FIELDS WITH SEVERAL COMMUTING DERIVATIONS

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Abstract. The existentially closed models of the theory of fields (of arbitrary characteristic) with a given finite number of commuting derivations can be characterized geometrically, in several ways. In each case, the existentially closed models are those models that contain points of certain differential varieties, which are determined by certain ordinary varieties.

How can we tell whether a given system of partial differential equations has a solution? An answer given in this paper is that, if we differentiate the equations enough times, and no contradiction arises, then it never will, and the system is soluble. Here, the meaning of 'enough times' can be expressed *uniformly;* this is one way of showing that the theory, m-DF, of fields with a finite number m of commuting derivations has a model-companion. In fact, this theorem is worked out here (as Corollary 4.6, of Theorem 4.5), not in terms of polynomials, but in terms of the varieties that they define, and the function-fields of these: in a word, the treatment is *geometric.*

The model-companion of m-DF_0 (in characteristic 0) has been axiomatized before, explicitly in terms of differential polynomials: see § 3. I attempted in [11] to characterize its models (namely, the existentially closed models of m-DF_0) in terms of differential forms, but I made a mistake. Here I correct the mistake, but also work in arbitrary characteristic. The existence of a model-companion of m-DF (with no specified characteristic) appears to be a new result when $m > 1$ (despite a remark by Saharon Shelah [22, p. 315]: 'I am quite sure that for characteristic p as well, [making m greater than 1] does not make any essential difference').

The theory of model-companions and model-completions was worked out decades ago; perhaps for that very reason, it may be worthwhile to review the theory here, as I do in § 1. I try to give the original references, when I have been able to consult them. In § 2, I review the various known characterizations of existentially closed fields with single derivations. Then §§ 3 and 4 consider how those characterizations can be generalized to allow for several commuting derivations.

§1. Model-theoretic background. Let $\mathfrak{M}$ be an arbitrary (first-order) structure; its theory is $\mathrm{Th}(\mathfrak{M})$. Let T be an arbitrary consistent (first-order) theory; its models compose the class $\mathrm{Mod}(T)$. Every class $\boldsymbol{K}$ of structures in some signature has a theory, $\mathrm{Th}(\boldsymbol{K})$. Then $\boldsymbol{K} \subseteq \mathrm{Mod}(\mathrm{Th}(\boldsymbol{K}))$; in case of equality, $\boldsymbol{K}$ is **elementary.** Always, $\mathrm{Th}(\mathrm{Mod}(T)) = T$.

The structure $\mathfrak{M}$ has the universe M. The structure denoted by $\mathfrak{M}_M$ is the expansion of $\mathfrak{M}$ that has a name for every element of M. Then $\mathfrak{M}$ embeds in $\mathfrak{N}$ if and only if $\mathfrak{M}_M$

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embeds in an expansion of $\mathfrak{N}$. However, although the class of structures in which $\mathfrak{M}$ embeds need not be elementary, the class of structures in which $\mathfrak{M}_M$ embeds *is* elementary. The theory of the latter class is the **diagram** of $\mathfrak{M}$, or $\operatorname{diag}(\mathfrak{M})$: it is axiomatized by the quantifier-free sentences in $\operatorname{Th}(\mathfrak{M}_M)$ [16, Thm 2.1.3, p. 24]. The class of structures in which $\mathfrak{M}_M$ embeds *elementarily* is also elementary, and its theory is just $\operatorname{Th}(\mathfrak{M}_M)$. The class of substructures of models of T is elementary, and its theory is denoted by $T_\forall$: this is axiomatized by the universal sentences of T [16, Thm 3.3.2, p. 71].

By a **system over** $\mathfrak{M}$, I mean a finite conjunction of atomic and negated atomic formulas in the signature of $\mathfrak{M}_M$; likewise, a system **over** T is in the signature of T. A structure $\mathfrak{M}$ **solves** a system $\varphi(\boldsymbol{x})$ if $\mathfrak{M} \models \exists \boldsymbol{x}\ \varphi(\boldsymbol{x})$. Note well here that $\boldsymbol{x}$, in boldface, is a *tuple* of variables, perhaps $(x^0, \dots, x^{n-1})$. By an **extension** of a model of T, I mean another model of T of which the first is a substructure. Two systems over a model $\mathfrak{M}$ of T are **equivalent** if they are soluble in the same extensions.

An **existentially closed** model of T is a model of T that solves every system over itself that is soluble in some extension. So a model $\mathfrak{M}$ of T is existentially closed if and only if $T \cup \operatorname{diag}(\mathfrak{M}) \vdash \operatorname{Th}(\mathfrak{M}_M)_\forall$, that is, every extension of $\mathfrak{M}$ is a substructure of an elementary extension ([5, § 7] or [23, § 2]).

A theory is **model-complete** if its every model is existentially closed. An equivalent formulation explains the name: T is model-complete if and only if $T \cup \operatorname{diag}(\mathfrak{M})$ is complete whenever $\mathfrak{M} \models T$ [17, Ch. 2].

Suppose every model of T has an existentially closed extension. Such is the case when T is **inductive,** that is, $\operatorname{Mod}(T)$ is closed under unions of chains [5, Thm 7.12]: equivalently, $T = T_{\forall\exists}$ [8, 3]. Suppose further that we have a uniform first-order way to tell when systems over models of T are soluble in extensions: more precisely, suppose there is a function

$$\varphi(\boldsymbol{x}, \boldsymbol{y}) \longmapsto \widehat{\varphi}(\boldsymbol{x}, \boldsymbol{y}), \tag{1}$$

where $\varphi(\boldsymbol{x}, \boldsymbol{y})$ ranges over the systems over T (with variables analyzed as shown), such that, for every model $\mathfrak{M}$ of T and every tuple $\boldsymbol{a}$ of parameters from M, the system $\varphi(\boldsymbol{x}, \boldsymbol{a})$ is soluble in some extension of $\mathfrak{M}$ just in case $\widehat{\varphi}(\boldsymbol{x}, \boldsymbol{a})$ is soluble in $\mathfrak{M}$. Then the existentially closed models of T compose an elementary class, whose theory T^* is axiomatized by T together with the sentences

$$\forall \boldsymbol{y}\ (\exists \boldsymbol{x}\ \widehat{\varphi}(\boldsymbol{x}, \boldsymbol{y}) \to \exists \boldsymbol{x}\ \varphi(\boldsymbol{x}, \boldsymbol{y})). \tag{2}$$

Immediately, T^* is model-complete, so $T^* \cup \operatorname{diag}(\mathfrak{M})$ is complete when $\mathfrak{M} \models T^*$. What is more, $T^* \cup \operatorname{diag}(\mathfrak{M})$ is complete whenever $\mathfrak{M} \models T$ [16, Thm 5.5.1].

In general, T^* is a **model-completion** of T if $T^*{}_\forall \subseteq T \subseteq T^*$ and $T^* \cup \operatorname{diag}(\mathfrak{M})$ is complete whenever $\mathfrak{M} \models T$. Model-completions are unique [15, (2.8)]. We have sketched the proof of part of the following (the rest is [15, (3.5)]):

Lemma 1.1 (Robinson's Criterion). *Let T be inductive. Then T has a model-completion if and only if a function $\varphi(\boldsymbol{x}, \boldsymbol{y}) \mapsto \widehat{\varphi}(\boldsymbol{x}, \boldsymbol{y})$ exists as in* (1)*. In this case, the model-completion is axiomatized* modulo T *by the sentences in* (2)*.*

If $T_\forall = T^*{}_\forall$ and T^* is model-complete, then T^* is a **model-companion** of T ([1, § 5]; *cf.* [5, § 2]). Model-completions are model-companions, and model-companions are unique [1, Thm 5.3]. If T has a model-companion, then its models are just the existentially closed models of T [5, Prop. 7.10]. Conversely, if T is inductive, and the class of existentially closed models of T is elementary, then the theory of this class is the model-companion of T [5, Cor. 7.13].

§2. Fields with one derivation. Let DF be the theory of fields with a derivation, and let DPF be the theory of models of DF that, for each prime ℓ, satisfy also

$$\forall x\ \exists y\ (\underbrace{1+\cdots+1}_{\ell}=0 \wedge Dx=0 \to \underbrace{y\cdots y}_{\ell}=x).$$

So models of DPF are **differentially perfect.** A subscript on the name of one of these theories will indicate a required characteristic for the field. In particular, we have DPF_0, which is the same as DF_0.

Abraham Seidenberg [21] shows the existence of the function in Lemma 1.1 in case T is DPF_p, where p is prime or 0. Consequently:

Theorem 2.1 (Robinson). DF_0 *has a model-completion, called* DCF_0.

Theorem 2.2 (Wood [26]). *If p is prime, then* DF_p *has a model-companion,* DCF_p*, which is the model-completion of* DPF_p.

2.1. Single variables. Since it involves *all* systems over a given theory, Robinson's criterion yields the crudest possible axiomatization for a model-completion. By contrast, though the theory ACF of algebraically closed fields is the model-completion of the theory of fields, its axioms (*modulo* the latter theory) can involve only systems in one variable (indeed, single equations in one variable). A generalization of this observation is the following, which can be extracted from the proof of [19, Thm 17.2, pp. 89–91] (see also [2]):

Lemma 2.3 (Blum's Criterion). *Say* $T^*{}_\forall \subseteq T \subseteq T^*$.

(i) *The theory T^* is the model-completion of T if and only if the commutative diagram*

$$\begin{tikzcd} \mathfrak{M} & \\ \mathfrak{A} \arrow[u] \arrow[r] & \mathfrak{B} \arrow[ul, dotted] \end{tikzcd}$$

of structures and embeddings can be completed as indicated when $\mathfrak{A}$ and $\mathfrak{B}$ are models of T and $\mathfrak{M}$ is a $|B|^+$-saturated model of T^*.

