: "one runs into difficulty in trying to say in a noncircular way what a given positive integer is." Moderns, it seems, avoid this difficulty by characterizing numbers axiomatically: they are generated from a unique unit 1 by repeated application of a one-to-one "successor operation" that does not have 1 itself in its range. In particular then, there is a unique 2, which is the successor of the unique 1. In this case, it seems to me that one might worry about how number theory can apply to this or that number *of things.* This is a theme of Mazur's article [64], quoted on page 57.

[17] Many people writing about the Axiom of Choice give the shoes-and-socks illustration, attributing it to Russell, but without giving a more precise reference. Wikiquote (`en.wikiquote.org/wiki/Bertrand_Russell`) was lacking a precise reference, until I supplied one to the late work *My Philosophical Development* (1959) on July 8, 2015, having been directed to it by Google. Perhaps there is an earlier reference; but the discussion of the "multiplicative axiom" in *Principia Mathematica* [97, Part II, Summary of Section D, pp. 500–4] is not it.

3.4 Parts

In considering Proposition VII.8, we have sketched a proof of one case in which a number is found to be the *same parts* of another number that a third is of a fourth. But what really does this conclusion mean?

When the phrase is "same part" rather than "parts," the meaning seems clear enough. If A, B, C, and D are numbers, or possibly units, and A is the *same part* of B that C is of D, then A must *measure B the same number of times* that C measures D: that is, for some multiplier n,

$$A \cdot n = B \ \ \& \ \ C \cdot n = D,$$

or (as on p. 43),

$$\underbrace{A + \cdots + A}_{n} = B \ \ \& \ \ \underbrace{C + \cdots + C}_{n} = D.$$

Here we can read $A \cdot n$ and $\underbrace{A + \cdots + A}_{n}$ as "A composed n times" (see page 98 on the terminology). We have now also that B is the same multiple of A that D is of C. Thus, by Definition VII.20, we have both of the proportions

$$A : B :: C : D, \qquad\qquad B : A :: D : C.$$

It remains to understand what being the "same parts" means in the definition.

By Definition VII.4 (p. 25), if A is less than B, but does not measure B, then A is not part of B, but is "parts" of B. Heath's entire comment on the definition of "parts" is the following.

> By the expression *parts* (μέρη, the plural of μέρος) Euclid denotes what we should call a *proper fraction.* That is, a *part* being a submultiple, the rather inconvenient term *parts* means any number of such submultiples making up a fraction less than unity. I have not found the word used in this special sense elsewhere, e.g. in

> Nicomachus, Theon of Smyrna or Iamblichus, except in one place of Theon (p. 79, 26) where it is used of a proper fraction, of which $\frac{2}{3}$ is an illustration.

This note is misguided in two ways. It ignores the fact that "parts" is defined by simple negation, as when we define an irrational number to be a real number that is not rational. We might then say that an irrational number is one whose decimal expansion is neither finite nor repeating; but this would have to be proved. In commenting on the definition of "parts," Heath is probably thinking ahead to Proposition VII.4, whose enunciation we also saw in §1.3 (p. 25): "Every number is of every number, the less of the greater, either a part or parts." The demonstration of this proposition is not a simple appeal to the definition of "parts." Thus that definition at the head of Book VII is not really a definition, but is a kind of summary of what is to come in the book.

We shall consider the demonstration of Proposition VII.4 in §3.7 (p. 90) below. Meanwhile, let us note that, *pace* Heath, the proposition is not likely to be about fractions. According to David Fowler [37, §7.1(b), pp. 227–9],

> We have no evidence for any conception of common fractions p/q and their manipulations such as, for example, $p/q \times r/s = pr/qs$ and $p/q + r/s = (ps + qr)/qs$, in Greek mathematical, scientific, financial, or pedagogical texts before the time of Heron and Diophantus; and even the fractional notations and manipulations found in the Byzantine manuscripts of these late authors may have been revised and introduced during the medieval modernization of their minuscule script. Among the thousands, possibly the tens of thousands, of examples of fractions to be found in *contemporary* Egyptian (hieroglyphic, hieratic, and demotic), Greek, and Coptic texts, all but a few isolated examples in five texts . . . use throughout the following 'Egyptian['] system for expressing fractions . . . [18]
>
> We take the basic sequence of the *arithmoi*:
>
> two, three, four, five, ...,

[18] Widely spaced dots " . . . " are my ellipses; narrowly spaced dots "..." denoting continuations of sequences are Fowler's.

represented in Greek by the letters $\beta, \gamma, \delta, \epsilon, \ldots$, and convert it to the sequence

half, third, quarter, fifth, ...,

where, after the exceptional cases of the first few terms, for which special symbols . . . are assigned, the derived symbol is . . . usually transcribed as an accent, $\acute{\gamma}, \acute{\delta}, \acute{\epsilon}, \ldots$, or a prime, $\gamma', \delta', \epsilon', \ldots$

The sequence of parts starts with $\acute{\beta}$, 'the two parts', τὰ δύο μέρη, an expression for 'two-thirds'. However extraordinary it may seem to us, it is an incontrovertible fact that the sequence of parts starts with two-thirds . . . (This is an additional reason for avoiding the name 'unit fractions' for the sequence $\acute{\beta}, \angle, \acute{\gamma}, \acute{\delta}, \ldots$) . . .

More complicated fractions than simple parts are expressed as sums of an integer and *different* simple parts . . .

Fowler goes on to describe "division tables" [37, §7.2(a), pp. 234 f.]:

our common fraction $\frac{12}{17}$ . . . would be expressed in a division table . . . as

$\tau\hat{\omega}\nu$	$\iota\beta$	[$\tau\grave{o}$ $\acute{\iota}\acute{\zeta}$]	$\angle\acute{\iota}\acute{\beta}\acute{\iota}\acute{\zeta}\acute{\lambda}\acute{\delta}\acute{\nu}\acute{\alpha}\acute{\xi}\acute{\eta}$
of the	12	[the 17th is]	$\acute{2}\acute{1}\acute{2}\acute{1}\acute{7}\acute{3}\acute{4}\acute{5}\acute{1}\acute{6}\acute{8}$

for what we would write as

$$\frac{12}{17} = \frac{1}{2} + \frac{1}{12} + \frac{1}{17} + \frac{1}{34} + \frac{1}{51} + \frac{1}{68}.$$

I think a better "translation" of the Greek $\angle\acute{\iota}\acute{\beta}\acute{\iota}\acute{\zeta}\acute{\lambda}\acute{\delta}\acute{\nu}\acute{\alpha}\acute{\xi}\acute{\eta}$ would be something like

$$\acute{2}\ \acute{10}\ \acute{2}\ \acute{10}\ \acute{7}\ \acute{30}\ \acute{4}\ \acute{50}\ \acute{1}\ \acute{60}\ \acute{8}$$

(or the initial $\acute{2}$ might be replaced with the Greek symbol $\angle$ for a half). In any case, possibly the definition of "parts" in the *Elements* serves as a reminder to the reader that, even though 12 is not a *part* of 17, it is still *parts* of 17, namely the half, and the twelfth, and the seventeenth, and the thirty-fourth, and the fifty-first,[19] and the

[19] I note the oddity of English in using the exceptional form "first" here, rather than "oneth." First means foremost, but "fifty-foremost" makes no sense. Another oddity is that "foremost" was originally *formest,* a superlative form of FORME, which has an article in the *OED,* though its latest illustrative quotation is from 1523. The *-me* ending is already superlative [65, p. 163],

sixty-eighth. In this case, Euclid's Proposition VII.4 will give a new meaning to the notion of "parts."

3.5 Music

As noted above (p. 76), in the diagrams of the propositions in Books VII–IX, numbers are bounded straight lines, or today's line segments. Heath remarks on this after Proposition VII.1:

> the representation in Books VII. to IX. of numbers by straight lines is adopted by Heiberg from the MSS. The method of those editors who substitute *points* for lines is open to objection because it practically necessitates, in many cases, the use of specific numbers, which is contrary to Euclid's manner.

Heath is right that points should not be used, but if I understand him correctly, he is wrong about the reason. I think he means that Euclid's propositions are *general,* but that this would be belied by diagrams that indicated specific numbers. However, as they are, Euclid's proofs are often *not* general in form, regardless of the diagram. One example is the proof of Proposition VII.8 sketched above (p. 76): the proposition is about being any number of parts, but the proof is about being *two* parts. In the proof of Proposition VII.1, showing one use of what we call the Euclidean algorithm, there are *three* alternations of subtractions, not some unspecified number of them. One must simply understand that there is nothing special about three alternations as being three. Likewise, throughout Book I of the *Elements,* specific triangles in the diagrams are to be understood as general. Specific numbers of dots in a diagram could be understood as general.

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like the *-st* in "first," making *forme* cognate with the Latin *prīmus.* Thus *forme* also means first, and "foremost" is a double superlative, as "lesser" is a double comparative (see note 6, p. 25). There was similarly an Old English word *hindema,* with a superlative ending *-dema* as in the Latin *optimus* [88, Hind]; but "hindmost" is hind + most, not derived from *hindema,* though they have the same meaning.

However, if unities could always be thought of as points, then it would be obvious that multiplication is commutative. Four rows of dots, of three dots each, obviously have as many dots as three rows, of four dots each (Figure 3.2). With diagrams of dots, there

Figure 3.2. Multiplication of points

would be no need to prove commutativity of multiplication, which is Proposition VII.16. But it is *not* obvious that four straight lines, of 3 units each, will have together the same length as three straight lines, of 4 units each (Figure 3.3).

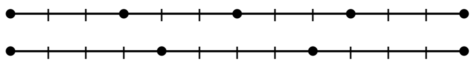

Figure 3.3. Multiplication of straight lines

Another possible reason why Euclid's numbers are straight lines is that lyre strings are like straight lines, and Euclid's study of numbers is inspired by music. According to Andrew Barker in *Greek Musical Writings* [8, p. 190], the ancient work called *Sectio Canonis* (Κατατομὴ κανόνος) is

> attributed to Euclid in the manuscripts, and by Porphyry, who quotes from it at length . . . Parts of the treatise are also quoted by Boethius . . . [T]he attribution has been debated . . . There are no good reasons, however, for denying Euclid's authorship of the main part of the treatise, at least as much as Porphyry quotes . . .

Fowler is more skeptical. He discusses the *Sectio Canonis* thoroughly [37, pp. 138–46]; I would just note that the chief burden of the treatise seems to be the following. An interval of musical notes somehow corresponds to a ratio of numbers, and those ratios are of three possible kinds, as are ratios in the *Elements.* In the

Elements, a ratio can be multiple, part, or parts. According to the *Sectio Canonis* [96, p. 149],

> πάντα δὲ τὰ ἐκ μορίων συγκείμενα ἀριθμοῦ λόγῳ λέγεται πρὸς ἄλληλα, ὥστε καὶ τοὺς φθόγγους ἀναγκαῖον ἐν ἀριθμοῦ λόγῳ λέγεσθαι πρὸς ἀλλήλους· τῶν δὲ ἀριθμῶν οἱ μὲν **ἐν πολλαπλασίῳ λόγῳ** λέγονται, οἱ δὲ **ἐν ἐπιμορίῳ,** οἱ δὲ **ἐν ἐπιμερεῖ,** ὥστε καὶ τοὺς φθόγγους ἀναγκαῖον ἐν τοῖς τοιούτοις λόγοις λέγεσθαι πρὸς ἀλλήλους. // All things composed of parts are said to be to one another in the ratio of a number [to a number]; so notes too must be said to be to one another in the ratio of a number. Some numbers are said to be **in multiple ratio,** others in **part-again,** still others in **parts-again** [ratio]; so notes too must be said to be in in these ratios to one another.

Numbers A and B are in

1) *multiple* ratio, if A is a multiple of B;
2) *part-again* ratio, if $A - B$ measures B, so that A is the whole of B with a part added;
3) *parts-again* ratio, otherwise.

The latter two kinds of ratio—ἐπιμόριος and ἐπιμερής—are also called in English "superparticular" and "superpartient," as for example in D'Ooge's translation of Nicomachus [73, pp. 52–3, 215, & 220]; but Barker uses the words "epimoric" and "epimeric" [8, p. 192, p. 43 n. 63]. The quoted passage from the *Sectio Canonis* is given in the big Liddell–Scott–Jones lexicon [62] as an example of the use of ἐπιμόριος. This word is derived from μόριον rather than μέρος; but either of these means "part."

The passage continues somewhat obscurely.

> τούτων δὲ οἱ μὲν πολλαπλάσιοι καὶ ἐπιμόριοι ἑνὶ ὀνόματι λέγονται πρὸς ἀλλήλους. // Of these, the multiple [ratios] and the part-again [ratios] to one another are said with a single name.

Barker's translation is, "of these, the multiple and the epimoric are spoken of in relation to one another under a single name"; Fowler changes "in relation" to "with respect." Barker suggests that "single name" refers to the single words like τριπλάσιος ("triple") and ἐπίτριτος ("third-again") that name specific multiple and part-again

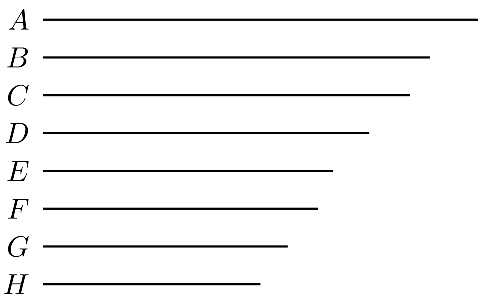

Figure 3.4. Lyre strings as numbers

ratios. Fowler goes further, suggesting that "single name" alludes to the single number that specifies one of these ratios, as the number three specifies the triple and third-again ratios. Fowler's suggestion would seem to be partially corroborated by the last three propositions of Book VII of the *Elements,* which we shall consider in §4.1 (p. 109).

Meanwhile, suppose the strings of an eight-stringed lyre[20] are represented by the numbers A through H, as in Figure 3.4. As I understand Barker in *The Science of Harmonics in Classical Greece* [9, p. 13], we can consider A, D, E, and H as fixed, while the other strings can vary according to the chosen musical mode. The interval AD or EH is a fourth or diatesseron (διὰ τεσσάρων), spanning, as it does, four strings; AE or DH is a fifth or diapente (διὰ πέντε); AH, an octave or diapason (διὰ πασῶν). These intervals are all *concords,* and the octave is composed of a fourth and a fifth. The double octave is also a concord, but the double fourth and the double fifth are not.

So much is observed or at least assumed. Then, following Euclid, or whoever wrote the *Sectio Canonis*, we propose the axiom that

[20] Apparently such lyres existed, although the traditional lyre had seven strings [9, pp. 15 & 276].

a concord corresponds to a ratio of numbers that is either multiple or part-again. As the modern commentators point out, this axiom does not correspond exactly to the observed reality. Nonetheless, the author of the *Sectio Canonis* uses the axiom to argue as follows. Since the double octave has a mean (namely the octave), it cannot be part-again, since a part-again ratio can have no mean (in modern terms, $(n+1)/n$ is never a square). Thus the double octave must be a multiple ratio, and therefore the octave itself must be multiple (since n^2 never measures $(n+1)^2$). Similarly, the fifth and the fourth cannot be multiple, since their doubles are not; so the fifth and the fourth are part-again. The least multiple ratio, namely the double, is composed of the two greatest part-again ratios, namely the half-again (ἡμιόλιος, half-whole or "hemiolic") and the third-again (ἐπίτριτος, "epitritic"). Therefore these three ratios must correspond respectively to the octave, the fifth, and the fourth.

Again, it may be questioned whether there is a purely mathematical proof of this correspondence between musical intervals and numerical ratios. The point for now is the importance given to two kinds of numerical ratios: the ratio that obtains between the greater A and the less B when B measures A, or when B is almost equal to A, but the difference measures B. These are situations when the *Euclidean Algorithm* concludes in one or two steps.

3.6 The Euclidean Algorithm

The first proposition of Book VII of the *Elements* is a theorem in the sense of Pappus (§2.2.6, p. 54); the next two are problems. All three of these propositions involve the so-called Euclidean Algorithm. The enunciation of the theorem is,

> Δύο ἀριθμῶν ἀνίσων ἐκκειμένων, **ἀνθυφαιρουμένου** δὲ ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος, ἐὰν ὁ λειπόμενος μηδέποτε καταμετρῇ τὸν πρὸ ἑαυτοῦ, ἕως οὗ λειφθῇ μονάς, οἱ ἐξ ἀρχῆς ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ἔσονται. // Two numbers being given, and the less **being alternately subtracted** from the greater

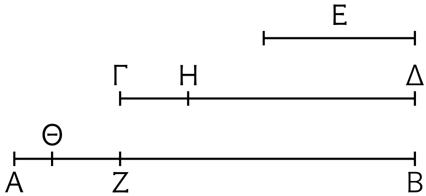

Figure 3.5. Proposition VII.1

> continually, if the remainder never measures the previous number until unity is left, the original numbers will be prime to one another.

In the Greek, the phrase being alternately subtracted is one word, which is a passive participle of the verb ἀνθυφαιρέ-ω. This is the source of the feminine noun ἀνθυφαίρεσις, which I shall render as **anthyphaeresis.**[21] In the exposition of the proposition (in the sense of Proclus, p. 46), unequal numbers AB and ΓΔ are given, as in Figure 3.5, which is Heiberg's diagram rotated counterclockwise through a right angle. In the demonstration, which depends on the diagram,

> ὁ μὲν ΓΔ τὸν BZ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ZA, ὁ δὲ AZ τὸν ΔH μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν HΓ, ὁ δὲ HΓ τὸν ZΘ μετρῶν λειπέτω μονάδα τὴν ΘA. // Let ΓΔ, measuring BZ, leave less than itself, ZA; then let AZ, measuring ΔH, leave less than itself, HΓ; then let HΓ, measuring ZΘ, leave a unit, ΘA.

It is not mentioned how many times ΓΔ measures BZ; or AZ, ΔH; or HΓ, ZΘ; but the unit ΘA measures the number HΓ that number of times. The four *numbers of times* here form a sequence, $[n_0, n_1, n_2, n_3]$, which we we may call the **anthyphaeretic sequence** of AB and ΓΔ.

Fowler calls the anthyphaeretic sequence a "ratio," or more precisely "anthyphairetic ratio" [37, §1.3, p. 24, & §10.1, p. 367]. He

[21] Other writers use the transliteration "anthyphairesis," but I follow the tradition of writing the Latin diphthong *ae* for the Greek diphthong αι. Presumably one will pronounce the word as something like "An thigh FEAR iss iss."

does this, because by one possible definition, A, B, C, and D are proportional just in case the pairs (A, B) and (C, D) have the same anthyphaeretic sequence. It does not matter whether A, B, C, and D are numbers or arbitary magnitudes. In modern terms, if we compute

$$n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3}}} = \frac{a}{b},$$

then the ratio of ΑΒ to ΓΔ above is that of a to b. Moreover, a and b will automatically be relatively prime (assuming we compute them in the obvious way, replacing $n_2 + 1/n_3$ with $(n_2n_3 + 1)/n_3$ and so forth). For arbitary magnitudes, the anthyphaeretic sequence might be infinite; however, if the magnitudes are lengths constructible with straightedge and compass, then the sequence will be periodic.

The historical situation is laid out by Thomas at the end of the first of the two Loeb Classical Library volumes, *Selections Illustrating the History of Greek Mathematics* [92, pp. 504–9]. According to the first proposition of Book VI of the *Elements,*

> Τὰ τρίγωνα καὶ τὰ παραλληλόγραμμα τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις. // Triangles and parallelograms that are under the same height are to one another as their bases.

In the *Topics,* Aristotle uses this result (or at least the part about parallelograms) as an example of something that is immediately clear, once one has the correct definition; and the definition of "same ratio" is "having the same *antanaeresis* (ἀνταναίρεσις)." In a comment on the passage, Alexander of Aphrodisias observes that Aristotle uses the word "antanaeresis" for anthyphaeresis. In 1933, Oskar Becker observed that Aristotle and Alexander could be alluding to the anthyphaeretic definition of proportion described above. It is thus reasonable to suppose that Euclid was aware of the possibility of an anthyphaeretic theory of proportion.

Proposition VII.2 is a *problem*:

> Δύο ἀριθμῶν δοθέντων μὴ πρώτων πρὸς ἀλλήλους τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν. // Two numbers not prime to one another being given, to find their greatest common measure.

We may ask why this proposition is separated from Proposition 1, since each of these propositions involves an application of the Euclidean Algorithm, that is, anthyphaeresis. But Proposition 2 itself considers two cases: when the the less of given numbers measures the greater, and when it does not. We see here the practice of using language as precisely as possible—a practice often abjured today in mathematics, in favor of simplicity. If a straight line is erected on another straight line, forming two angles, today we say that the sum of the angles is equal to that of two right angles; for Euclid, in Proposition I.13, either this is so, *or* each of the angles is already a right angle. Today we have an algorithm, the Euclidean Algorithm, for finding greatest common divisors; Euclid has an algorithm for verifying that two numbers are relatively prime, *and* for finding their greatest common measure when they are not.

There is now a break in the usual pattern of exposition. Proposition VII.2 has a πορίσμα, a porism: in the words of Proclus, "a kind of windfall or bonus in the investigation" [92, p. 481], namely:

> Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν ἀριθμὸς δύο ἀριθμοὺς μετρῇ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει. // From this then it is clear that, if a number measure two numbers, it will also measure their greatest common measure.

As a corollary follows from the enunciation of a theorem, a porism follows from the proof. The term could be used more widely today, as for example to label a generalization of a theorem, when it turns out that the same proof yields the generalization.

Proposition VII.3 is to find the greatest common measure of three given numbers. There seems to be no reason to look further at this before moving on to Proposition 4.

3.7 Part or Parts

We have seen the enunciation of Proposition VII.4 in §§1.3 & 3.4 (pp. 25 & 80); it is that the less number is part or parts of the greater number. If being *parts* of a number just meant not being *part* of the number, that is, not measuring the number, then Proposition 4 would be immediate. Euclid does not treat it as immediate, and so the real meaning of *parts* must be more subtle. We should consider what Euclid actually says. After the enunciation of Proposition 4 come the exposition and specification:

> Ἔστωσαν δύο ἀριθμοὶ οἱ Α, ΒΓ, καὶ ἔστω ἐλάσσων ὁ ΒΓ· λέγω, ὅτι ὁ ΒΓ τοῦ Α ἤτοι μέρος ἐστὶν ἢ μέρη. // Let the two numbers be A and ΒΓ, and let the less be ΒΓ. I say that ΒΓ is, of A, either part or parts.

The demonstration then considers two cases:

> Οἱ Α, ΒΓ γὰρ ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὔ. // For A and ΒΓ are either prime to one another, or not.

The two cases thus correspond to Propositions VII.1 and VII.2. Here is the first case:

> ἔστωσαν πρότερον οἱ Α, ΒΓ πρῶτοι πρὸς ἀλλήλους. διαιρεθέντος δὴ τοῦ ΒΓ εἰς τὰς ἐν αὐτῷ μονάδας ἔσται ἑκάστη μονὰς τῶν ἐν τῷ ΒΓ μέρος τι τοῦ Α· **ὥστε** μέρη ἐστὶν ὁ ΒΓ τοῦ Α. // First suppose A and ΒΓ are prime to one another. ΒΓ being divided into the units in itself, each unit of those that are in ΒΓ will be some part of A. **Thus** ΒΓ is parts of A.

The word thus here translates Euclid's ὥστε. As "thus" can mean either *therefore* or *in this way,* so can ὥστε. I propose that in the present instance it has the latter meaning. That is, since A and ΒΓ are coprime, ΒΓ is *parts* of A, *in the sense that* each of its units is a part of A.

Sometimes, indeed, ὥστε simply means *therefore,* only less emphatically; Heath translates it as "so that." Euclid's usual word to indicate a logical conclusion is ἄρα, used *postpositively*—not at the beginning of a clause; but occasionally ὥστε is used in place of this, only *prepositively*—at the beginning. For example, Proposition VII.34 is to find the least common multiple of two numbers, and Proposition VII.35 is that this least common multiple measures every common multiple. Proposition VII.36 is to find the least common multiple of three numbers, given as Α, Β, and Γ. One lets Δ be the least common multiple of Α and Β. In case Γ does not measure this, one lets Ε be the least common multiple of Γ and Δ, so that Ε is a common multiple of Α, Β, and Γ. If it is not the least such, one lets Ζ be less. Euclid's argument continues as follows.

> ἐπεὶ οἱ Α, Β, Γ τὸν Ζ μετροῦσιν, καὶ οἱ Α, Β **ἄρα** τὸν Ζ μετροῦσιν· καὶ ὁ ἐλάχιστος **ἄρα** ὑπὸ τῶν Α, Β μετρούμενος τὸν Ζ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Α, Β μετρούμενός ἐστιν ὁ Δ· ὁ Δ **ἄρα** τὸν Ζ μετρεῖ. μετρεῖ δὲ καὶ ὁ Γ τὸν Ζ· οἱ Δ, Γ **ἄρα** τὸν Ζ μετροῦσιν· **ὥστε** καὶ ὁ ἐλάχιστος ὑπὸ τῶν Δ, Γ μετρούμενος τὸν Ζ μετρήσει. // Since Α, Β, and Γ measure Ζ, **therefore** also Α and Β measure Ζ; **therefore** also the least [number] measured by Α and Β will measure Ζ. But the least [number] measured by Α and Β is Δ. **Therefore** Δ measures Ζ. But also Γ measures Ζ; **therefore** Δ and Γ measure Ζ; **thus** also the least [number] measured by Δ and Γ will measure Ζ.

The last ὥστε replaces ἄρα, perhaps because the indicated kind of inference (that a least common multiple measures every common multiple) has already been seen in the proof. Then ὥστε indicates an easy result. This is seen also in Book I, where ὥστε is used three times, in Propositions 3, 35, and 36, to indicate a result of the form

$$A = C \text{ and } B = C, \textbf{ thus } A = B;$$

and ὥστε is used in Propositions 7, 41, and 48, for similarly easy results.

But in Proposition I.4, which we looked at in §2.3 (p. 54), ὥστε is used to indicate the conclusions that Γ falls on Ζ, and ΒΓ on ΕΖ, and triangle ΑΒΓ on ΔΕΖ; and in Proposition I.8, ὥστε indicates again that ΑΒΓ coincides with ΔΕΖ. In short, ὥστε indicates that something happens, *thus*.