(ii) *If $T = T_\forall$, it is enough to assume that $\mathfrak{B}$ is generated over $\mathfrak{A}$ by a single element.*

This allows a refinement of Lemma 1.1 in a special case:

Lemma 2.4. *Suppose $T = T_\forall$. Then Lemma 1.1 still holds when $\varphi(\boldsymbol{x}, \boldsymbol{y})$ is replaced with $\varphi(x, \boldsymbol{y})$ (where x is a single variable).*

From Lemma 2.3, Lenore Blum obtains Theorem 2.5 below in characteristic 0, in which case the first two numbered conditions amount to $K \models$ ACF ([19, pp. 298 ff.] or [2]). If $p > 0$, then DF_p is not universal, so part (ii) of Blum's criterion does *not* apply; Carol Wood instead uses a primitive-element theorem of Seidenberg [20] to obtain new axioms for DCF_p [27]. These can be combined with Blum's axioms for DCF_0 to yield the following. (Here SCF is the theory of separably closed fields.)

Theorem 2.5 (Blum, Wood). *A model (K, D) of* DF *is existentially closed if and only if*

(i) $K \models \mathrm{SCF}$*;*
(ii) $(K, D) \models \mathrm{DPF}$*;*
(iii) $(K, D) \models \exists x\ (f(x, Dx, \dots, D^{n+1}x) = 0 \wedge g(x, Dx, \dots, D^n x) \neq 0)$ *whenever* f *and* g *are ordinary polynomials over* K *in tuples* $(x^0, \dots, x^{n+1})$ *and* $(x^0, \dots, x^n)$ *of variables respectively such that* $g \neq 0$ *and* $\partial f/\partial x^{n+1} \neq 0$.

Hence DF *has a model-companion,* DCF.

There is a similar characterization of the existentially closed *ordered* differential fields [24].

2.2. First derivatives. Alternative simplified axioms for DCF are parallel to those found for the model-companion ACFA of the theory of fields with an automorphism [9, 4]. Suppose $(K, D) \models \mathrm{DPF}$ and $K \models \mathrm{SCF}$. Every system over (K, D) can be written as

$$\bigwedge_f f(\boldsymbol{x}, D\boldsymbol{x}, \dots, D^n\boldsymbol{x}) = 0 \wedge g(\boldsymbol{x}, D\boldsymbol{x}, \dots, D^n\boldsymbol{x}) \neq 0, \tag{3}$$

where g and the f are ordinary polynomials over K. This system is equivalent to one that involves only first derivatives, namely

$$\bigwedge_f f(\boldsymbol{x}_0, \dots, \boldsymbol{x}_{n-1}, \boldsymbol{y}_{n-1}) = 0 \wedge g(\boldsymbol{x}_0, \dots, \boldsymbol{x}_{n-1}, \boldsymbol{y}_{n-1}) \neq 0 \wedge \\ \wedge \bigwedge_{i+1<n} \boldsymbol{x}_{i+1} = \boldsymbol{y}_i \wedge \bigwedge_{i<n} D\boldsymbol{x}_i = \boldsymbol{y}_i.$$

This system, in form, is a special case of the system

$$\bigwedge_f f(\boldsymbol{x}, \boldsymbol{y}) = 0 \wedge g(\boldsymbol{x}, \boldsymbol{y}) \neq 0 \wedge D\boldsymbol{x} = \boldsymbol{y}, \tag{4}$$

which can also be written as $\bigwedge_f f(\boldsymbol{x}, D\boldsymbol{x}) = 0 \wedge g(\boldsymbol{x}, D\boldsymbol{x}) \neq 0$. Suppose (4) has the solution $(\boldsymbol{a}, \boldsymbol{b})$. Then $K(\boldsymbol{a}, \boldsymbol{b})/K$ is separable [12, Lem. 1.5, p. 1328]. Let V and W be the varieties over K with generic points $\boldsymbol{a}$ and $(\boldsymbol{a}, \boldsymbol{b})$ respectively, let $T_D(V)$ be the twisted tangent bundle of V, and let U be the open subset of W defined by the inequation $g \neq 0$; then the situation can be depicted thus:

$$\begin{tikzcd} U \arrow[r, tail] \arrow[dr, dashed, two heads] & W \arrow[r, tail] \arrow[d, dashed, two heads] & T_D(V) \arrow[dl, dashed, two heads, "(\boldsymbol{x},\boldsymbol{y})\mapsto\boldsymbol{x}"] \\ & V & \end{tikzcd}$$

In characteristic 0, the model (K, D) of DF is existentially closed if and only if, in every such geometric situation, U contains a K-rational point $(\boldsymbol{c}, D\boldsymbol{c})$; this yields the so-called geometric axioms for DCF_0 found with Anand Pillay [13]. In positive characteristic, it is still true that, if $(\boldsymbol{a}, \boldsymbol{b})$ is a generic point of V, then D extends to $K(\boldsymbol{a})$ so that $D\boldsymbol{a} = \boldsymbol{b}$. However, an additional condition is needed to ensure that D extends to all of $K(\boldsymbol{a}, \boldsymbol{b})$; it is enough to require that the projection of $T_D(W)$ onto $T_D(V)$ contain a generic point of W; this yields Piotr Kowalski's geometric axioms for DCF_p [7].

In an alternative geometric approach to DCF, instead of (3), it is enough to look at an arbitrary system of *equations,* $\bigwedge_f f(\boldsymbol{x}, D\boldsymbol{x}, \dots, D^n\boldsymbol{x}) = 0$. As before, we can take the derivations out of the polynomials, this time getting the equivalent system

$$\bigwedge_f f(\boldsymbol{x}_0, \dots, \boldsymbol{x}_n) = 0 \wedge \bigwedge_{i<n} D\boldsymbol{x}_i = \boldsymbol{x}_{i+1}.$$

This is a special case of

$$\bigwedge_f f(x^0, \dots, x^{n-1}) = 0 \wedge \bigwedge_{i<k} Dx^i = g^i(x^0, \dots, x^{n-1}), \tag{5}$$

where $k \leqslant n$ and the g^i are rational functions over K. Suppose this has solution $\boldsymbol{a}$, which is a generic point of a variety V. It is enough to assume that $(a^0, \dots, a^{k-1})$ is a separating transcendence-basis of $K(\boldsymbol{a})/K$. Then we have a dominant, separable rational map $\boldsymbol{x} \mapsto (x^0, \dots, x^{k-1})$ or φ from V onto $\mathbb{A}^k$, and another rational map $\boldsymbol{x} \mapsto (g^0(\boldsymbol{x}), \dots, g^{k-1}(\boldsymbol{x}))$ or ψ from V to $\mathbb{A}^k$. So (K, D) is existentially closed if and only if V always has a K-rational point P such that $D(\varphi(P)) = \psi(P)$ [12, Thm 1.6, p. 1328].

§3. Fields with several derivations. Let m-DF be the theory of fields with m commuting derivations. Tracey McGrail [10] axiomatizes the model-completion, m-DCF$_0$, of m-DF$_0$. Alternative axiomatizations arise as special cases in work of Yoav Yaffe [28] and Marcus Tressl [25]. There is a common theme: A differential ideal has a generating set of a special form; in the terminology of Joseph Ritt [14, § I.5, p. 5] (when $m = 1$) and Ellis Kolchin [6, § I.10, pp. 81 ff.], this is a *characteristic set.* There is a first-order way to tell, uniformly in the parameters, whether a given set of differential polynomials is a characteristic set of some differential ideal, and then to tell, if it *is* a characteristic set, whether it has a root. In short, the function $\varphi \mapsto \widehat{\varphi}$ in Robinson's criterion (Lemma 1.1) is defined for sufficiently many systems φ. (Applying Blum's criterion, McGrail and Yaffe consider only systems in one variable, so they must include inequations in these systems; Tressl uses only equations, in arbitrarily many variables.)

I do not give the definition of a characteristic set, as not all ingredients of the definition are needed for the arguments presented in § 4. However, some of the ingredients *are* needed; these are in 4.2.

3.1. Differential forms. In [11] I attempted to apply the geometric approach described in 2.2 to m-DF$_0$. I worked more generally with DF$_0^m$, where DF^m is the theory of structures $(K, D_0, \dots, D_{m-1})$ such that $(K, D_i) \models$ DF for each i, and each bracket $[D_j, D_k]$ is a K-linear combination of the D_i. (This is roughly what Yaffe did too.) In [12, § 2] I made some minor corrections and otherwise adapted the argument to arbitrary characteristic. Nonetheless, in May, 2006, Ehud Hrushovski showed me a counterexample to [11, Thm A, p. 926], a theorem that was an introductory formulation of [11, Thm 5.7, p. 942]. Then I found an error at the end of the proof of the latter theorem. Before offering a resolution of the problem, let me review the general situation.

Let $(K, D_0, \dots, D_{m-1}) \models$ DF^m, and let E be the K-linear span of the D_i. Then E is a Lie-ring, as well as a vector-space over K. As a vector-space, E has a dual, E^*; and there is a derivation d from K into E^* given by

$$D(\mathrm{d}\, x) = Dx. \tag{6}$$

Then E^* has a basis $(\mathrm{d}\, t^i : i < \ell)$ for some t^i in K and some ℓ no greater than m [11, Lem. 4.4, p. 932], and this basis is dual to a basis $(\partial_i : i < \ell)$ of E, where $[\partial_i, \partial_j] = 0$ in each case, and d is then given by

$$\mathrm{d}\,x = \sum_{i<\ell} \mathrm{d}\,t^i \cdot \partial_i x \tag{7}$$

[11, Lem. 4.7, p. 934]. We can use these ideas to find an extension of $(K, D_0, \dots, D_{m-1})$ in which the named derivations are linearly independent over the field. Indeed, if $\ell < m$, then let $L = K(\alpha^\ell, \dots, \alpha^{m-1})$, where $(\alpha^\ell, \dots, \alpha^{m-1})$ is algebraically independent over K. Extend the original ∂_i (where $i < \ell$) to L so that they are 0 at the α^j; then, if $\ell \leqslant k < m$, define ∂_k to be 0 on K and to be δ^j_k at α^j. Then $(\partial_i \colon i < m)$ is a linearly independent m-tuple of commuting derivations on L; from this, we obtain linearly independent extensions of the D_i to L such that the brackets $[D_j, D_k]$ are the same linear combinations of the D_i as before ([11, Lem. 5.2, p. 937]—or [24, Lem. 2.1, p. 1930], by a different method—in characteristic 0; generally, [12, Lem. 2.4, p. 1334]).