We see ὥστε again in the second case of Proposition VII.4, or at least in its second subcase (corresponding to the second case of Proposition VII.2). The case begins:

> Μὴ ἔστωσαν δὴ οἱ Α, ΒΓ πρῶτοι πρὸς ἀλλήλους· ὁ δὴ ΒΓ τὸν Α ἤτοι μετρεῖ ἢ οὐ μετρεῖ. // Now suppose Α and ΒΓ are not prime to one another. Then ΒΓ either measures Α or does not

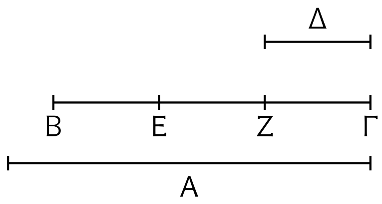

Figure 3.6. Proposition VII.4

measure.

εἰ μὲν οὖν ὁ ΒΓ τὸν Α μετρεῖ, μέρος ἐστὶν ὁ ΒΓ τοῦ Α. // If ΒΓ measures Α, ΒΓ is a part of Α.

The second subcase is diagrammed in Figure 3.6.

εἰ δὲ οὔ, εἰλήφθω τῶν Α, ΒΓ μέγιστον κοινὸν μέτρον ὁ Δ, καὶ διῃρήσθω ὁ ΒΓ εἰς τοὺς τῷ Δ ἴσους τοὺς ΒΕ, ΕΖ, ΖΓ. καὶ ἐπεὶ ὁ Δ τὸν Α μετρεῖ, μέρος ἐστὶν ὁ Δ τοῦ Α· ἴσος δὲ ὁ Δ ἑκάστῳ τῶν ΒΕ, ΕΖ, ΖΓ· καὶ ἕκαστος **ἄρα** τῶν ΒΕ, ΕΖ, ΖΓ τοῦ Α μέρος ἐστίν· **ὥστε** μέρη ἐστὶν ὁ ΒΓ τοῦ Α. // If not, let the greatest common measure Δ of Α and ΒΓ be taken, and let ΒΓ be divided into ΒΕ, ΕΖ, and ΖΓ, equal to Δ. Since Δ measures Α, Δ is a part of Α. And Δ is equal to each of ΒΕ, ΕΖ, and ΖΓ. **Therefore** each of ΒΕ, ΕΖ, and ΖΓ is a part of Α; **thus** ΒΓ is parts of Α.

The conclusion of the proposition repeats the enunciation, with—as is usual—the addition of ἄρα and a QED:

Ἅπας **ἄρα** ἀριθμὸς παντὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος ἤτοι μέρος ἐστὶν ἢ μέρη· ὅπερ ἔδει δεῖξαι. // **Therefore** every number is of every number, the less of the greater, either a part or parts; which is just what was to be shown.

The last verb, δεῖξαι "to be shown," is the normal ending for a *theorem,* as opposed to a *problem*, which would end with ποιῆσαι "to be done." Nonetheless, strictly speaking, Proposition VII.4 is neither a theorem nor a problem in the usual sense, because its enunciation alone gives us nothing that we can either contemplate or use. I read the *whole* proposition as an explanation of the definition of proportion. The proposition tacitly explains how to tell when a number A is the *same parts* of B that C is of D. Assuming that A

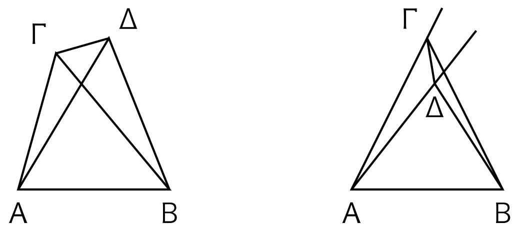

Figure 3.7. Proposition I.7, with the missing case

is not in fact *part* of B, and C is not part of D, we may conclude that A is the **same parts** of B that C is of D, just in case A and B are, respectively, the same multiples of their greatest common measure that C and D are of *their* greatest common measure.

If Euclid spelled out this definition, he might distinguish the case where, say, A and B are prime to one another. In this case, since the only common measure of A and B is unity, A is the same parts of B that C is of D, just in case $C = E \times A$ and $D = E \times B$, where E is the greatest common measure of C and D. (We shall look at the multiplications here symbolized in §3.8.)

Euclid does *not* spell out the details of the definition of having the same parts. This need not be considered unusual. In Book I of the *Elements,* there are several propositions where a thorough account of all possible cases is lacking. Euclid does not even bother to mention that there are other cases than the one he considers. We have no reason to think that he is not aware of the other cases. We looked earlier (p. 58) at Proposition I.5, which is that, in an isosceles triangle, not only (1) the base angles, but also (2) the exterior angles at the base, are equal to one another. A consequence of this is Proposition 7, which, as suggested earlier (p. 60), is that two different points on the same side of a straight line cannot have the same respective distances from its endpoints. In his proof by contradiction, Euclid considers only the case shown on the left of Figure 3.7, where neither of the two points Γ and Δ lies inside the triangle determined by the other point and the given straight line AB. The proof of this case needs only the first conclusion of Proposition 5: if $\mathsf{A\Gamma} = \mathsf{A\Delta}$, then

$$\angle\mathsf{B\Gamma\Delta} < \angle\mathsf{A\Gamma\Delta} = \angle\mathsf{A\Delta\Gamma} < \angle\mathsf{B\Delta\Gamma}, \qquad \mathsf{B\Gamma} \neq \mathsf{B\Delta}.$$

The other case, on the right of Figure 3.7, goes unmentioned, but is proved by means of the second conclusion of Proposition 5.

As discussed in §3.2 (p. 71), by definition, a number A is to B as C is to D, provided A is the same (1) multiple or (2) part or (3) parts of B that C is of D. This means, symbolically, according to my reading of Proposition 4,

1) for some multiplier n, both $A = B \cdot n$ and $C = D \cdot n$, or
2) for some multiplier n, both $A \cdot n = B$ and $C \cdot n = D$, or
3) for some multipliers k and ℓ, and some numbers E and F,

$$A = E \cdot k, \qquad B = E \cdot \ell, \qquad C = F \cdot k, \qquad D = F \cdot \ell,$$

where

$$E = \operatorname{gcm}(A, B), \qquad F = \operatorname{gcm}(C, D) \tag{3.4}$$

(where gcm means greatest common measure). Again, condition (3.4) is made as explicit as need be by Proposition 4.

Pengelley and Richman [77, p. 199] leave out Condition (3.4) when they formulate Euclid's definition of proportion. (The omission is repeated in [76, p. 870].) Then they observe that later proofs make use of the transitivity of "equality" of ratios without having proved it. However, the transitivity of this "equality" is immediate from the definition of proportion, properly understood. By definition, A is to B as C is to D, provided that something about pairs (X, Y) of numbers is the *same,* whether the pair be (A, B) or (C, D). The "something" that is the same: it cannot be the mere *existence* of a number Z and multipliers k and ℓ such that

$$X = Z \cdot k, \qquad Y = Z \cdot \ell. \tag{3.5}$$

Such Z, k, and ℓ exist for all pairs of numbers X and Y. We could try letting the desired "something" be the set of *all* pairs (k, ℓ) of multipliers such that, for some number Z, (3.5) holds. That is, if we introduce the notation

$$[X : Y] = \{(k, \ell) \colon \exists Z \ (X = Z \cdot k \ \ \& \ \ Y = Z \cdot \ell)\},$$

then we could try defining

$$A : B :: C : D \iff [A : B] = [C : D]. \tag{3.6}$$

This is close to the option labelled (ii') that Mueller proposes [69, p. 65], except that Mueller's proposal would replace the equality above with an inclusion:

$$A : B :: C : D \iff [A : B] \subseteq [C : D]. \tag{3.7}$$

This certainly cannot be the right definition. As Mueller observes, it would falsify the proportion $6 : 8 :: 3 : 4$, since $[6 : 8] = \{(6, 8), (3, 4)\}$, while $[3 : 4] = \{(3, 4)\}$. For the same reason, (3.6) is not right either. By the way though, (3.7) cannot be right for the more fundamental reason that the symmetry of sameness of ratio must be obvious. It must be immediate from the definition that

$$A : B :: C : D \iff C : D :: A : B.$$

By the account of Pengelley and Richman, which agrees with Mueller's option (i'), the definition of proportion is

$$A : B :: C : D \iff [A : B] \cap [C : D] \neq \varnothing. \tag{3.8}$$

With a bit less symbolism, this is that $A : B :: C : D$ if and only if $[A : B]$ and $[C : D]$ contain *the same element.* This formulation does preserve the reference to sameness found in Euclid's original definition (p. 71). It also has the benefit over (3.6) and (3.7) of being true. However, it does not specify *which* element is the same element of $[A : B]$ and $[C : D]$. As Euclid implies in Proposition VII.4, that same element must be *least* element (in the obvious ordering): the one pair (k, ℓ) such that

$$X = \operatorname{gcm}(X, Y) \cdot k, \qquad\qquad Y = \operatorname{gcm}(X, Y) \cdot \ell,$$

whether (X, Y) be (A, B) or (C, D).

If (3.8) were the definition of proportion, then Euclid would have to prove the transitivity of sameness of ratio. Pengelley and Richman resolve this problem by proving this transitivity. Mueller proposes to resolve it by proving the equivalence of his (i') and (ii'), our (3.7) and (3.8); and he says that, according to Zeuthen, this is in effect what Euclid does in Proposition VII.4. There is some confusion here, since by my reading of the logical notation of Mueller's (ii'), it is false. Seeing this did however bring out my own earlier confusion: I had earlier been thinking (wrongly) of (3.6) as being

$$A : B :: C : D \iff \{(k, \ell) \colon A \cdot \ell = B \cdot k\} = \{(k, \ell) \colon C \cdot \ell = D \cdot k\}.$$

This equivalence is true, and it obviously makes sameness of ratio an equivalence relation; but apparently it is not Euclid's actual definition.

To the modern reader accustomed to manipulating and reducing fractions, Euclid's meaning may not be entirely clear. But I think it is clear enough to his intended audience, to suit Euclid's purposes at least. He may not have *wanted* to tell them everything; but again, as noted in §3.4 (p. 79), he did not have to worry that they would be confused by the modern notion of a fraction.

Euclid is aware that he can define proportionality—sameness of ratio—by means of anthyphaeresis. He prefers to avoid this, perhaps in order to make proofs easier. Proposition VII.13 will be an example of this (p. 102). Meanwhile, for the arbitrary magnitudes considered in Book V, Euclid defines proportions as quoted in §3.2 (p. 72). Looking at arbitrary equimultiples is conceptually easier than looking at the potentially infinite sequences of numbers generated by anthyphaeresis. For one thing, even if we say A measures B, k times, with remainder C, Euclid has no notation for the multiplier k. Indeed, k here is not a *thing.* It is not, strictly speaking, a *number,* but a *numeral*; not a noun, but an adjective. It can be turned into a noun, as it will be in Book VII, when multiplication of numbers is defined; but there is no need to do this in Book V.

When anthyphaeresis is applied to *numbers,* finite sequences are generated. We apply anthyphaeresis to numbers A and B, we get a greatest common measure C. Then $A = C \cdot k$ and $B = C \cdot \ell$ for some k and ℓ, as Euclid knows. He does not say much about k and ℓ directly though. He does not say, for example, that they are relatively prime. He does not have good notation for them. We can write multipliers as we are doing, with minuscules; Euclid cannot.

Now, notation is not like house-building supplies. You cannot will bricks and mortar into existence; but if you find your notation inadequate, you just improve it. If Euclid was inhibited by a lack of good notation, it would seem the fault was his own.

On the other hand, Euclid's concern was not written expression as such. Writing supplies *are* like house-building supplies. Even if Euclid had as much scratch paper (or papyrus) to write on as we do, he could not easily have distributed a copy of his lectures to everybody who attended. I think of Phaedrus, in the eponymous dialogue of Plato, borrowing the text of a speech of Lysias, not so that he can copy it, but so that he can memorize it [83, 228b, p. 417]. Evidently Euclid wrote things out, and that is why we have his work; but his aim was not to come up with good *written* expressions as such. See however §4.1 (p. 109).

3.8 Multiplication

In the *Elements,* addition of numbers is implicitly understood to be what we call a commutative, associative operation. Thus numbers compose an additive semigroup. Moreover, each number is a sum or combination of units, all of these units being equal to one another.

By contrast, *multiplication* is explicitly defined. The fifteenth of the definitions at the head of Book VII reads,

> Ἀριθμὸς ἀριθμὸν **πολλαπλασιάζειν** λέγεται, ὅταν, ὅσαι εἰσὶν ἐν αὐτῷ μονάδες, τοσαυτάκις συντεθῇ ὁ πολλαπλασιαζόμενος, καὶ γένηταί τις. // A number is said to **multiply** a number when, however many units are in it, so many times is the multiplied number composed, and some number comes to be.