Consequently, a model-companion DCF^m for DF^m can be obtained from a model-companion m-DCF for m-DF ([11, Thm 5.3 and proof] or [24, § 3]). This much stands, and differential forms are convenient for establishing it.

But I tried also to obtain DCF^m independently as follows. Suppose now we have a separably closed field K, along with a Lie-ring and finite-dimensional space E of derivations of K; as a space, E has a basis $(\partial_i \colon i < m)$, whose dual is $(\mathrm{d}\,t^i \colon i < m)$, so that the ∂_i commute. We may assume that $(K, \partial_0, \dots, \partial_{m-1})$ is differentiably perfect [12, Lem. 2.4]. Every system over $(K, \partial_0, \dots, \partial_{m-1})$ is equivalent to a system of the form of (5), generalized to

$$\bigwedge_f f(x^0, \dots, x^{n-1}) = 0 \wedge \bigwedge_{j<k} \bigwedge_{i<m} \partial_i x^j = g^j_i(\boldsymbol{x}). \tag{8}$$

By means of (7), we can also write this as

$$\bigwedge_f f(x^0, \dots, x^{n-1}) = 0 \wedge \bigwedge_{j<k} \mathrm{d}\,x^j = \sum_{i<m} \mathrm{d}\,t^i \cdot g^j_i(\boldsymbol{x}). \tag{9}$$

If $\boldsymbol{a}$ is a solution (from some extension), it is enough to assume that $(a^0, \dots, a^\ell)$ is a separating transcendence-basis of $K(\boldsymbol{a})/K$ for some ℓ such that $k \leqslant \ell < m$. That we cannot generally assume $k = \ell$ is an important difference from the case of one derivation; it is what causes the difficulties in the case of several derivations. The solution $\boldsymbol{a}$ to (9) can be understood as follows. First we have the field $K(\boldsymbol{a})$, and then (9) can be be written as

$$\bigwedge_{j<k} \mathrm{d}\,a^j = \sum_{i<m} \mathrm{d}\,t^i \cdot g^j_i(\boldsymbol{a}). \tag{10}$$

A solution of this can be understood as a model $(L, \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1})$ of m-DF extending $(K, \partial_0, \dots, \partial_{m-1})$ such that $K(\boldsymbol{a}) \subseteq L$ and (10) holds when $\mathrm{d}\,a^j = \sum_{i<m} \mathrm{d}\,t^i \cdot \tilde{\partial}_i a^j$, that is,

$$\bigwedge_{j<k} \bigwedge_{i<m} \tilde{\partial}_i a^j = g^j_i(\boldsymbol{a}). \tag{11}$$

Since the $\tilde{\partial}_i$ commute, it is necessary that

$$\bigwedge_{j<k} \bigwedge_{h<i<m} \tilde{\partial}_h(g^j_i(\boldsymbol{a})) = \tilde{\partial}_i(g^j_h(\boldsymbol{a})) \tag{12}$$

[11, § 1, p. 926]. Any derivative with respect to $\tilde{\partial}_i$ of an element of $K(\boldsymbol{a})$ is a constant plus a linear combination of the derivatives $\tilde{\partial}_i a^j$, where $j < \ell$ (by [12, Fact 1.1 (0, 2)], for example); we know what these derivatives $\tilde{\partial}_i a^j$ are when $j < k$, by (11); so (12) becomes a linear system over $K(\boldsymbol{a})$ in the unknowns $\tilde{\partial}_i a^j$ where $k \leqslant j < \ell$.

If $k = \ell$, then this linear system has no variables, so it is true or false; its truth is a sufficient condition for (10) to have a solution. If $k < \ell$, then the linear system is soluble or not. If it is soluble, then it is possible to extend the ∂_i to derivations $\tilde{\partial}_i$ as required by (11) that commute on $K(a^0, \dots, a^{k-1})$; but these derivations need not commute on all of $K(\boldsymbol{a})$. In [11] I claimed that they could commute, and that the solubility of (12) was sufficient for solubility of (10) in the sense above. I was wrong.

3.2. A counterexample. In the counterexample supplied by Hrushovski, the constants (m, k, ℓ) of § 3.1 are $(2, 2, 3)$. Let (a, b, c) be an algebraically independent triple, and consider the system

$$\mathrm{d}\,a = \mathrm{d}\,t^0 \cdot c \cdot c + \mathrm{d}\,t^1 \cdot c, \qquad \mathrm{d}\,b = \mathrm{d}\,t^0 \cdot 2a + \mathrm{d}\,t^1 \cdot c; \tag{13}$$

equivalently, by (7), the system comprises the equations $\partial_0 a = c \cdot c$, $\partial_1 a = c$, $\partial_0 b = 2a$, and $\partial_1 b = c$. From these, we compute

$$\partial_1\partial_0 a = 2c \cdot \partial_1 c, \qquad \partial_0\partial_1 a = \partial_0 c, \qquad \partial_1\partial_0 b = 2 \cdot \partial_1 a = 2c, \qquad \partial_0\partial_1 b = \partial_0 c.$$

Equating $\partial_0\partial_1$ and $\partial_1\partial_0$ yields the linear system

$$\partial_0 c - 2c \cdot \partial_1 c = 0, \qquad \partial_0 c = 2c,$$

which has the solution $\partial_0 c = 2c$, $\partial_1 c = 1$. But then we must have $\partial_1\partial_0 c = 2 \cdot \partial_1 c = 2$, while $\partial_0\partial_1 c = \partial_0 1 = 0$, which means (13) has no solution, contrary to my claim in [11].

For the record, the mistake is at the end of the proof of [11, Thm 5.7, p. 942] and can be seen as follows. Write the system (13) as $\mathrm{d}\,a = \alpha$, $\mathrm{d}\,b = \beta$; then

$$\begin{aligned} \mathrm{d}\,\alpha &= \mathrm{d}(\mathrm{d}\,t^0 \cdot c \cdot c + \mathrm{d}\,t^1 \cdot c) \\ &= \mathrm{d}\,c \wedge \mathrm{d}\,t^0 \cdot 2c + \mathrm{d}\,c \wedge \mathrm{d}\,t^1 \\ &= \mathrm{d}\,c \wedge (\mathrm{d}\,t^0 \cdot 2c + \mathrm{d}\,t^1), \end{aligned} \qquad \begin{aligned} \mathrm{d}\beta &= \mathrm{d}(\mathrm{d}\,t^0 \cdot 2a + \mathrm{d}\,t^1 \cdot c) \\ &= \mathrm{d}\,a \wedge \mathrm{d}\,t^0 \cdot 2 + \mathrm{d}\,c \wedge \mathrm{d}\,t^1 \\ &= (\mathrm{d}\,t^0 \cdot c \cdot c + \mathrm{d}\,t^1 \cdot c) \wedge \mathrm{d}\,t^0 \cdot 2 + \mathrm{d}\,c \wedge \mathrm{d}\,t^1 \\ &= -\,\mathrm{d}\,t^0 \wedge \mathrm{d}\,t^1 \cdot 2c + \mathrm{d}\,c \wedge \mathrm{d}\,t^1 \\ &= (\mathrm{d}\,c - \mathrm{d}\,t^0 \cdot 2c) \wedge \mathrm{d}\,t^1. \end{aligned} \tag{14}$$

Since also $\mathrm{d}\beta = \mathrm{d}^2\,b = 0$, (14) imposes a condition on $\mathrm{d}\,c \wedge \mathrm{d}\,t^1$, hence on $\partial_0 c$; in particular, $\partial_0 c = 2c$, which is what we found above. But there is no apparent condition on $\partial_1 c$, so I try introducing a new transcendental, d, for this. Then $\mathrm{d}\,c = \mathrm{d}\,t^0 \cdot 2c + \mathrm{d}\,t^1 \cdot d$. But this causes a problem, since it allows a substitution in (14), yielding

$$\begin{aligned} \mathrm{d}\,\alpha &= (\mathrm{d}\,t^0 \cdot 2c + \mathrm{d}\,t^1 \cdot d) \wedge (\mathrm{d}\,t^0 \cdot 2c + \mathrm{d}\,t^1) \\ &= \mathrm{d}\,t^0 \wedge \mathrm{d}\,t^1 \cdot 2c(1 - d), \end{aligned}$$

which means $d = 1$, contrary to assumption. In short, the next to last sentence of the proof of [11, Thm 5.7] (beginning "This ideal is linearly disjoint from") is simply wrong. (I had given no argument that it was correct.)

§4. Resolution. To resolve the problem, it is better not to introduce differential forms from the beginning, but to allow equations to involve any number of applications of the derivations. Examples (as in 4.1) may involve only one variable; but in contrast to 2.1, there does not seem to be an advantage in restricting attention to this case.