I use the verb to be composed here for Euclid's συντίθημι,[22] because from this verb is derived the adjective σύνθετος, and we translate this as composite (as in Definition VII.13, p. 75). Heath uses "to be added to itself" for Euclid's συντίθημι; but taken literally, this is misleading.[23] To multiply A by B means to lay down a copy of A for each unit in B; it does *not* mean to add all of those copies to the A that already exists. If we multiply A by B—or as Euclid says, if B multiplies A—, we can describe the result as "A, B times," or "B times A": more precisely, "A, the number of times that there are units in B." We can write this result as

$$A \times B,$$

although Euclid has no such notation. As we noted in §3.5 (p. 83), it is not obvious that $A \times B = B \times A$, and this will actually be the result of Proposition 16. Nonetheless, the definition of multiplication is followed by:

> Ὅταν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος **ἐπίπεδος** καλεῖται, **πλευραὶ** δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί. // And whenever two num-

[22] The form συντεθῇ appears to be an aorist subjunctive, like the next verb γένηται. I have sometimes used the English subjunctive to translate Euclid's subjunctives, but not here.

[23] Jonathan Crabtree [27] traces this misleading translation to Billingsley (mentioned on page 67) who renders Euclid's definition of multiplication as, "A number is sayd to multiply a number, when the number multiplyed, is so oftentimes added to it self, as there are in the number multiplying unities: and an other number is produced."

bers, multiplying one another, make some [number], the [number] produced is called **plane,** and its **sides** are the numbers multiplying one another.

This suggests an understanding that multiplication is indeed commutative. *This* contributes to the feeling that the "definitions" at the head of Book VII are more of a summary or introduction than a list of formal definitions.

Given numbers A and B, we can write

$$A \times B = A \cdot n, \tag{3.9}$$

where n is the number or rather *numeral* of units in B. Here B is a noun, but n is an adjective. We can draw B in a diagram, but not n itself. Rather, n would be a feature of the diagram.[24] Instead of (3.9), we can write

$$A \times B = \underbrace{A + \cdots + A}_{B},$$

or less precisely

$$A \times B = A + \cdots + A;$$

another way to say this last is that $A \times B$ belongs to the semigroup generated by A.

Euclid distinguishes between products $A \times B$ and multiples $A \cdot n$ in the propositions of Book VII. The enunciation of **Proposition 5** is:

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρος ᾖ, καὶ ἕτερος ἑτέρου τὸ αὐτὸ μέρος ᾖ, καὶ συναμφότερος συναμφοτέρου τὸ αὐτὸ μέρος ἔσται, ὅπερ ὁ εἷς τοῦ ἑνός. // If a number be part of a number, and another be the same part of another, then the combination will be the very same part of the combination that the one is of the one.

[24]This observation is reminiscent of the passage of Sextus Empiricus quoted above (§3.1, p. 71).

Heath has "sum" instead of my combination; but the latter seems closer to Euclid's συναμφότερος.[25] If the numbers in question are respectively A, B, C, and D, we assume that, for some n, both $A \cdot n = B$ and $C \cdot n = D$; the conclusion is that $(A+C) \cdot n = B+D$. In one line then,

$$(A + C) \cdot n = A \cdot n + C \cdot n; \tag{3.10}$$

this can be understood as a consequence of the commutativity of addition.

Proposition 6 is the same as 5, but with "part" replaced by "parts," so that if

$$A = E \cdot k, \qquad B = E \cdot n, \qquad C = F \cdot k, \qquad D = F \cdot n,$$

then

$$A + C = (E + F) \cdot k, \qquad B + D = (E + F) \cdot n.$$

There is no need here to assume that k and n are relatively prime, or equivalently that E is the greatest common measure of A and

[25] The original meaning of συναμφότερος is both together [62], while the English "sum" comes from the Latin feminine *summa* of the superlative of *superus,* meaning above [51]. Αμφότερος is an alternative to ἄμφω [89, 349e, p. 105], which has the meaning of, and is cognate with, the Latin *ambō* and the English "both" [88, 66]. In "combination," the Latin prefix *com-* corresponds (in meaning, if not in etymology) to the συν- of συναμφότερος, while "-bin-" is from *bīnī* "two together" [51]. Actually *bīnī* is a *distributive* numeral, something that Turkish also has [74, p. 254]. English and Greek do not have distributives as such, so they can say only "two by two, two each" [67], κατὰ δύο, or σύνδυο [89, 354a, p. 106]. In fact Euclid uses the latter word in Proposition VII.14 (which we shall not otherwise consider): If numbers A_1, A_2, …, A_n, and numbers B_1, B_2, …, B_n, taken *two by two* (σύνδυο), are in the same ratio, so that $A_1 : A_2 :: B_1 : B_2$ and so on, then $A_1 : A_n :: B_1 : B_n$. Remarkably enough, the B of the Latin *bīnī* was originally D, thus corresponding to the Δ of the Greek δύο, whereas the B of *ambō* corresponds to the Φ of ἄμφω. None of my sources suggest an etymological connection between *bīnī* and *ambō*, despite the fact that each is about *two* things together.

B. Nonetheless, from Propositions 5 and 6, Euclid immediately obtains **Proposition 12:**

Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ ἀνάλογον, ἔσται ὡς εἷς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένον, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἑπομένους. // If however many numbers be proportional, one of the leading terms will be to one of the following terms as all of the leading terms are to all of the following terms.

Thus

$$A : B :: C : D \implies A : B :: A + C : B + D,$$

or more generally

$$A_1 : B_1 :: A_2 : B_2 :: \cdots :: A_n : B_n \\ \implies A_1 : B_1 :: A_1 + \cdots + A_n : B_1 + \cdots + B_n. \quad (3.11)$$

Note that this hardly needs proof, if one understands and accepts the anthyphaeretic definition of proportion: If (A, B) and (C, D) have the same anthyphaeretic sequence, then $(A + C, B + D)$ has the same sequence, since we shall have something like

$$A = B \cdot n + E, \qquad B > E, \qquad C = D \cdot n + F, \qquad D > F,$$

so that, by Proposition VII.5, as expressed by (3.10),

$$A + C = (B + D) \cdot n + E + F, \qquad B + D > E + F.$$

By assumption, (B, E) and (D, F) have the same anthyphaeretic sequence, so we can continue as before.

Meanwhile, Euclid has already used the idea of the general form (3.11) of Proposition 12 in proving **Proposition 9.** The enunciation of this proposition is:

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρος ᾖ, καὶ ἕτερος ἑτέρου τὸ αὐτὸ μέρος ᾖ, καὶ ἐναλλάξ, ὃ μέρος ἐστὶν ἢ μέρη ὁ πρῶτος τοῦ τρίτου, τὸ αὐτὸ μέρος ἔσται ἢ τὰ αὐτὰ μέρη καὶ ὁ δεύτερος τοῦ τετάρτου. // If a number be a part of a number, and another be the same part of another, also alternately, what part or parts the first is of the third, the same part or parts will also the second be of the fourth.

Thus if the four numbers are A, B, C, and D, and $A \cdot n = B$ and $C \cdot n = D$, then whatever part or parts A is of C, the same part or parts will B be of D, that is, $A : C :: B : D$. Without introducing B and D, we can write the proposition as

$$A : C :: A \cdot n : C \cdot n. \tag{3.12}$$

Thus VII.12 is a more general form of VII.9. In Proposition 9, A is contemplated as being part or parts of C, but not as being a *multiple* of C, presumably because, without loss of generality, we may assume $A < C$ (the case $A = C$ being trivial).

Proposition 10 is the same as 9, but again with the first two instances of "part" in the enunciation replaced with "parts." Thus if the four numbers again are A, B, C, and D, but now $E \cdot k = A$ and $B = E \cdot n$ for some E, while $F \cdot k = C$ and $D = F \cdot n$ for some F, then by (3.12)

$$E : F :: A : C, \qquad E : F :: B : D,$$

and so, by transitivity of sameness of ratio, which should be obvious (p. 94),

$$A : C :: B : D.$$

Briefly, the proposition is $E \cdot k : F \cdot k :: E \cdot n : F \cdot n$.

Proposition 13 expresses 9 and 10 in the language of proportion:

Ἐὰν τέσσαρες ἀριθμοὶ ἀνάλογον ὦσιν, καὶ ἐναλλὰξ ἀνάλογον ἔσονται. If four numbers be proportional, they will also be proportional alternately.

That is, **alternation** of a ratio is possible:

$$A : B :: C : D \implies A : C :: B : D.$$

This is so, because we may assume $A < C$, and then A is either part or parts of C, as explained in VII.4. These two cases have been covered in Propositions 9 and 10.

Proposition 9, like VII.12, is immediate from the anthyphaeretic definition of proportion, and then VII.10 follows easily. The combination of Propositions 9 and 10 into VII.13 by means of VII.4 is *not* immediate from the anthyphaeretic definition. But we shall use alternation to establish commutativity of multiplication.

Proposition 15 can be read as a special case of 13, though Euclid enunciates it in terms of measurement rather than proportion:

> Ἐὰν μονὰς ἀριθμόν τινα μετρῇ, ἰσακις δὲ ἕτερος ἀριθμὸς ἄλλον τινὰ ἀριθμὸν μετρῇ, καὶ ἐναλλὰξ ἰσάκις ἡ μονὰς τὸν τρίτον ἀριθμὸν μετρήσει καὶ ὁ δεύτερος τὸν τέταρτον. // If unity measure some number, and equally a different number measure some other number, also, alternately, equally will unity measure the third number, and the second the fourth.

Thus, if unity measures A as B measures C, this means

$$A = \underbrace{1 + \cdots + 1}_{A}, \qquad C = \underbrace{B + \cdots + B}_{A}, \tag{3.13}$$

and therefore

$$1 : A :: B : C.$$

Our 1, A, B, and C are Euclid's Α, ΒΓ, Δ, and ΕΖ, and by assumption,

> ὅσαι . . . εἰσὶν ἐν τῷ ΒΓ μονάδες, τοσοῦτοί εἰσι καὶ ἐν τῷ ΕΖ ἀριθμοὶ ἴσοι τῷ Δ. // however many units are in ΒΓ, so many numbers equal to Δ are also in ΕΖ.

These units and numbers are not identical to one another:

> διῃρήσθω ὁ μὲν ΒΓ εἰς τὰς ἐν ἑαυτῷ μονάδας τὰς ΒΗ, ΗΘ, ΘΓ, ὁ δὲ ΕΖ εἰς τοὺς τῷ Δ ἴσους τοὺς ΕΚ, ΚΛ, ΛΖ. // Let ΒΓ have been divided into the units ΒΗ, ΗΘ, and ΘΓ in it, and ΕΖ into the numbers ΕΚ, ΚΛ, and ΛΖ equal to Δ.

However, the *number* of units is equal to the number of numbers equal to Δ; and the latter numbers are equal to one another, as are the units:

> ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΘ, ΘΓ τῷ πλήθει τῶν ΕΚ, ΚΛ, ΛΖ, // The multitude of ΒΗ, ΗΘ, and ΘΓ will be equal to the multitude of ΕΚ, ΚΛ, and ΛΖ,

and also

> ἴσαι εἰσὶν αἱ ΒΗ, ΗΘ, ΘΓ μονάδες ἀλλήλαις, εἰσὶ δὲ καὶ οἱ ΕΚ, ΚΛ, ΛΖ ἀριθμοὶ ἴσοι ἀλλήλοις. // the BH, HΘ, and ΘΓ units are equal to one another, and the EK, KΛ, and ΛZ numbers are equal to one another.

Thus (3.13) is justified. By Proposition 12,

$$1 : B :: \underbrace{1 + \cdots + 1}_{n} : \underbrace{B + \cdots + B}_{n}.$$

Letting $A = 1 \cdot n$, we have

$$1 : B :: A : C,$$

or unity measures B as A measures C.

In the argument above, initially, C is $B \times A$; by the conclusion, $C = A \times B$; thus

$$A \times B = B \times A.$$

This is made explicit in **Proposition 16,** whose enunciation is

> Ἐὰν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινας,
> οἱ γενόμενοι ἐξ αὐτῶν ἴσοι ἀλλήλοις ἔσονται.
> If two numbers, multiplying one another, make some [number],
> the products will be equal to ane another.

This would indeed appear to be a "real" theorem.

We can understand Euclid's situation as follows. His numbers compose a set $\mathscr{S}$ that is equipped with (1) a commutative, associative operation of addition, which we denote by $+$; (2) an associative multiplication, $\times$, which distributes over addition; (3) a multiplicative identity, 1; (4) a linear ordering, $<$, such that

$$A < B \iff \exists X\ A + X = B.$$

We may define

$$\mathscr{R} = \mathscr{S} \cup \{0\} \cup \{-X : X \in \mathscr{S}\},$$

where $0 \notin \mathscr{S}$, and $X \mapsto -X$ is just some injection on $\mathscr{S}$ whose range is disjoint from $\mathscr{S} \cup \{0\}$. Then we can turn $\mathscr{R}$ into an ordered ring, just as, in school, one obtains the integers from the natural numbers. However, not every ordered ring is commutative like the ring of integers. For example, the free group F_2 on two letters can be made into an ordered group, and then the noncommutative group-ring $\mathbb{Z}F_2$ can be ordered.[26]

To apply the Euclidean Algorithm, Euclid tacitly assumes what we call the Well Ordering Property of $\mathscr{S}$. I think Euclid *also* tacitly assumes that $A \times B$ always belongs to the semigroup generated by A. Either of these assumptions implies the other, although Euclid does not prove this or perhaps even contemplate a proof.