4.1. Another example. Over a differential field $(K, \partial_0, \partial_1)$, consider the system

$$\partial_0{}^n\partial_1{}^n x - x = 0, \qquad \partial_0{}^n\partial_1 x - \partial_1{}^n x = 0, \tag{15}$$

where $n \geqslant 2$. Think of the derivatives $\partial_0{}^i\partial_1{}^j x$ as matrix entries:

$$\begin{array}{cccc} x & \partial_1 x & \partial_1{}^2 x & \dots \\ \partial_0 x & \partial_0\partial_1 x & \partial_0\partial_1{}^2 x & \dots \\ \partial_0{}^2 x & \partial_0{}^2\partial_1 x & \partial_0{}^2\partial_1{}^2 x & \dots \\ \dots & \dots & \dots & \dots \end{array}$$

Treat each derivative $\partial_0{}^i\partial_1{}^j x$ as a new variable, $x^{(i,j)}$. In case $n = 3$, the system (15) determines a 4×4 matrix $(a^{(i,j)})^{i<4}_{j<4}$ in which $a^{(3,3)} = a^{(0,0)}$ and $a^{(3,1)} = a^{(0,3)}$; we can depict this as follows:

$$\begin{array}{cccc} a & * & * & b \\ * & * & * & * \\ * & * & * & * \\ * & b & * & a \end{array} \tag{16}$$

We first ask whether the derivations can be extended to commuting derivations on the field $K(a^{(i,j)} \colon (i, j) \leqslant (3, 3))$, where $\leqslant$ is the product order on ω^2, so that $\partial_0 a^{(i,j)} = a^{(i+1,j)}$ when $(i + 1, j) \leqslant (3, 3)$ and $\partial_1 a^{(i,j)} = a^{(i,j+1)}$ when $(i, j + 1) \leqslant (3, 3)$. That is, writing $a^{(4,j)}$ for $\partial_0 a^{(3,j)}$, and $a^{(i,4)}$ for $\partial_1 a^{(i,3)}$, we ask whether ∂_0 and ∂_1 can map $K(a^{(i,j)} \colon (i, j) \leqslant (3, 3))$ into $K(a^{(i,j)} \colon (i, j) \leqslant (4, 3))$ and $K(a^{(i,j)} \colon (i, j) \leqslant (3, 4))$ respectively, in the manner suggested by the notation; and we ask further whether ∂_0 and ∂_1 can still commute, which means there should be $a^{(4,4)}$ so that ∂_0 maps $K(a^{(i,j)} \colon (i, j) \leqslant (3, 4))$, and ∂_1 maps $K(a^{(i,j)} \colon (i, j) \leqslant (4, 3))$, into $K(a^{(i,j)} \colon (i, j) \leqslant (4, 4))$. In short, can the matrix (16) be extended by one row and column? Since $a^{(3,3)} = a^{(0,0)}$, we must have $a^{(4,3)} = a^{(1,0)}$, and so forth; none of this causes any problem, and the new matrix can be depicted:

$$\begin{array}{cccc|c} a & d & * & b & g \\ c & e & * & f & h \\ * & * & * & * & * \\ * & b & g & a & d \\ \hline * & f & h & c & e \end{array} \tag{17}$$

To extend the derivations further to $K(a^{(i,j)} \colon (i, j) \leqslant (4, 4))$, we require a new condition on the original matrix $(a^{(i,j)})^{i<4}_{j<4}$, namely, $a^{(3,0)} = a^{(0,2)}$; this comes out when we try to extend (17) by one column:

$$\begin{array}{cccc|cc} a & d & j & b & g & a \\ c & e & * & f & h & c \\ i & * & * & * & * & i \\ j & b & g & a & d & j \\ \hline * & f & h & c & e & * \end{array} \tag{18}$$

Indeed, as $a^{(0,4)} = a^{(3,2)}$, so $a^{(0,5)} = a^{(3,3)} = a^{(0,0)}$, so column 5 must be the same as column 0; also $a^{(0,1)} = a^{(3,4)}$, whence $a^{(0,2)} = a^{(3,5)} = a^{(3,0)}$. This does not mean that (15) is insoluble; it is. But the additional condition found in (18) does mean that the associated non-homogeneous system

$$\partial_0{}^n\partial_1{}^n x - x = 0, \qquad \partial_0{}^n\partial_1 x - \partial_1{}^n x = 1 \tag{19}$$

is insoluble, although this is not clear from (16) or (17). Alternatively, we can work with the differential ideal, looking for a characteristic set as mentioned at the beginning of this section:

$$\begin{aligned}
&[\partial_0{}^n\partial_1{}^n x - x,\ \partial_0{}^n\partial_1 x - \partial_1{}^n x - 1]\\
&= [\partial_1{}^{2n-1} x - x,\ \partial_0{}^n\partial_1 x - \partial_1{}^n x - 1]\\
&= [\partial_1{}^{2n-1} x - x,\ \partial_0{}^n\partial_1 x - \partial_1{}^n x - 1,\ \partial_1{}^{3n-2} x - \partial_0{}^n x]\\
&= [\partial_1{}^{2n-1} x - x,\ \partial_0{}^n\partial_1 x - \partial_1{}^n x - 1,\ \partial_0{}^n x - \partial_1{}^{n-1} x]\\
&= [\partial_1{}^{2n-1} x - x,\ 1,\ \partial_0{}^n x - \partial_1{}^{n-1} x] = [1],
\end{aligned}$$

so (19) is insoluble; but higher-order derivatives were needed to discover this.

By the same computation, the differential ideal $[\partial_0{}^n\partial_1{}^n x - x,\ \partial_0{}^n\partial_1 x - \partial_1{}^n x]$ has the characteristic set $\{\partial_1{}^{2n-1} x - x, \partial_0{}^n x - \partial_1{}^{n-1} x\}$, which, in the manner described above, determines the following matrix in case $n = 2$:

$$\begin{matrix} a & b & * & a \\ * & * & * & * \\ b & * & * & * \end{matrix} \tag{20}$$

Therefore (15) has a solution, simply because the ordinary ideal $(x^{(0,2n-1)} - x, x^{(n,0)} - x^{(0,n-1)})$ has a zero. One way of justifying this conclusion is [10, Lem. 3.1.2]; another way will be Theorem 4.7, according to which it is enough to observe that the dependencies in (20) extend in a 'nice' way to a triangle:

$$\begin{array}{cccc|cc} a & b & c & a & b & c \\ d & e & f & d & e & \\ b & c & a & b & & \\ \cline{1-4} e & f & d & & & \\ c & a & & & & \\ f & & & & & \end{array} \tag{21}$$

4.2. Terminology. I shall now avoid working with differential polynomials as such, but shall work instead with the algebraic dependencies that they determine.

Let $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF. Higher-order derivatives with respect to the ∂_i can be indexed by elements of ω^m: so, for $\partial_0{}^{\sigma(0)} \cdots \partial_{m-1}{}^{\sigma(m-1)} x$, we may write $\partial^\sigma x$. Let $\leqslant$ be the product ordering of ω^m. Then the derivative $\partial^\sigma x$ is **below** $\partial^\tau x$ (and the latter is **above** the former) if $\sigma \leqslant \tau$. If $n \in \omega$, then two elements of $\omega^n \times n$ will be related by $\leqslant$ only if they agree in the last coordinate, so that

$$(\sigma, k) \leqslant (\tau, \ell) \iff \sigma \leqslant \tau \wedge k = \ell;$$

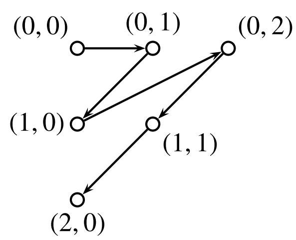
we may use the corresponding terminology of above and below, so that $\partial^\sigma x_k$ is below $\partial^\tau x_\ell$ if (and only if) $(\sigma, k) \leqslant (\tau, \ell)$.

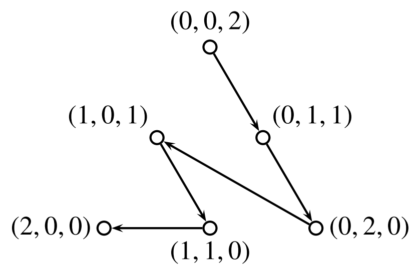
If $\sigma \in \omega^m$, let the sum $\sum_{i<m} \sigma(i)$ be denoted by $|\sigma|$: this is the **height** of σ or of $\partial^\sigma x$. (Kolchin [6, § I.1, p. 59] uses the word *order*.) If n is a positive integer, let $\omega^m \times n$ be

(totally) ordered by $\trianglelefteq$, which is taken from the left lexicographic ordering of ω^{m+1} by means of the embedding

$$(\xi, k) \longmapsto (|\xi|, k, \xi(0), \dots, \xi(m-2))$$

of $\omega^m \times n$ in ω^{m+1}. Then $(\omega^m \times n, \trianglelefteq)$ is isomorphic to $(\omega, \leqslant)$. In case $n = 1$, and m is 2 or 3, the picture is thus:





We may write $(\sigma, k) \lhd \infty$ for all (σ, k) in $\omega^m \times n$. Suppose $(x_h \colon h < n)$ is a tuple of indeterminates. By ordering the formal derivatives $\partial^\sigma x_k$ in terms of (σ, k) and $\trianglelefteq$, we have Kolchin's example of an *orderly ranking* of derivatives [6, § I.8, p. 75]. If $(\sigma, k) \lhd (\tau, \ell)$, I shall say that the derivative $\partial^\sigma x_k$ is **less** than $\partial^\tau x_\ell$ or is a **predecessor** of $\partial^\tau x_\ell$, and $\partial^\tau x_\ell$ is **greater** than $\partial^\sigma x_k$; likewise for the expressions a_k^σ and a_ℓ^τ, introduced in (23) below. (So, the terms just defined refer to the total ordering $\lhd$, while 'below' and 'above' refer to the partial ordering $\leqslant$.)

Addition and subtraction on ω induce corresponding operations on ω^m. Then

$$\begin{gathered}
\tau \leqslant \sigma + \tau, \\
\partial^\sigma \partial^\tau x_k = \partial^{\sigma+\tau} x_k, \\
(\sigma, k) \trianglelefteq (\sigma + \tau, k), \\
(\sigma, k) \trianglelefteq (\tau, \ell) \iff (\sigma + \rho, k) \trianglelefteq (\tau + \rho, \ell).
\end{gathered} \tag{22}$$

If $i < m$, let $\boldsymbol{i}$ denote the characteristic function of $\{i\}$ in ω^m, so that $\partial^{\boldsymbol{i}} = \partial_i$, and more generally $\partial_i \partial^\sigma = \partial^{\sigma+\boldsymbol{i}}$, and $\partial_i \partial^{\sigma-\boldsymbol{i}} = \partial^\sigma$.

Let L be an extension of K with generators that are indexed by an initial segment of $(\omega^m \times n, \trianglelefteq)$; that is,

$$L = K(a_h^\xi \colon (\xi, h) \lhd (\tau, \ell)), \tag{23}$$

where $(\tau, \ell) \in \omega^m \times n$, or possibly $(\tau, \ell) = \infty$, in which case $L = K(a_h^\xi \colon (\xi, h) \in \omega^m \times n)$. By (22), if $a_k^{\sigma+\boldsymbol{i}}$ is one of the generators of L/K, then so is a_k^σ. Let us say that L, with the given generators, meets the **differential condition** if there is no obstacle to extending each derivation ∂_i to a derivation D_i on $K(a_h^\xi \colon (\xi + \boldsymbol{i}, h) \lhd (\tau, \ell))$ such that

$$D_i a_k^\sigma = a_k^{\sigma+\boldsymbol{i}} \tag{24}$$

whenever $(\sigma + \boldsymbol{i}, k) \lhd (\tau, \ell)$. (So, if the right-hand member of (24) is not defined, then the left need not be defined.) Formally, the differential condition is that, if f is a rational function over K in variables $(x_h^\xi \colon (\xi, h) \trianglelefteq (\sigma, k))$, where $(\sigma + \boldsymbol{i}, k) \lhd (\tau, \ell)$ for some i in m, and if

$$f(a_h^\xi \colon (\xi, h) \trianglelefteq (\sigma, k)) = 0, \tag{25}$$

then we may apply D_i to this, assuming (24), to get

$$\sum_{(\eta,g)\trianglelefteq(\sigma,k)} \frac{\partial f}{\partial x_g^\eta}(a_h^\xi\colon (\xi,h)\trianglelefteq(\sigma,k))\cdot a_g^{\eta+\boldsymbol{i}} + f^{\partial_i}(a_h^\xi\colon (\xi,h)\trianglelefteq(\sigma,k)) = 0. \tag{26}$$

(Note well the assumption that $(\sigma+\boldsymbol{i},k)\triangleleft(\tau,\ell)$. In (26), each of the $a_g^{\eta+\boldsymbol{i}}$ must exist, even though the coefficient $(\partial f/\partial x_g^\eta)(a_h^\xi\colon (\xi,h)\trianglelefteq(\sigma,k))$ might be 0.) So the differential condition is *necessary* for the extensibility of the ∂_i as desired (see for example [12, Fact 1.1 (0)]); sufficiency is part of Lemma 4.1 below.

An extension $(M, D_0, \dots, D_{m-1})$ of $(K, \partial_0, \dots, \partial_{m-1})$ is **compatible** with the extension L of K given in (23) if $L \subseteq M$, and (24) holds whenever $(\sigma+\boldsymbol{i},k)\triangleleft(\tau,\ell)$.

Borrowing some terminology used for differential polynomials [14, § IX.1, p. 163], let us say that a generator a_k^σ of L/K is a **leader** if it is algebraically dependent over K on its predecessors, that is,

$$a_k^\sigma \in K(a_h^\xi\colon (\xi,h)\triangleleft(\sigma,k))^{\mathrm{alg}}.$$

Then a_k^σ is a **separable** leader if it is separably algebraic over $K(a_h^\xi\colon (\xi,h)\triangleleft(\sigma,k))$; otherwise, it is an **inseparable** leader. A separable leader a_k^σ is **minimal** if there is no separable leader strictly below it—no separable leader a_k^ρ such that $\rho<\sigma$.

For example, in the field $K(a^{(i,j)}\colon (i,j)\leqslant(3,3))$ depicted in (16) above, the generator $a^{(3,3)}$ is a (non-minimal) separable leader, and $a^{(3,1)}$ is a minimal separable leader. But here we wanted $a^{(3,3)}$ ultimately to be a derivative of $a^{(3,1)}$, namely ${\partial_1}^2 a^{(3,1)}$. Passing to a larger field in (18), we found the condition $a^{(3,0)} = a^{(0,2)}$; then $a^{(3,0)}$ became a new separable leader, strictly below than the formerly minimal separable leader $a^{(3,1)}$.

Lemma 4.1. *Suppose $(K,\partial_0,\dots,\partial_{m-1})\models m$-DF, and L is an extension $K(a_h^\xi\colon(\xi,h)\triangleleft(\tau,\ell))$ of K meeting the differential condition. Then the derivations ∂_i extend to derivations D_i from $K(a_h^\xi\colon(\xi+\boldsymbol{i},h)\triangleleft(\tau,\ell))$ into L such that* (24) *holds when* $(\sigma+\boldsymbol{i},k)\triangleleft(\tau,\ell)$*. If a_k^σ is a separable leader, and $(\sigma+\boldsymbol{i},k)\triangleleft(\tau,\ell)$, then*

$$a_k^{\sigma+\boldsymbol{i}} \in K(a_h^\xi\colon (\xi,h)\triangleleft(\sigma+\boldsymbol{i},k)) \tag{27}$$

(that is, $a_k^{\sigma+\boldsymbol{i}}$ is a rational function over K of its predecessors); in particular, $a_k^{\sigma+\boldsymbol{i}}$ is a separable leader. Therefore generators of L/K that are above separable leaders are themselves separable leaders.

Proof. The claim follows from the basic properties of derivations, such as are gathered in [12, § 1]. Let B comprise the generators of L/K that are not leaders. Then B is algebraically independent over K, so we are free to extend the ∂_i to D_i on $K(B)$ so that (24) holds whenever it applies. Then these extensions are uniquely determined, and there are further unique extensions of the D_i to the separable closure $K(B)^{\mathrm{sep}}$. In particular, suppose a_k^σ is a separable leader, and $(\sigma+\boldsymbol{i},k)\triangleleft(\tau,\ell)$. Then $D_i a_k^\sigma$ is obtained by differentiating the minimal polynomial of a_k^σ over $K(B)$. That is, $D_i a_k^\sigma$ is obtained by differentiating an equation like (25); by the differential condition, $D_i a_k^\sigma$ must be $a_k^{\sigma+\boldsymbol{i}}$ as given by (26); this shows that $a_k^{\sigma+\boldsymbol{i}}$ is a rational function over K of its predecessors.

Finally, in a positive characteristic p, there may be an inseparable leader a_k^σ. Then $(a_k^\sigma)^{p^r} \in K(a_h^\xi\colon (\xi,h)\triangleleft(\sigma,k))^{\mathrm{sep}}$ for some positive r. If $(\sigma+\boldsymbol{i},k)\triangleleft(\tau,\ell)$, then we are free to define $D_i a_k^\sigma$ as $a_k^{\sigma+\boldsymbol{i}}$, provided $D_i((a_k^\sigma)^{p^r}) = 0$. But again this condition is ensured by the differential condition. Indeed, we may suppose (25) shows the separable dependence of $(a_k^\sigma)^{p^r}$ over the predecessors of a_k^σ. That is, we can understand $f(a_h^\xi\colon (\xi,h) \trianglelefteqslant (\sigma,k))$ as $g((a_k^\sigma)^{p^r})$ for some separable polynomial g over $K(a_h^\xi\colon (\xi,h) \vartriangleleft (\sigma,k))$. Then $D_i((a_k^\sigma)^{p^r})$ is obtained from (26), provided we replace the term $(\partial f/\partial x_k^\sigma)(a_h^\xi\colon (\xi,h) \trianglelefteqslant (\sigma,k)) \cdot a_k^{\sigma+\boldsymbol{i}}$ with $g'((a_k^\sigma)^{p^r}) \cdot D_i((a_k^\sigma)^{p^r})$. But in the present case, the former term is 0. Since, after the replacement, the resulting equation still holds, we must have $D_i((a_k^\sigma)^{p^r}) = 0$. ⊣

4.3. An example. The differential ideal $[\partial^{(1,1)}x - \partial^{(0,2)}x, \partial^{(1,2)}x - \partial^{(2,0)}x]$, with two generators, is equal to $[f,g]$, where

$$f = \partial^{(1,1)}x - \partial^{(0,2)}x, \qquad g = \partial^{(0,3)}x - \partial^{(2,0)}x.$$

Then ${\partial_1}^2 f + \partial_1 g - \partial_0 g = \partial^{(1,3)}x - \partial^{(0,4)}x + \partial^{(0,4)}x - \partial^{(2,1)}x - \partial^{(1,3)}x + \partial^{(3,0)}x = h$, where

$$h = \partial^{(3,0)}x - \partial^{(2,1)}x.$$

Then $[f,g] = [f,g,h]$, and the conditions imposed by f, g, and h are as in the following triangle, where the minimal separable leaders in the present sense are underlined (though we have not yet checked the corresponding field meets the differential condition):

$$\begin{array}{cccc} * & * & a & \underline{b} \\ * & \underline{a} & * & \\ b & c & & \\ \underline{c} & & & \end{array} \tag{28}$$