Nonetheless, Euclid does not assume commutativity of multiplication. He proves it. This would seem to be more than number-theory textbooks do today. Landau [57] and Hardy & Wright [45] do not discuss foundations, but start right in, proving theorems about divisibility. I suppose they expect the reader to understand the natural numbers and all of the integers as lying among the so-called real numbers, which are understood as composing an *ordered field* (although this terminology may not be introduced). As far as the integers themselves are concerned, the authors evidently take for granted the axioms that LeVeque [60, pp. 8–10] makes explicit: in effect, the integers compose an ordered commutative ring whose positive elements are well-ordered.[27]

Leveque observes that the well-ordering axiom can be replaced with the induction axiom, once the axioms of a commutative ordered ring are assumed. Burton [16, pp. 1–2] proves the "Archimedean property" and the "First Principle of Finite Induction," using the "Well-Ordering Principle" as the only explicit property of the natural numbers; otherwise, he says,

> We shall make no attempt to construct the integers axiomatically,

[26] Textbook references are Lam [56, Exercise 17.1, p. 269] and Botto Mura and Rhemtulla [14, Thm 2.3.1, pp. 33 f.]. Every free product of ordered groups is orderable.

[27] Before introducing the ordering, LeVeque gives the axiom whereby a commutative ring becomes an integral domain; but this will be implied by the ordering.

assuming instead that they are already given and that any reader of this book is familiar with many elementary facts about them.

It is presently asserted [16, p. 5] that “Mathematical induction is often used as a method of definition as well as a method of proof.” This suggests that recursive definitions are justified by induction alone. They are not, and clarifying this point is of value to number theory itself, as we discussed above (p. 18).

Again, in our terms, Euclid proves that every ordered ring whose positive elements are well-ordered is commutative. Thus he would appear to be more sensitive to rigor and logical economy than are the corresponding texts of today.

3.9 Euclid’s Lemma

“Euclid’s Lemma” now follows from some straightforward manipulations. Thus **Proposition VII.17** is

$$C : D :: C \times A : D \times A, \tag{3.14}$$

which looks like a restatement of Proposition 9, when this is written as (3.12); but Euclid derives the result afresh, using that, by definition,

$$1 : A :: C : C \times A, \qquad 1 : A :: D : D \times A,$$

so that $C : C \times A :: D : D \times A$ (since sameness of ratio is immediately transitive), and therefore, alternately, (3.14). By applying Proposition 16 to this, we obtain **Proposition 18,**

$$A : B :: C \times A : C \times B. \tag{3.15}$$

From (3.14) and (3.15), along with the rule

$$E : F :: E : F' \implies F = F',$$

we obtain

$$A : B :: C : D \iff D \times A = C \times B, \tag{3.16}$$

which is **Proposition 19.** Euclid's argument is as we have put it. However, as we have already noted (p. 97), Euclid is not concerned with how mathematics appears when written on a page. When he writes out Propositions 17 and 18, he does *not* do as we have done, choosing the letters in (3.14) and (3.15) so that Proposition 19 follows almost visually in the form of (3.16).

In **Proposition 20,** we suppose C and D are the least X and Y such that $X : Y :: A : B$. By Alternation (Proposition 13), though Euclid does not make this explicit,

$$C : A :: D : B.$$

Thus if C is *parts* of A, so that D is the same parts of B, then for some parts E of A and F of B, we have $C = E \cdot k$ and $D = F \cdot k$; but then by Proposition 12,

$$E : F :: C : D :: A : B,$$

contrary to the minimality of C and D. Thus C is not *parts,* but is a *part* of A, and D is the same part of B; or as Euclid says, C measures A and D measures B *equally* (ἰσάκις).

Proposition 20 does not tell us *which* numbers are minimal as described. Proposition 21 establishes a sufficient condition for being such numbers. If A and B are prime to one another, then they must be the least Ξ and Υ such that Ξ : Υ :: A : B. For, suppose not, but Γ and Δ are least.

> ἰσάκις ἄρα ὁ Γ τὸν Α μετρεῖ καὶ ὁ Δ τὸν Β. ὁσάκις δὴ ὁ Γ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε. καὶ ὁ Δ ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας. καὶ ἐπεὶ ὁ Γ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, καί ὁ Ε ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας. // Therefore Γ measures A and Δ measures B equally. Then as many times as Γ measures A, so many units let there be in E. Therefore also Δ measures B according to the units in E. And since Γ measures A according to the units in E, also therefore E measures A according to the units in Γ.

This is not the language of the *enunciation* of Proposition 16, but is the language of its demonstration. At present, for some E we have $\mathsf{\Gamma} \times \mathsf{E} = \mathsf{A}$, and therefore $\mathsf{E} \times \mathsf{\Gamma} = \mathsf{A}$, that is, E measures A; and similarly the same E measures B, which is absurd, since A and B are coprime.

We jump ahead to **Proposition 29,** which is immediate from the definitions:

> Ἅπας πρῶτος ἀριθμὸς πρὸς ἅπαντα ἀριθμόν, ὃν μὴ μετρεῖ, πρῶτός ἐστιν. // Every prime number, to every number that it does not measure, is prime.

Finally then, in **Proposition 30,** we suppose P is prime and measures $A \times B$, but does not measure A, so that P is prime to A by Proposition 29. By hypothesis, for some C, $P \times C = A \times B$, so by Proposition 19,

$$P : A :: B : C.$$

By Proposition 21, P and A are minimal; then by Proposition 20, P measures B.

4 Symbolic mathematics

4.1 Analysis

The last three propositions of Book VII of the *Elements* hint at a *symbolic* mathematics, a mathematics of manipulating symbols that start out with no definite meaning. Such mathematics can be called *analytic,* as Descartes's geometry is called. It did not originate with Descartes, for Pappus had already described it, more than a thousand years earlier [93, pp. 596–7]:

> ἐν μὲν γὰρ τῇ ἀναλύσει τὸ ζητούμενον ὡς γεγονὸς ὑποθέμενοι τὸ ἐξ οὗ τοῦτο συμβαίνει σκοπούμεθα. // For in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about.

Pappus attributes the work of analysis that he describes to "Euclid the writer of the *Elements,* Apollonius of Perga and Aristaeus the elder."

Briefly, Proposition VII.37 is the theorem that a number *measuring* a number also *divides* it. Proposition 38 is the converse, that a number dividing a number also measures it. If it is A dividing B, this means B has, so to speak, an A'th part. Thus it seems A can stand for the "ratio" that we denote by $1/A$. We observed in §3.2 (p. 73) that a ratio is not *one* thing in a diagram, but two; and indeed, the number A is *not* in itself the ratio of unity to A. It is perhaps more easily confused with the ratio of itself to unity.

Finally, Proposition VII.39 is the problem of finding the least number having given parts. In the exposition (in the sense of Proclus, p. 46), these parts are given as A, B, and $\mathsf{\Gamma}$. In the construction, numbers $\mathsf{\Delta}$, E, and Z are found such that, in effect,

$$\mathsf{A} = 1/\mathsf{\Delta}, \qquad \mathsf{B} = 1/\mathsf{E}, \qquad \mathsf{\Gamma} = 1/\mathsf{Z}.$$

This is unorthodox. I think the orthodox Euclidean approach would be simply to *start* with the numbers Δ, E, and Z. The problem then is to find the least number that has Δ'th, E'th, and Z'th parts, and this means finding the least number H such that, for some numbers Θ, K, and Λ,

$$1 : \Delta :: \Theta : \mathsf{H}, \qquad 1 : \mathsf{E} :: \mathsf{K} : \mathsf{H}, \qquad 1 : \mathsf{Z} :: \Lambda : \mathsf{H}.$$

As it is, Proposition 39 starts not with the numbers Δ, E, and Z, but with the *partitive numerals* Δ'th, E'th, and Z'th, which are called A, B, and Γ. No number is given that has parts designated by these numerals.

It will be worth while to consider these last propositions of Book VII in more detail. **Proposition 37** has the enunciation,

> Ἐὰν ἀριθμος ὑπό τινος ἀριθμοῦ μετρῆται, ὁ μετρούμενος ὁμώνυμον μέρος ἕξει τῷ μετροῦντι. // If a number be measured by some number, the measured [number] has a part having the same name with the measuring [number].

Instead of having the same name, we could use "homonymous" for Euclid's ὁμώνυμος. If A is measured by the number three, that is, by a triad B, then $A = B \times C$ for some number C; but then also, by commutativity of multiplication, $A = C \times B$, or $A = C \cdot 3$, so C is the third part of A. Here the third part is said to have the same name as the triad, or to be homonymous with it.

The proposition seems trivial to us, who from a young age are given to understand that multiplication of numbers is commutative. It might seem trivial to Euclid's intended audience as well. At any rate, commutativity will perhaps be understood by anybody who has to find products of specific numbers. Nonetheless, Euclid works out a proof of the proposition; and the proof relies not on commutativity of multiplication as in Proposition 16, but on alternation of proportions as in Proposition 15. The diagram in Heiberg's text is as in Figure 4.1, and the Greek text itself begins on page 112, in parallel with that of Proposition 38. Meanwhile, I translate Euclid's argument for Proposition 37 as follows, paying attention to the adjectival and appositional uses of letters discussed earlier (p. 49).

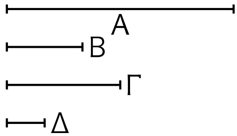

Figure 4.1. Proposition VII.37

For let the number A be measured by some number B.

I say that A has a part homonymous with B.

For as many times as B measures A, so many units let there be in Γ.

Since B measures A according to the units in Γ, and also the Δ unit measures the Γ number according to the units in it, equally, therefore, the Δ unit measures the Γ number, and B, A. Alternately, therefore, equally the Δ unit measures the B number, and Γ, A. What part, therefore, the Δ unit is of the B number, the same part is also Γ of A. And the Δ unit is a part of the B number homonymous with it; also, therefore, Γ is a part of A homonymous with B.

Thus A has a part, Γ, that is homonymous with B; which is just what was to be shown.

Thus if A is measured by B, it is measured a number Γ of times, and so

$$\begin{gathered} 1 : \mathsf{\Gamma} :: \mathsf{B} : \mathsf{A}, \\ 1 : \mathsf{B} :: \mathsf{\Gamma} : \mathsf{A}, \end{gathered} \tag{4.1}$$

by Proposition 15. Then Γ is the same part of A that unity is of B. In other words, *quâ* part of A, Γ is homonymous with B, or Γ is the B'th part of A. In short, if B measures A, then A has a B'th part, or as we may say (though Euclid does not), B divides A.

Conversely, if A has a B'th part—if B divides A—, then B measures A. This is **Proposition 38,** though Euclid's exposition will use Γ in place of our B. As usual, the enunciation uses no such letters at all:

If a number have any part whatever, it will be measured by a number having the same name with the part.

Heath calls this, "practically a restatement of the preceding proposition" [34, v. II, p. 343] and Euclid's subsequent lettering may suggest as much (as may the diagram in Heiberg, which is as for Proposition 37, though slightly bigger, and with the label A moved to the left of the corresponding straight line). But Proposition 38 is the converse of 37. In the hypothesis of the enunciation of Proposition 38, having a given part does *not* mean being measured by a given number as in the hypothesis of Proposition 37. It means being measured by a part a given number of times. For Euclid proceeds with 38 as follows.

> For let the number A have whatever part B, and let the [number] homonymous with the B part be Γ.
>
> I say that Γ measures B.
>
> For since B is a part of A homonymous with Γ, and also the Δ unit is a part of Γ homonymous with it, what part, therefore, the Δ unit is of the Γ number, the same part also is B of A. Equally, therefore, the Δ unit measures the Γ number, and B, A. Alternately, therefore, equally the Δ unit measures the B number, and Γ, A.
>
> Therefore Γ measures A; which is just what was to be shown.

Here, as in Proposition 37, A is measured Γ times by B, so the proportions (4.1) are known to hold again, in the same order, and therefore Γ measures A. Thus Proposition 38 is superficially similar to 37. Indeed, the very phrases expressing (4.1) in the Greek text are the same in both propositions. We can compare the two propositions side by side as follows.

Proposition 37	Proposition 38
Enunciation	
Ἐὰν ἀριθμος ὑπό τινος ἀριθμοῦ μετρῆται,	Ἐὰν ἀριθμος μέρος ἔχῃ ὁτιοῦν,
ὁ μετρούμενος ὁμώνυμον μέρος ἕξει τῷ μετροῦντι.	ὑπὸ ὁμωνύμου ἀριθμοῦ μετρηθήσεται τῷ μέρει.
Exposition	
Ἀριθμὸς γὰρ ὁ A ὑπὸ τινος ἀριθμοῦ τοῦ B μετρείσθω·	Ἀριθμὸς γὰρ ὁ A μέρος ἐχέτω ὁτιοῦν τὸν B, καὶ τῷ B μέρει ὁμώνυμος ἔστω [ἀριθμὸς] ὁ Γ·
Specification	
λέγω, ὅτι ὁ A ὁμώνυμον μέρος ἔχει τῷ B.	λέγω, ὅτι ὁ Γ τὸν A μετρεῖ.

Construction

Ὁσάκις γὰρ ὁ Β τον Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Γ.	