These leaders have common derivatives, which however impose no new conditions. (In the terminology introduced by Azriel Rosenfeld [18, § I.2, p. 397] in characteristic 0, the set $\{f,g,h\}$ is *coherent.*) For example,

$$\begin{aligned} \partial^{(3,0)}g - \partial^{(0,3)}h &= \partial^{(3,0)}(\partial^{(0,3)}x - \partial^{(2,0)}x) - \partial^{(0,3)}(\partial^{(3,0)}x - \partial^{(2,1)}x) \\ &= \partial^{(3,3)}x - \partial^{(5,0)}x - \partial^{(3,3)}x + \partial^{(2,4)}x = \partial^{(2,4)}x - \partial^{(5,0)}x, \end{aligned}$$

where we used derivatives as great as $\partial^{(3,3)}x$; but we already had

$$\begin{aligned} \partial^{(2,1)}g - \partial^{(2,0)}h &= \partial^{(2,1)}(\partial^{(0,3)}x - \partial^{(2,0)}x) - \partial^{(2,0)}(\partial^{(3,0)}x - \partial^{(2,1)}x) \\ &= \partial^{(2,4)}x - \partial^{(4,1)}x - \partial^{(5,0)}x + \partial^{(4,1)}x = \partial^{(2,4)} - \partial^{(5,0)}x, \end{aligned}$$

where no derivative is as great as $\partial^{(3,3)}x$. This checking did require some derivatives that were not *below* $\partial^{(3,3)}x$; but they do all belong to the following larger triangle, whose corresponding field meets the differential condition (though note how the c from (28) changes to b):

$$\begin{array}{cccc|ccc} * & * & a & \underline{b} & b & b & b \\ * & \underline{a} & b & b & b & b & \\ b & b & b & b & b & & \\ \underline{b} & b & b & b & & & \\ \cline{1-4} b & b & b & \multicolumn{1}{c}{} & & & \\ b & b & & \multicolumn{1}{c}{} & & & \\ b & & & \multicolumn{1}{c}{} & & & \end{array} \tag{29}$$

We shall see that extensibility to triangles as in (29), without introduction of new minimal separable leaders, is sufficient to guarantee solutions.

4.4. A solubility condition. If $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF, then this model has an extension whose underlying field is the separable closure of K (as by [11, Lem. 3.4, p. 930] and [12, Lem. 2.4, p. 1334]). We shall need this in a more general form:

LEMMA 4.2. *Suppose a field M has two subfields L_0 and L_1, which in turn have a common subfield K. For each i in 2, suppose there is a derivation D_i mapping K into L_i and L_{1-i} into M. Then the bracket $[D_0, D_1]$ is a well-defined derivation on K. Suppose it is the 0-derivation, that is, the following diagram commutes.*

$$\begin{array}{ccc} K & \xrightarrow{D_1} & L_0 \\ {\scriptstyle D_0}\big\downarrow & & \big\downarrow{\scriptstyle D_0} \\ L_1 & \xrightarrow[D_1]{} & M \end{array}$$

Suppose also that a is an element of M that is separably algebraic over K. Then each D_i extends uniquely to $K(a)$, and $D_i a \in L_{1-i}(a)$, so $D_{1-i}D_i a$ is also well-defined. Moreover, $[D_0, D_1]a = 0$.

PROOF. The claim follows from standard facts, at least if $L_0 = K = L_1$; but the proof is the same in the general case. Indeed, though the derivations D_0 and D_1 are defined on K, their bracket $[D_0, D_1]$ need not be so, since the compositions D_0D_1 and D_1D_0 need not be so; but if they are, then $[D_0, D_1]$ is a *derivation* on K. A derivation on K extends uniquely to K^{sep}; if the derivation is 0 on K, then it is 0 on K^{sep} [12, Fact 1.1 (2)]. In the present case, as $a \in K^{\text{sep}}$, so $D_i a \in L_{1-i}(a)$, and therefore $D_i a \in {L_{1-i}}^{\text{sep}}$; hence $D_{1-i}D_i a$ is defined. Thus $[D_0, D_1]$ is defined on $K(a)$, where $a \in K^{\text{sep}}$; and if the bracket is 0 on K, then is 0 at a. ⊣

Another example illustrates the following theorem in positive characteristic. Over a differential field $(K, \partial_0, \partial_1)$, where $\operatorname{char}(K) = p > 0$, the differential equation

$$(\partial_0 x)^p + x = \partial_1 x$$

determines an extension $K(a^\xi : |\xi| \leqslant 2)$ of K that meets the differential condition: the generators form a triangle thus:

$$\begin{array}{lll} a & b^p + a & b^p + a \\ b & b & \\ c & & \end{array}$$

where (a, b, c) is algebraically independent over K. In particular, $a^{(1,0)}$ (which has the value b) is an inseparable leader, but none of the generators of height 2 (namely, $a^{(0,2)}$, $a^{(1,1)}$ and $a^{(2,0)}$) is an inseparable leader.

THEOREM 4.3. *Suppose $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF, and K has an extension $K(a_h^\xi : |\xi| \leqslant 2r \wedge h < n)$ meeting the differential condition for some positive integers r and n. Suppose further that, whenever a_k^σ is a minimal separable leader, then $|\sigma| \leqslant r$. Then $(K, \partial_0, \dots, \partial_{m-1})$ has an extension $(M, D_0, \dots, D_{m-1})$ compatible with $K(a_h^\xi : |\xi| < 2r \wedge h < n)$.*

Proof. The claim can be compared to and perhaps derived from a differential-algebraic lemma of Rosenfeld [18, § I.2], at least in characteristic 0. Here I give an independent argument, for arbitrary characteristic. We shall obtain M recursively as $K(a_h^\xi\colon (\xi,h)\in\omega^m\times n)$, at the same time proving inductively that the ∂_i can be extended to D_i so that (24) holds in all cases.

Let $L = K(a_h^\xi\colon |\xi|<2r\wedge h<n)$; this is $K(a_h^\xi\colon (\xi,h)\trianglelefteq((2r-1,0,\dots,0),n-1))$. Then by (26), the differential condition requires of the tuple $(a_h^\xi\colon |\xi|=2r\wedge h<n)$ only that it solve some linear equations over L. The hypothesis of our claim is that there *is* a solution, namely $(a_h^\xi\colon |\xi|=2r\wedge h<n)$. We may therefore assume that this tuple is a *generic* solution of these equations. In particular, no entry of this tuple is an inseparable leader. (If the entries of $(a_h^\xi\colon |\xi|=2r\wedge h<n)$ were chosen from the field L, then this field would be closed under the desired extensions D_i of ∂_i, and the derivations D_i would commute on the subfield $K(a_h^\xi\colon |\xi|+1<2r\wedge h<n)$; but they might not commute on all of L.)

Now, as an inductive hypothesis, suppose we have the extension $K(a_h^\xi\colon (\xi,h)\triangleleft(\tau,\ell))$ of K meeting the differential condition, so that there are derivations D_i as given by Lemma 4.1; suppose also that

(i) if a_k^σ is a minimal separable leader, then $|\sigma|\leqslant r$;
(ii) if a_k^σ is an inseparable leader, then $|\sigma|<2r$.

We need to choose a_ℓ^τ in such a way that these conditions still hold for $K(a_h^\xi\colon (\xi,h)\trianglelefteq(\tau,\ell))$. The inductive hypothesis is correct when $|\tau|\leqslant 2r$, and then the desired conclusion follows; so we may assume $|\tau|>2r$. Hence, if $\tau(i)>0$, so that $\tau-\boldsymbol{i}$ is defined, then $|\tau-\boldsymbol{i}|\geqslant 2r$, so $a_\ell^{\tau-\boldsymbol{i}}$ is not an inseparable leader.

If $a_\ell^{\tau-\boldsymbol{i}}$ is not a leader at all, for any i in m, then we may let a_ℓ^τ be a new transcendental, and we may define each derivative $D_ia_\ell^{\tau-\boldsymbol{i}}$ as this [12, Fact 1.1 (1)].

In the other case, $a_\ell^{\tau-\boldsymbol{i}}$ is a separable leader for some i. Then $D_ia_\ell^{\tau-\boldsymbol{i}}$ is determined (Lemma 4.1). We want to let a_ℓ^τ be this derivative. However, possibly also $a_\ell^{\tau-\boldsymbol{j}}$ is a separable leader, where $i\neq j$. In this case, we must check that

$$D_ja_\ell^{\tau-\boldsymbol{j}} = D_ia_\ell^{\tau-\boldsymbol{i}}, \tag{30}$$

that is, $[D_i,D_j]a_\ell^{\tau-\boldsymbol{i}-\boldsymbol{j}}=0$.

There are minimal separable leaders a_ℓ^π and a_ℓ^ρ below $a_\ell^{\tau-\boldsymbol{i}}$ and $a_\ell^{\tau-\boldsymbol{j}}$ respectively. Let ν be $\pi\vee\rho$, the least upper bound of $\{\pi,\rho\}$ with respect to $\leqslant$. Then $\nu\leqslant\tau$. But $|\nu|\leqslant|\pi|+|\rho|\leqslant 2r<|\tau|$; so $\nu<\tau$. Hence $\nu\leqslant\tau-\boldsymbol{k}$ for some k in m, which means a_ℓ^ν is below $a_\ell^{\tau-\boldsymbol{k}}$. Consequently,

(i) a_ℓ^π is below both $a_\ell^{\tau-\boldsymbol{i}}$ and $a_\ell^{\tau-\boldsymbol{k}}$;
(ii) a_ℓ^ρ is below both $a_\ell^{\tau-\boldsymbol{j}}$ and $a_\ell^{\tau-\boldsymbol{k}}$.