Demonstration

ἐπεὶ ὁ Β τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδες,	Ἐπεὶ γὰρ ὁ Β τοῦ Α μέρος ἐστὶν ὁμώνυμον τῷ Γ,
μετρεῖ δὲ καὶ ἡ Δ μονὰς τὸν Γ ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδες,	ἔστι δὲ καὶ ἡ Δ μονὰς τοῦ Γ μέρος ὁμώνυμον αὐτῷ,
	ὃ ἄρα μέρος ἐστὶν ἡ Δ μονὰς τοῦ Γ ἀριθμοῦ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Β τοῦ Α·
ἰσάκις ἄρα ἡ Δ μονὰς τὸν Γ ἀριθμὸν μετρεῖ καὶ ὁ[1] Β τὸν Α.	ἰσάκις ἄρα ἡ Δ μονὰς τὸν Γ ἀριθμὸν μετρεῖ καὶ ὁ Β τὸν Α.
ἐναλλάξ ἄρα ἰσάκις ἡ Δ μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Γ τὸν Α·	ἐναλλὰξ ἄρα ἰσάκις ἡ Δ μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Γ τὸν Α.
ὃ ἄρα μέρος ἐστὶν ἡ Δ μονὰς τοῦ Β ἀριθμοῦ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Α. ἡ δὲ Δ μονὰς τοῦ Β ἀριθμοῦ μέρος ἐστὶν ὁμώνυμον αὐτῷ· καὶ ὁ Γ ἄρα τοῦ Α μέρος ἐστὶν ὁμώνυμον τῷ Β.	

Conclusion

ὥστε ὁ Α μέρος ἔχει τον Γ ὁμώνυμον ὄντα τῷ Β·	ὁ Γ ἄρα τὸν Α μετρεῖ·
ὅπερ ἔδει δεῖξαι.	ὅπερ ἔδει δεῖξαι.

In Proposition 37, Γ is found in the construction, and it is named in the conclusion only *quâ* B'th part of A; it does not appear in the specification. In Proposition 38, Γ is part of the exposition: it is one of the givens. Morever, in 38, B is given only as the Γ'th part of A, and B is not mentioned in the specification or conclusion. Thus Propositions 37 and 38 are logically distinct. Again, they are logical converses of one another. Together they establish the logical

[1] In the `pdf` scan that I have of Heiberg's text, only ο appears, without the rough breathing mark; I assume this is a misprint, if not a defect of the scan itself.

equivalence of measuring and dividing finite sets. We noted in §3.3 (p. 74) that measuring and dividing finite sets are logically distinct.

It may seem that Euclid could have phrased the exposition and specification of Proposition 38 as,

For let A have a part homonymous with Γ.
I say that Γ measures A.

Then there would have to be a construction:

For let the part of A homonymous with Γ be B.

But this construction would be to *divide* A by Γ (to divide A into Γ-many parts). We do not not know to do this: we only know how to *measure.* Thus A must be given as already divided into Γ-many parts, these being equal to B.

So far everything seems properly "Euclidean," even painstakingly so. Various ways of understanding the commutativity of multiplication have been worked out in the established style. But then we reach **Proposition 39.** This is a complement and corollary of 34 and 36, which we looked at earlier (p. 90). The enunciation of 34 is

Δύο ἀριθμῶν δοθέντων εὑρεῖν,
ὅν ἐλάχιστον μετροῦσιν ἀριθμόν.
Two numbers being given,
to find the least number that they measure.

Proposition 36 is the same, with "Three" for "Two" (Τριῶν for Δύο). The two propositions thus represent the problem of finding least common multiples. Then 39 is to find the least number having given parts:

Ἀριθμὸν εὑρειν, ὃς ἐλάχιστος ὢν ἕξει τὰ δοθέντα μέρη.
To find the number that—being least—will have given parts.

Here the word order is different, the givens now being mentioned last, and not in the genitive absolute construction used in Propositions 34 and 36. Formed in parallel to these, Proposition 39 would begin something like,

Μερῶν δοθέντων εὑρεῖν, ὅς ἐλάχιστος ταῦτα ἕξει ἀριθμόν.
Parts being given, to find the least number that has them.

It might be objected that this formulation introduces parts, without any number that they are a part of; but this is what the exposition of the proposition will go on to to anyway. Indeed, the exposition, specification, and demonstration of 39 proceed as follows.

Ἔστω τὰ δοθέντα μέρη τὰ Α, Β, Γ·

δεῖ δὴ ἀριθμὸν εὑρεῖν, ὃς ἐλάχιστος ὢν ἕξει τὰ Α, Β, Γ μέρη.

Ἔστωσαν γὰρ τοῖς Α, Β, Γ μέρεσιν ὁμώνυμοι ἀριθμοὶ οἱ Δ, Ε, Ζ, καὶ εἰλήφθω ὑπὸ τῶν Δ, Ε, Ζ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ Η.

Ὁ Η ἄρα ὁμώνυμα μέρη ἔχει τοῖς Δ, Ε, Ζ. τοῖς δὲ Δ, Ε, Ζ ὁμώνυμα μέρη ἐστὶ τὰ Α, Β, Γ· ὁ Η ἄρα ἔχει τὰ Α, Β, Γ μέρη.

Let the given parts be A, B, and Γ.

We must find a number that will *minimally* have A, B, and Γ parts.

So let the numbers having the same names as the A, B, and Γ parts be Δ, E, and Z; and let the least number measured by Δ, E, and Z have been taken, namely H.

Then H has parts having the same names as Δ, E, and Z. But the parts having the same names as Δ, E, and Z are A, B, and Γ. Therefore H has A, B, and Γ parts.

That is, if A, B, and Γ are Δ'th, E'th, and Z'th respectively, using VII.36 we let H be the least number measured by Δ, E, and Z. By VII.37, H has Δ'th, E'th, and Z'th parts, that is, it has A, B, and Γ parts.

Having used the Latin A, B, C, and G for A, B, Γ and H, Heath says "therefore G has the parts A, B, C," but this is misleading. In the Greek, the letters are used adjectivally, not substantively:

ὁ Η ἄρα ἔχει τὰ Α, Β, Γ μέρη.

Therefore H has A, B, and Γ parts.

A contrasting substantival use can be seen in the *construction* of VII.8 (see p. 76), after HΘ has been made equal to AΔ, which has been given as being parts of ΓΔ. The command is given,

διῃρήσθω ὁ μὲν ΗΘ εἰς τὰ τοῦ ΓΔ μέρη
τὰ ΗΚ, ΚΘ.

Let HΘ have been divided into the parts of ΓΔ,
[namely] HK and KΘ.

Thus HK and KΘ have been given the definite article, making them nouns, placed in apposition to "the parts of ΓΔ." This feature of the Greek is pointed out by Netz [72, §2.3, p. 43–4], who considers, from Proposition XIII.4, the sentence ἔστω εὐθεῖα ἡ ΑΒ: Heath translates it as "Let AB be a straight line," but it ought to be, "Let there be a straight line, [viz.] *AB*." Proposition VII.39, Α, Β, and Γ are not parts of Η in the way that ΗΚ and ΚΘ are parts of ΓΔ in Proposition VII.8: again, they are not numbers, but partitive numerals.

For the record, the demonstration of Proposition VII.39 continues, establishing that Η must be the least number having Δ'th, Ε'th, and Ζ'th parts; for, if Θ has these parts and is less, then, by 38, it also is measured by Δ, Ε, and Ζ, which is absurd. Such is the argument of the last proposition of Book VII.

The diagram for this proposition consists of eight separate line segments, labelled Α through Θ, as in Figure 4.2. What can the

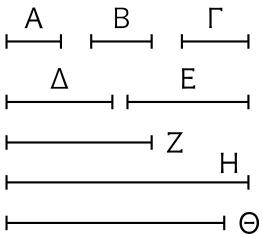

Figure 4.2. Proposition VII.39

first three of these be—the ones that are called the given parts? At the beginning, they are not known to be parts *of* anything; they are just parts, simply, like a half or a third or a fourth. Or are they parts of some definite, though unspecified, segment? This segment would then be like Descartes's unit length (mentioned on p. 44), introduced so that every length can be understood as its own ratio to that unit length. Under this interpretation, Α, Β, and Γ are close to being ratios as such, rendered concrete for the moment so that they can be talked about, but not intended for comparison with particular numbers. This would be the only instance I know of in Euclid where a ratio can actually be pointed to in a diagram; elsewhere, a ratio is a relation between *two* things in a diagram. Nonetheless, Heath remarks on Proposition 39,

> This again is practically a restatement in another form of the problem of finding the L.C.M.

This is actually more correct than Heath's similar remark on Proposition 38; for 39 is a *problem,* whose solution is indeed effected by finding a least common multiple, just as in 35. Proposition 39 may be the more practical solution to this problem, if one is more likely to want a number X of cubits that can be divided into two, three, and five parts, than a number Y of cubits that can be measured by two, three, and five cubits. On the other hand, in practical life, perhaps everybody knows that these X and Y are the same.

Though Heath and Heiberg give no suggestion of the possibility, I wonder whether Proposition 39 is not a later addition to Book VII, put there by somebody who thought Proposition 36 should have a formal correlate (or a more "practical" correlate, in the sense just suggested).[2]

The position of Proposition 39 at the end of Book VII may contribute to the plausibility of its being a later addition to the *Elements.* Perhaps 37 and 38 are also later additions.[3] All three

[2]Of Propositions 37–9, Taisbak considers only the last, and then only to remark on the "abstract character of the line segment notation" that Euclid uses generally [91, p. 15]. According to Taisbak, "When Euclid represents a number by a line segment, he is *not* thinking *geometrically.* No arithmetical theorem is ever proved by means of theorems from geometry; but by using a line segment as an abstract symbol he produces the same sort of generalisation as we do by our 'alphabetic' algebra." I have suggested however (on page 76) that Euclid *is* thinking *visually,* as when in proving Proposition vii.8 for the sake of visual clarity, he introduces a new number that is equal to one that has already been drawn. In any case, it is just the abstract character of Proposition 39 in particular that suggests to me that the proposition is not Euclidean. Further evidence might be Taisbak's own observation that in the diagram as we have it (as in Figure 4.2), A is less than $\mathsf{\Gamma}$, but $\mathsf{\Delta}$ is less than Z, when it ought to be greater if everything here is a magnitude.

[3]In a note, Mueller remarks, "Itard (p. 128 [of *Les Livres arithmetiques d'Euclide,* Paris, 1961]) suggests that 37–39 are a vestige of an earlier arithmetic treatise somehow connected with the use of auxiliary numbers in Egyptian calculation. The suggestion rests on treating auxiliary numbers as least common denominators. For arguments against treating them in this way see Neugebauer, *Vorgriechische Mathematik,* p. 137ff. or van der

propositions follow easily by means of the commutativity of multiplication, and this commutativity is relied on implicitly throughout the ensuing Books VIII and IX. Again, Proposition VII.39 in particular seems to be due to somebody who is thinking "formally." Since measuring is correlated with dividing by means of Proposition VII.16, for every proposition like 36 about measuring, one may think that a proposition like 39 about dividing can and should be written. But we have seen that 39 does not really fit the preceding propositions. Similar additions to *Wikipedia* are inserted today by people who want to make their contribution: their contributions do not always reflect a full understanding of the surrounding text.

4.2 Symbolic commutativity

In §3.8 (p. 97), on the basis of Propositions 1, 2, 4, 5, 6, 12, and 15 of Book VII of the *Elements,* we reviewed Euclid's proof of the commutativity of multiplication given in Proposition VII.16. Using modern symbolism freely now, we can summarize the argument as follows. We shall denote numbers by small letters. Multiplication is "defined" by

$$a \cdot b = \underbrace{a + \cdots + a}_{b}. \tag{4.2}$$

Then in particular, by the "definition" of number,

$$a = \underbrace{1 + \cdots + 1}_{a} = 1 \cdot a. \tag{4.3}$$

If the Euclidean algorithm (illustrated in Propositions 1 and 2) has the same steps for a and b that it does for c and d, this means, by definition (inferred from Proposition 4), that the proportion

$$a : b :: c : d$$

Waerden, [*Science Awakening* (trans. A. Dresden), Groningen, 1954,] pp. 26–27."

holds. In particular, we always have the proportion

$$1 : a :: b : b \cdot a, \tag{4.4}$$

as well as the implication

$$a : b :: c : d \implies a : b :: a + c : b + d \tag{4.5}$$

(which is Proposition 12, obtained from 5 and 6). Repeated application of this, starting with $1 : a :: 1 : a$, gives

$$1 : a :: \underbrace{1 + \cdots + 1}_{b} : \underbrace{a + \cdots + a}_{b}, \tag{4.6}$$

that is,

$$1 : a :: b : a \cdot b$$

(which is Proposition 13). Consequently, because of (4.4),

$$b \cdot a = a \cdot b.$$

If that is Euclid's proof, a streamlined proof seems possible. The proof of (4.5) uses the distribution axiom

$$(a + b) \cdot c = a \cdot c + b \cdot c, \tag{4.7}$$

for which Euclid gives intuitive justification. The way he generalizes (4.5) to get (4.6), he might generalize the distribution axiom to get

$$(\underbrace{a + \cdots + a}_{b}) \cdot c = \underbrace{a \cdot c + \cdots + a \cdot c}_{b},$$

that is,

$$(a \cdot b) \cdot c = (a \cdot c) \cdot b.$$

Letting a be unity yields commutativity of multiplication. This second proof may seem simpler; but then we are using symbolism that Euclid does not. At this stage, he may prefer to avoid talking about products of three (or more) factors.

In any case, (4.2) and (4.3) are not up to modern standards for definitions. We shall bring them up to this standard and, on Euclid's cues, develop a nontrivial proof of the commutativity of multiplication, in the remainder of this chapter.

4.3 Euclid's Arithmetical Structure

In his arithmetical books, we can understand Euclid as working in a structure

$$(\mathbb{N}, 1, +, \times, <).$$

Here 1 is **unity,** or the **unit,** as discussed in §3.1 (p. 67). Then elements of $\mathbb{N}$ are **numbers,** as in §3.3 (p. 74).

For Euclid, there are many units, but they are all equal to one another, as noted explicitly in the proof of Proposition VII.15 (p. 103). Equality is not identity (§2.3, p. 54), though treating it as such should not cause us any problem.

The binary operation $+$ of **addition** is undefined, but is used in the *Elements* from the beginning. We can read the expression $a+b$ in the conventional way, as a **plus** b, meaning a with the addition of b. For Euclid, this will be the same as $b+a$; we call it the **sum** of a and b, or b and a.

The binary operation $\times$ of **multiplication** is defined, after a fashion (§3.8, p. 97). When in use between two numbers, our symbol $\times$ will become a dot. In case

$$a \cdot b = c, \tag{4.8}$$

we understand this to mean that when a is the **multiplicand,** and b the **multiplier,** then c is resulting **product.** We may write (4.8) informally as

$$\underbrace{a + \cdots + a}_{b} = c,$$

meaning c is the sum of b copies of a; but we should be careful to justify anything derived from this "intuitive" formulation.

In Euclid's terminology, a **measures** the product $a \cdot b$. Measuring is an undefined notion in the text of the *Elements* as we have it; but by Definitions VII.3 and 5, there are two more ways to express it (§3.2, p. 71):

- a is **part** of $a \cdot b$,

- $a \cdot b$ is a **multiple** of a.

We might also say

- b **divides** $a \cdot b$ (into parts, each of which is equal to a),
- $a \cdot b$ is b **times** a (or a, b times).

The main point is that multiplication is initially presented in an asymmetrical way.

As the status of unity as a number is ambiguous in Euclid (p. 75), we may refer to the elements of $\mathbb{N}$ other than 1 as **proper** numbers.

The binary relation $<$ is of course the (undefined) notion of being **less than.** There is the converse relation $>$ of being **greater than,** which may be more common in the *Elements*: Common Notion 5 (p. 61) is "The whole is greater than the part," although the geometric sense of "part" meant here is not the more precise arithmetical sense given above.

I propose now that, for Euclid, the structure $(\mathbb{N}, 1, +, \times, <)$ tacitly has just the properties that make it isomorphic to a structure

$$(\alpha \smallsetminus \{0\}, 1, +, \times, \in),$$

where α is an ordinal, and so $1 = \{0\}$, and also $+$ and $\times$ are addition and multiplication of ordinals, and α is closed under these operations. In particular,

$$\alpha = \omega^{\omega^{\beta}}$$

for some ordinal β, as will be shown below. Then Euclid will make additional assumptions, from which it can be proved that $\beta = 0$, so $\alpha = \omega$. In particular then, multiplication will be commutative.

4.4 Euclid's Implicit Axioms

Presumably Euclid is not actually thinking in terms of our ordinals. But his work suggests that he understands the following axioms to be true in the structure $(\mathbb{N}, 1, +, \times, <)$.

A1. The less-than relation is a linear ordering.
A2. Every non-empty subset has a least element.
A3. 1 is the least element of the whole set:

$$1 \leqslant a.$$

A4. Addition is associative:

$$a + (b + c) = (a + b) + c.$$

A5. Addition makes greater:

$$a + b > a.$$

A6. Being greater is achieved by addition:

$$\exists x \; (b > a \implies xb = a + x).$$

A7. To multiply by a sum is to add the multiples:

$$c \cdot (a + b) = c \cdot a + c \cdot b.$$

A8. Multiplication by unity is identical:

$$a \cdot 1 = a.$$

A9. Division with remainder is always possible:

$$\exists x \; \exists y \; \bigl(b \geqslant a \implies b = a \cdot x \;\text{ OR }\; (b = a \cdot x + y \;\; \& \;\; a > y)\bigr).$$

A10. Multiplication is associative:

$$c \cdot (b \cdot a) = (c \cdot b) \cdot a.$$

A11. Multiplication by a proper number makes greater:

$$b > 1 \implies a \cdot b > a.$$

A12. The multiple of unity by a number is the number:

$$1 \cdot a = a.$$

A13. A multiple of a sum is the sum of the multiples:

$$(a + b) \cdot c = a \cdot c + b \cdot c.$$

A14. Addition is commutative:

$$b + a = a + b.$$

All of these but **A2** belong to first-order logic. Most of the axioms would seem to be "obvious" properties of numbers. **A13** is a modern interpretation of Euclid's Proposition VII.5; this proposition then should be understood as an "intuitive" argument for why the axiom is true. **A9** is used implicitly in the Euclidean Algorithm. We shall see below how the other axioms arise in Euclid's work.

4.5 Modern Analysis of the Axioms

Meanwhile, let us note that the first twelve axioms in §4.4 are indeed true in $\left(\omega^{\omega^\beta} \smallsetminus \{0\}, 1, +, \times, \in\right)$: see for example [61, Ch. IV]. It is a straightforward exercise to show that the converse is true as follows.

Theorem 1. *If a structure $(A, 1, \oplus, \otimes, <)$ satisfies the first nine axioms in §4.4, then it is isomorphic to some structure*

$$(\alpha \smallsetminus \{0\}, 1, +, \times, \in),$$

where $\alpha = \omega^{\omega^\beta}$ for some ordinal β, and $+$ and $\times$ are the ordinal operations.

Proof. Since $(A, <)$ is a well-ordered set by **A1** and **A2**, we may assume from the start that it is a nonempty ordinal with 0 removed, and $<$ is $\in$. By **A3**, the element of A called 1 is indeed the ordinal 1. Showing that $\oplus$ and $\otimes$ are the ordinal operations is now equivalent to showing the following, for all α and β in A:

1. $\alpha \oplus 1$ is the successor α' of α with respect to $<$.
2. $\alpha \oplus \beta' = (\alpha \oplus \beta)'$.
3. If β is a limit ordinal, then $\alpha \oplus \beta = \sup_{\xi<\beta}(\alpha \oplus \beta)$.
4. $\alpha \otimes 1 = \alpha$.
5. $\alpha \otimes \beta' = (\alpha \otimes \beta) \oplus \alpha$.
6. If β is a limit, then $\alpha \otimes \beta = \sup_{\xi<\beta}(\alpha \otimes \xi)$.

Details are as follows. I shall make explicit only the *first* use of an axiom.

1. By **A5**, $\alpha' \leqslant \alpha \oplus 1$. Suppose $\alpha < \beta$. We shall show $\alpha \oplus 1 \leqslant \beta$. By **A6**, for some γ in A, $\alpha \oplus \gamma = \beta$. Thus it is enough to show that the operation $\xi \mapsto \alpha \oplus \xi$ on A is strictly increasing. Since α is arbitrary, we consider δ in its place. By **A4**,

$$\delta \oplus \alpha < (\delta \oplus \alpha) \oplus \gamma = \delta \oplus (\alpha \oplus \gamma) = \delta \oplus \beta,$$

as desired. Thus $\alpha \oplus 1 = \alpha'$.

2. $\alpha \oplus \beta' = \alpha \oplus (\beta \oplus 1) = (\alpha \oplus \beta) \oplus 1 = (\alpha \oplus \beta)'$.

3. Now suppose β is a limit ordinal in A. Then $\alpha \oplus \beta$ is an upper bound of $\{\alpha \oplus \xi \colon \xi < \beta\}$. Let γ be the least upper bound. Then for some δ in A, $\alpha \oplus \delta = \gamma$. We must have $\delta \leqslant \beta$. If $\delta < \beta$, then $\delta' < \beta$, so $\alpha \oplus \delta' \leqslant \gamma = \alpha \oplus \delta < \alpha \oplus \delta'$, which is absurd. Therefore $\delta = \beta$.

4. $\alpha \otimes 1 = \alpha$ by **A8**.

5. By **A7**, $\alpha \otimes \beta' = \alpha \otimes (\beta \oplus 1) = \alpha \otimes \beta \oplus \alpha \otimes 1 = \alpha \otimes \beta \oplus \alpha$.

6. The operation $\xi \mapsto \delta \otimes \xi$ is always strictly increasing, because if again $\alpha < \beta$ in A, so that $\alpha \oplus \gamma = \beta$ for some γ, then

$$\delta \otimes \alpha < \delta \otimes \alpha \oplus \delta \otimes \gamma = \delta \otimes (\alpha \oplus \gamma) = \delta \otimes \beta.$$

Now suppose again β is a limit ordinal in A, so $\alpha \otimes \beta$ is an upper bound of $\{\alpha \otimes \xi \colon \xi < \beta\}$. Let γ be the least upper bound. By **A9**, for some δ and θ, we have either $\alpha \otimes \delta = \gamma$ or $\alpha \otimes \delta \oplus \theta = \gamma$, where $\theta < \alpha$. Again $\delta \leqslant \beta$, but $\delta < \beta$ yields absurdity, so $\delta = \beta$, and there is no θ.

Thus $\oplus$ and $\otimes$ are $+$ and $\times$. The ordinal $A \cup \{0\}$ has a Cantor normal form

$$\omega^{\alpha_0} \cdot b_0 + \cdots + \omega^{\alpha_n} \cdot b_n,$$

where $\alpha_0 > \cdots > \alpha_n$ and $\{b_0, \dots, b_n\} \subseteq \omega \smallsetminus 1$. If $n > 0$, then A contains $\omega^{\alpha_0} \cdot b_0$, but not its double, namely $\omega^{\alpha_0} \cdot b_0 + \omega^{\alpha_0} \cdot b_0$ or $\omega^{\alpha_0} \cdot (b_0 \cdot 2)$: this is absurd, since A is closed under addition. Thus $n = 0$, and we may write $A \cup \{0\} = \omega^{\alpha} \cdot b$. If $b > 1$, then $b = c'$ for some c in $\omega \smallsetminus 1$, and A contains $\omega^{\alpha} \cdot c$, but not its double, which again is absurd. Thus $b = 1$, and $A \cup \{0\} = \omega^{\alpha}$. Since A is closed under multiplication, α must be closed under addition, so as before, α must be ω^{β} for some β. □

Corollary 1. *The first nine axioms in §4.4 entail the next three.*

Corollary 2. *With the first nine axioms in §4.4, either of **A13** and **A14** entails*

$$(\omega \smallsetminus 1, +, \times, \in) \cong (\mathbb{N}, 1, +, \times, <),$$

and therefore $\times$ on $\mathbb{N}$ is commutative.

Proof. We prove the contrapositive. In the statement of Theorem 1, if $\beta > 0$, then α contains $\omega + 1$; but

$$\begin{aligned}(\omega + 1) \cdot \omega = \omega \cdot \omega &< \omega \cdot \omega + 1 \cdot \omega,\\ 1 + \omega = \omega &< \omega + 1.\end{aligned}$$

□

Of course Euclid's argument for the commutativity of multiplication does not proceed as above. We consider it again in the next section. Meanwhile, there is yet another approach:

Theorem 2. *The first six axioms in §4.4, along with Axiom 14, entail*

$$(\omega \smallsetminus 1, 1, {}', \in) \cong (\mathbb{N}, 1, x \mapsto x + 1, <).$$

Proof. Without the use of any axioms about $\mathbb{N}$ at all, there is a unique homomorphism from $(\omega \smallsetminus 1, 1, ')$ to $(\mathbb{N}, 1, x \mapsto x+1)$. Showing that the homomorphism is an isomorphism is equivalent to establishing the so-called Peano Axioms:

1. $(\mathbb{N}, 1, x \mapsto x+1)$ allows proof by induction.
2. The operation $x \mapsto x+1$ on $\mathbb{N}$ is not surjective.
3. The operation $x \mapsto x+1$ on $\mathbb{N}$ is injective.

See for example [79, Thm 2].

1. In $\mathbb{N}$, if $a \neq 1$, then $a > 1$ by **A3**, so $a = 1 + b$ for some b by **A6**, and then $a = b + 1$ by **A14**, so $b < a$ by **A5**. Therefore the structure $(\mathbb{N}, 1, x \mapsto x+1)$ allows proof by induction by the standard argument: if $B \subseteq \mathbb{N}$, and $1 \in B$, and $a + 1 \in B$ whenever $a \in B$, then the complement $\mathbb{N} \smallsetminus B$ can contain no least element, so by **A2** it is empty.

2. Since $1 \leqslant a$ and $a < a+1$, so that $1 < a+1$ by **A1**, the operation $x \mapsto x+1$ on $\mathbb{N}$ is not surjective.

3. Using also **A4** as in the proof of Theorem 1, if $a < b$, then we have $a+1 = 1+a < 1+b = b+1$, and so the operation $x \mapsto x+1$ on $\mathbb{N}$ is injective.