If $k=j$, then a_ℓ^π is below $a_\ell^{\tau-\boldsymbol{i}-\boldsymbol{j}}$, so this is a separable leader. As D_i and D_j commute on $K(a_h^\xi\colon (\xi,h)\triangleleft(\tau-\boldsymbol{i}-\boldsymbol{j},\ell))$ by the differential condition, they must commute also at $a_\ell^{\tau-\boldsymbol{i}-\boldsymbol{j}}$ (Lemma 4.2), so (30) is established. The argument is the same if $k=i$. If k is different from i and j, then again the same argument yields $D_ja_\ell^{\tau-\boldsymbol{j}}=D_ka_\ell^{\tau-\boldsymbol{k}}$ and $D_ka_\ell^{\tau-\boldsymbol{k}}=D_ia_\ell^{\tau-\boldsymbol{i}}$, so (30) holds.

In no case did we introduce a new minimal separable leader or an inseparable leader. This completes the induction and the proof. ⊣

The claim at the end of the last subsection (4.3) is now justified.

In terms of differential polynomials and ideals, the theorem can be understood as follows. Given the hypothesis of the theorem, let S be the set of differential polynomials $f(\partial^\xi x_h : |\xi| < 2r \wedge h < n)$, where f ranges over the ordinary polynomials over K such that $f(a_h^\xi : |\xi| < 2r \wedge h < n) = 0$. Then S includes a characteristic set for the differential ideal that it generates.

We can now characterize the existentially closed models of m-DF by means of the following lemma. The lemma follows from unproved statements in [6, § 0.17, p. 49]; let's just prove it here.

Lemma 4.4. *For every m in ω and positive integer n, every antichain of $(\omega^m \times n, \leqslant)$ is finite.*

Proof. The general case follows from the case when $n = 1$, since if S is an antichain of $(\omega^m \times n, \leqslant)$, then

$$S = \bigcup_{j<n} \{(\xi, h) \in S : h = j\},$$

and each component of the union is in bijection with an antichain of $(\omega^m, \leqslant)$. As an inductive hypothesis, suppose every antichain of $(\omega^\ell, \leqslant)$ is finite; but suppose also, if possible, that there is an infinite antichain S of $(\omega^{\ell+1}, \leqslant)$. Then S contains some σ. By inductive hypothesis, the subset

$$\bigcup_{j\leqslant\ell} \bigcup_{i\leqslant\sigma(j)} \{\xi \in S : \xi(j) = i\}$$

of S is a finite union of finite sets, so its complement in S has infinitely many elements τ; but then $\sigma < \tau$, so S was not an antichain. ⊣

Theorem 4.5. *Suppose $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF. Then the following are equivalent:*

(i) *The model $(K, \partial_0, \dots, \partial_{m-1})$ of m-DF is existentially closed.*

(ii) *For all positive integers r and n, if K has an extension $K(a_h^\xi : |\xi| \leqslant 2r \wedge h < n)$ meeting the differential condition such that $|\sigma| \leqslant r$ whenever a_k^σ is a minimal separable leader, then the tuple $(a_h^\xi : |\xi| < 2r \wedge h < n)$ has a specialization $(\partial^\xi b_h : |\xi| < 2r \wedge h < n)$ for some tuple $(b_h : h < n)$ of elements of K.*

Proof. Assume (i) and the hypothesis of (ii). Let S be a (finite) generating set of the ideal of $(a_h^\xi : |\xi| < 2r \wedge h < n)$ over K. By Theorem 4.3, the system

$$\bigwedge_{f\in S} f(\partial^\xi x_h : |\xi| < 2r \wedge h < n) = 0$$

has a solution in some extension, hence it has a solution in K itself, which means the conclusion of (ii) holds. So (ii) is necessary for (i).

Every system over $(K, \partial_0, \dots, \partial_{m-1})$ is equivalent to a system of equations. Suppose such a system has a solution $(a_h : h < n)$ in some extension. Then the extension $K(\partial^\xi a_h : (\xi, h) \in (\omega^m \times n))$ has a *finite* set of minimal separable leaders, by Lemma 4.4, since this set is indexed by an antichain of $(\omega^m \times n, \leqslant)$. Hence there is r large enough that all of these minimal separable leaders are also generators of $K(a_h^\xi : |\xi| \leqslant r \wedge h < n)$. We may assume also that r is large enough that $|\sigma| \leqslant r$ for every derivative $\partial^\sigma x_k$ that appears in the original system. The hypothesis of (ii) is now satisfied when each a_k^σ is taken as $\partial^\sigma a_k$. If the conclusion of (ii) follows, then $(b_h\colon h < n)$ is a solution of the original system. Thus, (ii) is sufficient for (i). ⊣

Corollary 4.6. *The theory m-*DF *has a model-companion, m-*DCF.

Proof. We consider all possible situations in which the hypothesis of (ii) in the theorem is satisfied. In each case, there is a finite tuple $\boldsymbol{c}$ of parameters from K, there is a universal formula $\varphi(\boldsymbol{y})$, and there is a system $\psi(\boldsymbol{x}, \boldsymbol{y})$ such that:

(i) $(K, \partial_0, \dots, \partial_{m-1}) \models \varphi(\boldsymbol{c})$;
(ii) $(K, \partial_0, \dots, \partial_{m-1}) \models \psi(\boldsymbol{b}, \boldsymbol{c})$ if and only if $\boldsymbol{b}$ is a tuple $(b_h\colon h < n)$ as in the conclusion of (ii);
(iii) for all models $\mathfrak{M}$ of m-DF and all $\boldsymbol{d}$ from M, if $\mathfrak{M} \models \varphi(\boldsymbol{d})$, then the system $\psi(\boldsymbol{x}, \boldsymbol{d})$ is soluble in some extension of $\mathfrak{M}$.

Then m-DCF is axiomatized *modulo* m-DF by all of the sentences

$$\forall \boldsymbol{x}\ (\varphi(\boldsymbol{x}) \to \exists \boldsymbol{y}\ \psi(\boldsymbol{y}, \boldsymbol{x}))$$

that can arise in this way. To check this in detail, we note that $\varphi(\boldsymbol{c})$ and $\psi(\boldsymbol{x}, \boldsymbol{c})$ can be such that:

(i) the system $\psi(\boldsymbol{x}, \boldsymbol{c})$ includes equations

$$f(\partial^\xi x_h\colon |\xi| < 2r \wedge h < n) = 0, \tag{31}$$

where the (ordinary) polynomials f generate the ideal of $(a_h^\xi\colon |\xi| < 2r \wedge h < n)$ over K;
(ii) for each leader of $K(a_h^\xi\colon |\xi| < 2r \wedge h < n)$, there is an equation (31) in $\psi(\boldsymbol{x}, \boldsymbol{c})$ such that the variables appearing in f correspond only to that leader and its predecessors that are *not* leaders, so that (since these non-leading predecessors are algebraically independent) f is in effect a constant multiple of the minimal polynomial of the leader over (the extension of K generated by) those predecessors;
(iii) then $\varphi(\boldsymbol{c})$ says that those f are irreducible (this is why φ is universal: quantification is over the coefficients in potential factors of f);
(iv) also, $\varphi(\boldsymbol{c})$ shows explicitly how nothing new results when the equations (31) are differentiated: if this differentiation does not introduce a new derivative (namely, some $\partial^\sigma x_k$ where $|\sigma| = 2r$), then the new equation is an explicit linear combination of the original equations; in the cases where new derivatives are introduced, the resulting linear system has an explicit solution.

This can all have been done so that $\boldsymbol{c}$ can be extracted to yield $\varphi(\boldsymbol{x})$ and $\psi(\boldsymbol{y}, \boldsymbol{x})$ as desired. ⊣

4.5. Differential forms again. The condition in Theorem 4.3 can be adjusted to yield the following:

Theorem 4.7. *Suppose* $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF*, and K has an extension* $K(a_h^\xi\colon |\xi| \leqslant |\mu| \wedge h < n)$ *meeting the differential condition for some* μ *in* ω^m *and some positive integer n. Suppose further that, if* a_k^σ *is a minimal separable leader, then* $\sigma \leqslant \mu$*. Then* $(K, \partial_0, \dots, \partial_{m-1})$ *has an extension compatible with* $K(a_h^\xi\colon |\xi| < |\mu| \wedge h < n)$.

Proof. The proof is as for Theorem 4.3, *mutatis mutandis.* What needs adjusting is the choosing of a_ℓ^τ in case both $a_\ell^{\tau-\boldsymbol{i}}$ and $a_\ell^{\tau-\boldsymbol{j}}$ are separable leaders. Again we have minimal separable leaders a_ℓ^π and a_ℓ^ρ below $a_\ell^{\tau-\boldsymbol{i}}$ and $a_\ell^{\tau-\boldsymbol{j}}$ respectively. Since we may assume $|\mu| < |\tau|$, there is some k in m such that $\mu(k) < \tau(k)$. If $k = j$, then $\pi(j) \leqslant \mu(j) < \tau(j)$, so $\pi(j) \leqslant (\tau - \boldsymbol{j})(j) = (\tau - \boldsymbol{i} - \boldsymbol{j})(j)$. Then $\pi \leqslant \tau - \boldsymbol{i} - \boldsymbol{j}$, so a_ℓ^π is below $a_\ell^{\tau-\boldsymbol{i}-\boldsymbol{j}}$. Now we can proceed as before. ⊣

As Theorem 4.3 yields Theorem 4.5, so Theorem 4.7 yields a characterization of the existentially closed models of m-DF. Moreover, Theorems 4.3 and 4.7 can be combined in the following way:

Theorem 4.8. *Suppose* $(K, \partial_0, \dots, \partial_{m-1}) \models m\text{-DF}$, *and K has an extension* $K(a_h^\xi : |\xi| \leqslant 2r \wedge h < n)$ *meeting the differential condition for some positive integers n and r. Suppose further that, for each k in m, either $\sigma \leqslant r$ whenever a_k^σ is a minimal separable leader, or else there is some τ in ω^m such that $|\tau| = 2r$, and $|\sigma| \leqslant |\tau|$ whenever a_k^σ is a minimal separable leader. Then $(K, \partial_0, \dots, \partial_{m-1})$ has an extension compatible with* $K(a_h^\xi : |\xi| < 2r \wedge h < n)$.