Finally, since the ordering of $\mathbb{N}$ is determined by addition according to the rule

$$a < b \iff \exists x\; a + x = b, \tag{4.9}$$

and a similar rule holds for $\omega \smallsetminus 1$, we have the desired isomorphism. □

Conversely, the Peano Axioms entail all of the axioms of §4.4, in the following sense. Assuming only that the structure $(\mathbb{N}, 1, x \mapsto x+1)$ allows proof by induction, Landau shows implicitly in *Foundations of Analysis* [58] that there are unique operations $+$ and $\times$ on $\mathbb{N}$ such that

$$x + (y + 1) = (x + y) + 1,$$

$$x \cdot 1 = x, \qquad x \cdot (y+1) = x \cdot y + x.$$

By induction too, these operations respect the axioms in §4.4 that govern the operations alone and 1. In the same way, multiplication is commutative. Under the additional assumption that $x \mapsto x+1$ is injective, but not surjective, one shows that the relation $<$ defined by (4.9) satisfies the remaining axioms.

4.6 Euclid's Argument

Finally we cast Euclid's argument for the commutativity of multiplication in modern terms, using the axioms above. We might take the following as being obvious for numbers, as Euclid seems to; but we *can* prove it using the axioms:

Lemma 1. *In* $\mathbb{N}$*, if* $a > b$*, then the equation*

$$a = b + x \tag{4.10}$$

has a unique *solution.*

Proof. There is a solution by **A6**. Any two solutions are comparable, by **A1**. But then there cannot be two solutions, since the operation $x \mapsto b+x$ is strictly increasing, as in the proof of Theorem 1, which uses also **A5** and **A4**. □

The unique solution to (4.10) is the **difference** of a from b, denoted by

$$a - b.$$

Lemma 2. *In* $\mathbb{N}$*, if* $a > b$*, then*

$$c \cdot (a - b) = c \cdot a - c \cdot b.$$

Proof. By **A7**,

$$\begin{aligned} b + (a - b) &= a, \\ c \cdot \big(b + (a - b)\big) &= c \cdot a, \end{aligned}$$

$$c \cdot b + c \cdot (a - b) = c \cdot a,$$
$$c \cdot (a - b) = c \cdot a - c \cdot b. \qquad \square$$

Lemma 3. *If a measures b, then $a \leqslant b$.*

Proof. If a measures b, this means $a \cdot c = b$ for some c. Since $c \geqslant 1$ by **A3**, we have $a \leqslant a \cdot c$ by **A11** (in case $c > 1$) or **A8** (in case $c = 1$). $\square$

The **Euclidean Algorithm** is given in **Propositions 1 and 2** of Book VII. If $a > b$, then by **A9** either $a = b \cdot k$ or $a = b \cdot k + c$ for some k and c such that $b > c$. In the latter case, either $b = c \cdot \ell$ or $b = c \cdot \ell + d$ for some ℓ and d such that $c > d$, and so on. We can understand **A2** to mean simply that the sequence $(a, b, c, d, \dots)$ must terminate; and then its last entry will be measured by all common measures of a and b and will be one itself. The following theorem spells out the details. We do not really make the argument more rigorous by replacing $(a, b, c, d, \dots)$ with $(a_1, a_2, a_3, a_4, \dots)$ unless we actually have a theory of the indices.

Theorem 3. *In $\mathbb{N}$, if $a_1 > a_2$, there are sequences $(b_1, b_2, \dots)$ and $(a_1, a_2, a_3, \dots)$ given by*

$$a_k = a_{k+1} \cdot b_k + a_{k+2} \;\;\&\;\; a_{k+1} > a_{k+2}.$$

The sequences must terminate: some a_{n+2} does not exist, but

$$a_n = a_{n+1} \cdot b_n. \tag{4.11}$$

Then

$$a_1 > a_2 > a_3 > \dots > a_{n+1}, \tag{4.12}$$

and a_{n+1} is a common measure of a_1 and a_2, and a_{n+1} is measured by every common measure of a_1 and a_2, so a_{n+1} is greater than every other common measure of a_1 and a_2.

Proof. By **A9**, from a_k and a_{k+1}, we can obtain b_k and perhaps a_{k+2}. Then $a_1 > a_2 > a_3 > \cdots$ by **A1**. By **A2** then, for some n, there is no a_{n+2}, so we have (4.11) and (4.12). We can now compute

$$\begin{aligned}
a_{n-1} &= a_n \cdot b_{n-1} + a_{n+1} \\
&= (a_{n+1} \cdot b_n) \cdot b_{n-1} + a_{n+1} \\
&= a_{n+1} \cdot (b_n \cdot b_{n-1}) + a_{n+1} && [\mathbf{A10}] \\
&= a_{n+1} \cdot (b_n \cdot b_{n-1}) + a_{n+1} \cdot 1 && [\mathbf{A8}] \\
&= a_{n+1} \cdot (b_n \cdot b_{n-1} + 1). && [\mathbf{A7}]
\end{aligned}$$

Continuing in this way, we obtain a_{n+1} as a common measure of a_1 and a_2. If d is a common measure, then

$$\begin{aligned}
d \cdot x &= (d \cdot y) \cdot b_1 + a_3 \\
&= d \cdot (y \cdot b_1) + a_3,
\end{aligned}$$

so by Lemma 2,

$$a_3 = d \cdot (x - y \cdot b_1).$$

Continuing in this way, we have d as a common measure of all of the a_k, including a_{n+1}. By Lemma 3 then, a_{n+1} is greater than every other common measure of a_1 and a_2. □

The sequence $(b_1, b_2, \dots)$ in the theorem is the **anthyphaeretic sequence** of (a_1, a_2) as defined on p. 87. The number a_{n+1} in the theorem is the **greatest common measure** of a_1 and a_2, and we may write

$$a_{n+1} = \operatorname{gcm}(a_1, a_2).$$

Two numbers are **prime to one another,** as in Definition 12 of Book VII, if their only (and therefore their greatest) common measure is unity.

As noted earlier (p. 123), **Proposition 5** of Book VII is our **A13**. Meanwhile, though *proportion* is mentioned in Definition 4, the real meaning is suggested by **Proposition 4,** as in §3.7 (p. 90): four numbers a, b, c, and d are **proportional,** and we shall write this as

$$a : b :: c : d, \tag{4.13}$$

just in case, for some e and f,

$$\begin{aligned} a &= \operatorname{gcm}(a,b) \cdot e, & c &= \operatorname{gcm}(c,d) \cdot e, \\ b &= \operatorname{gcm}(a,b) \cdot f, & d &= \operatorname{gcm}(c,d) \cdot f. \end{aligned} \tag{4.14}$$

Though we shall not actually use it, the following observation can be taken as the motivation for Proposition 4:

Theorem 4. *Assuming $a > b$ and $c > d$,* (4.13) *holds if and only if (a, b) or (c, d) have the same anthyphaeretic sequence.*

Proof. The proof of Theorem 3 shows that the numbers e and f such that $a = \gcd(a, b) \cdot e$ and $b = \gcd(a, b) \cdot f$ depend only on the anthyphaeretic sequence of (a, b). Conversely, by its definition in the statement of the theorem, the sequence depends only on e and f. □

The "definition" of proportion at the head of Book VII suggests the following.

Theorem 5. *If a measures b, so that $a \cdot f = b$ for some f, then*

$$a : b :: c : d \iff c \cdot f = d.$$

Proof. Under the hypothesis, $\operatorname{gcm}(a, b) = a$ by Theorem 3, and so

$$\begin{aligned} a : b :: c : d &\iff \begin{cases} c = \operatorname{gcm}(c,d), \\ d = \operatorname{gcm}(c,d) \cdot f \end{cases} \\ &\iff d = c \cdot f. \end{aligned}$$ □

Euclid seems not to make the following two lemmas explicit. Lemma 4, like Lemma 1, might be considered as axiomatic. Lemma 5 might be taken as an obvious consequence of the axioms, although writing out a proof in modern fashion is tedious.

Lemma 4. *If $a \cdot b = a \cdot c$, then $b = c$.*

Proof. If $b \neq c$, then we may assume $b < c$, by **A1**. But then $a \cdot b < a \cdot c$, since $x \mapsto a \cdot x$ is strictly increasing, as in the proof of Theorem 1, which uses **A6**, **A5**, and **A7**. □

Lemma 5. *Under the conditions (4.14), e and f must be prime to one another. Conversely, if this is so, and*

$$\begin{aligned} a &= g \cdot e, & c &= h \cdot e, \\ b &= g \cdot f, & d &= h \cdot f \end{aligned} \tag{4.15}$$

for some g and h, then (4.13) holds.

Proof. Given (4.14) and **A10**, we find that $\operatorname{gcm}(a, b) \cdot \operatorname{gcm}(e, f)$ is a common measure of a and b. Then

$$\operatorname{gcm}(a, b) \cdot \operatorname{gcm}(e, f) \leqslant \operatorname{gcm}(a, b)$$

by Theorem 3; but if $\operatorname{gcm}(e, f) > 1$, then

$$\operatorname{gcm}(a, b) \cdot \operatorname{gcm}(e, f) > \operatorname{gcm}(a, b)$$

by **A11**; therefore $\operatorname{gcm}(e, f) = 1$ by **A1** and **A3**.

Conversely, if (4.15) holds, then g is a common measure of a and b, so for some k,

$$g \cdot k = \operatorname{gcm}(a, b).$$

But for some e' and f',

$$\begin{aligned} g \cdot e = a = \operatorname{gcm}(a, b) \cdot e' = (g \cdot k) \cdot e' = g \cdot (k \cdot e'), \\ g \cdot f = b = \operatorname{gcm}(a, b) \cdot f' = (g \cdot k) \cdot f' = g \cdot (k \cdot f') \end{aligned}$$

by **A10**, so

$$e = k \cdot e', \qquad\qquad f = k \cdot f'$$

by Lemma 4, and so k is a common divisor of e and f. Suppose these are prime to one another. then $g = \mathrm{gcm}(a, b)$ by **A8**, and likewise $h = \mathrm{gcm}(c, d)$. We thus obtain (4.14), and therefore (4.13). □

I suppose it is just possible that Euclid overlooked the need to prove the last lemma. It seems to me more likely that he would just reason as follows. Suppose a and b are as in (4.14). Applying the Euclidean algorithm to a and b of course yields $\mathrm{gcm}(a, b)$. But if we now consider $\mathrm{gcm}(a, b)$ as a unit, this shows that the same algorithm applied to e and f yields unity. Conversely, if we know this, but a and b are as in (4.15), then the algorithm applied to these will obviously yield g.

In any case, the lemma yields the following, which is **Proposition 12** of Book VII:

Theorem 6. *If $a : b :: c : d$, then*

$$a : b :: a + c : b + d.$$

Proof. Suppose (4.13) holds, so that (4.14) holds. By **A13**,

$$\begin{aligned} a + c &= \bigl(\mathrm{gcm}(a, b) + \mathrm{gcm}(c, d)\bigr) \cdot e, \\ b + d &= \bigl(\mathrm{gcm}(a, b) + \mathrm{gcm}(c, d)\bigr) \cdot f, \end{aligned}$$

and therefore $a : b :: a + c : b + d$ by Lemma 5. □

Euclid's **Proposition 15** is that if b measures a as many times as unity measures c, then c measures a as many times as unity measures b. Since $c = 1 \cdot c$ by **A12**, the conclusion is

$$b \cdot c = a \implies c \cdot b = a,$$

or simply the following theorem, which is Euclid's **Proposition 16.**

Theorem 7. *In $\mathbb{N}$, multiplication is commutative:*

$$a \cdot b = b \cdot a.$$

Proof. Since $a = a \cdot 1$ by **A8**, we have

$$1 : a :: 1 : a \cdot 1.$$

Suppose

$$1 : a :: c : a \cdot c.$$

Then

$$\begin{aligned} 1 : a &:: 1 + c : a + a \cdot c && [\text{Theorem 6}] \\ &:: c + 1 : a \cdot c + a && [\mathbf{A14}] \\ &:: c + 1 : a \cdot c + a \cdot 1 && [\mathbf{A8}] \\ &:: c + 1 : a \cdot (c + 1) && [\mathbf{A7}]. \end{aligned}$$

By induction (which we established in the proof of Theorem 2, by means of **A3**, **A6**, **A14**, **A5**, and **A2**),

$$1 : a :: b : a \cdot b. \tag{4.16}$$

Since $a = 1 \cdot a$ by **A12**, by Theorem 5 we must have $b \cdot a = a \cdot b$. □

Euclid does not use induction as we have. But we can apply Theorem 6 as many times as we like to get

$$\begin{aligned} 1 : a &:: 1 + \cdots + 1 : a + \cdots + a \\ &:: 1 + \cdots + 1 : a \cdot 1 + \cdots + a \cdot 1 && [\mathbf{A8}] \\ &:: 1 + \cdots + 1 : a \cdot (1 + \cdots + 1) && [\mathbf{A7}]. \end{aligned}$$

For Euclid, every number is a sum $1 + \cdots + 1$, and so (4.16) follows.[4]

[4] According to Mueller [69, p. 69], " it is important to realize that in every instance where Euclid proceeds in this quasi-inductive way, the implicit induction is on the number of terms involved in a construction or assertion and not on the integers themselves. He does not, for example, prove the commutativity of multiplication (VII,16) by induction, but by a general example."

We can just use the axioms to show that there is no *least* number that is not of the form $1 + \cdots + 1$.

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