Proof. Combine the proofs of Theorems 4.3 and 4.7. ⊣

Again there is a corresponding characterization of the models of m-DCF.

Suppose $K(a_h^\xi : |\xi| \leqslant |\mu| \wedge h < n)$ is as in the hypothesis of Theorem 4.7, and $|\mu| > 2$. If $|\tau| \leqslant 2$ and $|\sigma + \tau| \leqslant |\mu|$, let $a_{(\sigma,k)}^\tau$ denote $a_k^{\sigma+\tau}$. Then the extension of K can be written as

$$K(a_{(\xi,h)}^\eta : |\eta| \leqslant 2 \wedge (\xi, h) \lhd ((0, \dots, 0, |\mu| - 1), 0)), \tag{32}$$

but this no longer need satisfy the conditions in Theorem 4.8. For example, in case $m = 2$ and $n = 1$ and $\mu = (3, 0)$, the original extension might determine the following triangle, with the single leader underlined:

$$\begin{array}{cccc} a & b & d & g \\ c & e & h & \\ f & k & & \\ \underline{a} & & & \end{array} \tag{33}$$

Then the derived extension in (32) determines three triangles thus, with minimal separable leaders underlined:

$$\begin{array}{ccc} a & \underline{b} & d \\ \underline{c} & e & \\ f & & \end{array} \qquad \begin{array}{ccc} b & d & g \\ e & h & \\ k & & \end{array} \qquad \begin{array}{ccc} c & \underline{e} & h \\ f & k & \\ \underline{a} & & \end{array}$$

We know that the two minimal separable leaders in the last triangle will not cause a problem; but we don't know this directly from the tree small triangles. In the present context, the system (10) is

$$\begin{aligned} &\mathrm{d}a = \mathrm{d}t^0 \cdot c + \mathrm{d}t^1 \cdot b, &&\mathrm{d}b = \mathrm{d}t^0 \cdot e + \mathrm{d}t^1 \cdot d, &&\mathrm{d}c = \mathrm{d}t^0 \cdot f + \mathrm{d}t^1 \cdot e, \\ &\mathrm{d}d = \mathrm{d}t^0 \cdot h + \mathrm{d}t^1 \cdot g, &&\mathrm{d}e = \mathrm{d}t^0 \cdot k + \mathrm{d}t^1 \cdot h, &&\mathrm{d}f = \mathrm{d}t^0 \cdot a + \mathrm{d}t^1 \cdot k, \end{aligned}$$

and then (11) consists of $\partial_0 g = \partial_1 h$, $\partial_0 h = \partial_1 k$, and $\partial_0 k = b$. As linear equations, these are soluble, but they don't carry the information in (33) that lets us know that the corresponding differential system is soluble.

4.6. Another sufficient condition. If $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF, and $K(a^\xi_h \colon |\xi| \leqslant |\pi| \wedge h < n)$ is an extension L of K meeting the differential condition, this by itself is not enough to ensure that $(K, \partial_0, \dots, \partial_{m-1})$ has an extension compatible with L. However, if such an extension does exist, then its existence can be shown by means of the Theorem 4.3, provided $|\pi|$ can be made large enough: this is Theorem 4.10 below, which relies on the existence of bounds as in the following.

Lemma 4.9. *For all positive integers m and n, for all sequences $(a_i \colon i \in \omega)$ of positive integers, there is a bound on the length of strictly increasing chains*

$$S_0 \subset S_1 \subset S_2 \subset \cdots \tag{34}$$

of antichains S_k of $(\omega^m \times n, \leqslant)$, where also $S_k \subseteq \{(\xi, h) \colon |\xi| \leqslant a_k\}$.

Proof. Divide and conquer. First reduce to the case when $n = 1$. Indeed, suppose the claim does hold in this case. Suppose also, as an inductive hypothesis, that the claim holds when $n = \ell$. Now fix m and the sequence $(a_i \colon i \in \omega)$ or rather $(a(i) \colon i \in \omega)$, and consider arbitrary chains as in (34), where $n = \ell + 1$. Analyze each S_k as $S'_k \cup S''_k$, where

$$S'_k = \{(\xi, h) \in S_k \colon h < \ell\},$$
$$S''_k = \{(\xi, h) \in S_k \colon h = \ell\}.$$

For each k such that S_{k+1} exists, at least one of the inclusions $S'_k \subseteq S'_{k+1}$ and $S''_k \subseteq S''_{k+1}$ is strict; also, by our assumption, there is an upper bound $f(k)$ on those r such that

$$S''_k \subset S''_{k+1} \subset \cdots \subset S''_{r-1}. \tag{35}$$

The function f depends only on m, n, and $(a_i \colon i \in \omega)$), not on the choice of chain in (34).

Let $k(0) = 0$, and if $k(i)$ has been chosen, let $k(i + 1)$ be the least r, if it exists, such that $S'_{k(i)} \subset S'_r$. Here k does depend on the chain. But if r is maximal in (35), and S'_r exists, then $S'_k \subset S'_r$. Hence $k(i + 1) \leqslant f(k(i))$. Since the function f is not necessarily increasing, we derive from it the increasing function g, where $g(k) = \max_{i \leqslant k} f(i)$. Then $x \leqslant y \Longrightarrow g(x) \leqslant g(y)$, so

$$k(r) \leqslant f(k(r-1)) \leqslant g(k(r-1)) \leqslant g \circ g(k(r-2)) \leqslant \cdots \leqslant \overbrace{g \circ \cdots \circ g}^{r}(0) = g^r(0). \tag{36}$$

In particular, $S_{k(r)} \subseteq \{(\xi, h) \colon |\xi| \leqslant a(g^r(0))\}$. The sequence $(a(g^i(0)) \colon i \in \omega)$ does not depend on the original chain. Hence the inductive hypothesis applies to the chain

$$S'_{k(0)} \subset S'_{k(1)} \subset \cdots , \tag{37}$$

showing that there is s (independent of the original chain) such that $k(s)$ is defined, and $r \leqslant s$ for all entries $S'_{k(r)}$ in (37). Hence also, by (36), if S_r is an entry in (37), then $r \leqslant k(s) \leqslant g^s(0)$.

Now suppose S'_r is the final entry in (37). Then $S''_r \subset S''_{r+1} \subset \cdots$; but if S''_t is an entry of this chain, then $t < f(r) \leqslant g(r) \leqslant g(g^s(0)) = g^{s+1}(0)$.

Therefore the original chain in (34) has a final entry S_t, where $t < g^{s+1}(0)$. Thus the claim holds when $n = \ell + 1$. By induction, the claim holds for all positive n, provided it holds when $n = 1$.

It remains to show that, for all positive m, for all sequences $(a_i \colon i \in \omega)$, there is a bound on the length of chains

$$S_0 \subset S_1 \subset S_2 \subset \cdots \tag{38}$$

of antichains S_k of $(\omega^m, \leqslant)$, where $S_k \subseteq \{\xi \colon |\xi| \leqslant a_k\}$. The claim is trivially true when $m = 1$. Suppose it is true when $m = \ell$. Now let $m = \ell + 1$, and suppose we have a chain as in (38). We may assume that S_0 contains some σ. If $i < m$ and $j \in \omega$, let

$$S_k^{i,j} = \{\xi \in S_k \colon \xi(i) = j\}.$$

Then the inductive hypothesis applies to chains of the form

$$S_{k(0)}^{i,j} \subset S_{k(1)}^{i,j} \subset S_{k(2)}^{i,j} \subset \cdots.$$

Moreover, if $\tau \in S_k$, then $\tau(i) \leqslant \sigma(i)$ for some i in m (since σ is also in S_k, and this is an antichain). Hence

$$S_k = \bigcup_{i<m} \bigcup_{j \leqslant \sigma(i)} S_k^{i,j},$$

a union of no more than $|\sigma| + m$-many sets, hence no more than $a_0 + m$-many sets. So the proof can proceed as in the reduction to $n = 1$: for each k such that S_{k+1} exists, one of the inclusions $S_k^{i,j} \subseteq S_{k+1}^{i,j}$ is strict, and so forth. ⊣

Theorem 4.10. *Suppose* $(K, \partial_0, \dots, \partial_{m-1}) \models m\text{-DF}$*, and r and n are positive integers. Then there is a positive integer s such that* $r \leqslant s$*, and if* $K(a_h^\xi \colon |\xi| \leqslant s \wedge h < n)$ *meets the differential condition, then* $(K, \partial_0, \dots, \partial_{m-1})$ *has an extension that is compatible with* $K(a_h^\xi \colon |\xi| \leqslant r \wedge h < n)$.

Proof. Suppose $K(a_h^\xi \colon |\xi| \leqslant 2^t r \wedge h < n)$ meets the differential condition for some t. When $u \leqslant t$, let $K_u = K(a_h^\xi \colon |\xi| \leqslant 2^u r \wedge h < n)$, and let S_u be the set of minimal separable leaders of K_u. Then we have an increasing chain $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_t$. By the preceding lemma, there is a value of t, depending only on m, r, and n, large enough that this chain cannot be strictly increasing. Then $S_u = S_{u+1}$ for some u less than this t. Then K_{u+1} satisfies the hypothesis of Theorem 4.3. So $(K, \partial_0, \dots, \partial_{m-1})$ has an extension compatible with $K(a_h^\xi \colon |\xi| < 2^{u+1} r \wedge h < n)$, and *a fortiori* with $K(a_h^\xi \colon |\xi| \leqslant r \wedge h < n)$. In short, the desired s is $2^t r$. ⊣

This theorem yields another yet characterization of the models of m-DCF.

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