A course in projective and hyperbolic geometries

Projektif ve hiperbolik geometriler dersi üzerine

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Preface

This is my record of a course called *Geometriler,* that is, Geometries (in the plural). We studied first projective geometry, starting with Pappus of Alexandria as a source, but considering also Desargues's Theorem (albeit not in the original); then we turned to non-Euclidean or hyperbolic geometry, with Lobachevski as a source. Students presented results at the board as much as possible. Class was conducted in Turkish. The Pappus text was in Turkish, as translated by me from the Greek; the Lobachevski was in English, in the translation by Halstead. A summary of the projective geometry that we would cover is part of the record of the first day of class. Using Euclid's theory of proportion as Pappus did, we proved the Hexagon Theorem. From this we proved Desargues's Theorem. But this theorem could have been used to develop the theory of proportion in the first place (provided we assumed that, once a designated point at infinity was removed, the remaining points of a projective line were ordered).

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Introduction

In the mathematics department of MSGSÜ, the elective course called Geometries (*Geometriler,* MAT 441) has the following description on the department website:

> Euclidean Geometry and Coordinates. Introduction to Affine, Projective, and Hyperbolic Geometries.

The recommended text is H.S.M. Coxeter, *Introduction to Geometry* (second edition), John Wiley & Sons [5]. I taught the course in the fall semester of 2015–16. The present document is my record of what happened. Briefly, we covered "only" projective and hyperbolic geometry, with little use of Coxeter's text.

The Geometries course had been last offered in the fall semester of 2010–11 by Özer Öztürk. I was then still at METU in Ankara, but Özer and I were collaborating to translate the 48 propositions of first book of Euclid's *Elements* from Greek into Turkish [6]. When I moved to Mimar Sinan in the following year, Özer and I led the two sections of a new course for incoming students,[1] called Introduction to Euclidean Geometry (*Öklid Geometrisine Giriş,* MAT 113). Here the students themselves presented Euclid's propositions at the board, in the manner that I knew from my own *alma mater,* St John's College [18].

[1]See Appendix E.1, page 147, on referring to first-year students as freshmen.

The Euclid course had three sections in the following year, and four sections after that, for a total of about sixty students each year. It is fortunate that we have been able to make the sections of the Euclid course small, so that each student both has more opportunities to go to the board and is less able to hide in the back of the room. Being part of a "fine arts" university, rather than a "technical" university, our department has few service courses to teach; thus our teaching energies can be focussed on our "own" students. (The METU mathematics department taught calculus to just about every student at the university; such is not the case at MSGSÜ.)

The language of instruction at METU was English. At Mimar Sinan it is Turkish. This was one reason why it was desirable for me to move here: I would learn Turkish better. In the Euclid course, the students would do most of the talking, to ease my transition to using Turkish. This was the expectation, at least. In the event, my spoken intervention in the class was required from the beginning, in unexpected ways.

Meanwhile, in translating Euclid, I had broken the Greek into phrases, which I translated quite literally into English; Özer then rendered them in Turkish. Our first edition of Euclid was in three parallel columns: English, Greek, and Turkish. Later I removed the English tried to harmonize the Turkish better with the Greek. Turkish nouns being declined like Greek nouns, one can usually maintain Euclid's word order in Turkish. The result may not be good Turkish style. This is not necessarily bad, since it should induce students to come up with their own words for the mathematics.

Eventually I wrote up more than fifty exercises keyed to Euclid. These were statements for which students should find proofs, using just the propositions up to specified points in the first book of the *Elements.* I led a section of the Euclid course for four years. In the fifth year, I took a break. Given in that year, the Geometries course was a chance to see how the students had developed since their first year. I had them go to the board again to present propositions.

We started the Geometries course with several propositions from Book VII of the *Collection* of Pappus of Alexandria. These are the propositions that establish what is now known as Pappus's Hexagon Theorem of projective geometry. I had translated into Turkish the first nineteen of the thirty-eight lemmas that Pappus gives as an aid for reading Euclid's (now-lost) *Porisms.* These lemmas are numbered consecutively with Roman numerals. For translating, I used Hultsch's edition [15] of the Greek text until I was almost finished. Only then did I learn about Jones's edition and English translation [16]. Professor Jones kindly made this available to me, and it helped to clarify some points.

With the Turkish version of Pappus (which should be available through my webpage), I provided an introduction, some of whose contents I would discuss on the first day of class. See then my record of that day for more information.

After projective geometry, we turned to what is now called hyperbolic geometry. We read Lobachevski in Halstead's English translation of 1891 [3].

I attempted to write the following record of each day of class in the present tense. I may sometimes have slipped into the past tense, especially if I was actually writing much later than the events described. Footnotes may be in the past tense. Chapters are numbered according to the week of the semester.

Class met every week for two hours, on Tuesday mornings, 9–11. Each hour included a ten-minute break, in principle, so that we were really in session 9–9:50 and 10–10:50. Moreover, students who arrived by mass transit (and this was all but one of them) were never on time. Most courses in the department meet three hours a week: so I had to remember not to expect from my students the same amount of work as in one of those courses.

Part I.

Projective Geometry

1. September 29

Though seven students are registered for my course, in the first class only Verity and Lucky are present.[1] Other departments in our university seem not to bother holding class in the first week; but ours does. Since Verity and Lucky are willing to listen, I talk a lot to them about what we are going to do. I give them printouts of the Pappus translation and arrange for presentations next week as follows:[2]

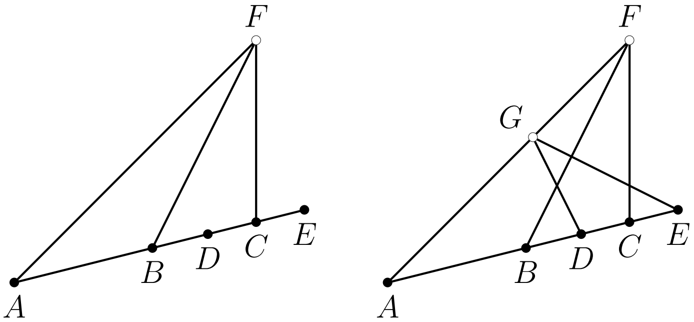
Verity: Lemma VIII;
Lucky: Lemma IV.

The remainder of this chapter, first drafted more than three weeks after the class, is based more on the written notes that I prepared for the day than on my memory of the day. It is an overview of what will be covered in the course, as far as projective geometry is concerned.

Suppose five points, A through E, fall on a straight line as in Figure 1.1a, and F is a random point not on the straight line. Join FA, FB, and FC. Now let G be a random point on FA, as in Figure 1.1b, and join GD and GE. Supposing these two

[1]The names are pseudonyms. In the underlying LaTeX file, the name of a student called Barış would be written as a command `\Baris`, which might be defined so as to print a capitalized word such as “Peace” (which happens to be what *Barış* means in Turkish).

[2]I added this information about who was to present what after the whole course is over. The information must be correct, since Verity and Lucky did in fact present these propositions in the following week. However, it made no logical difference which of VIII and IV was presented first.



(a) Point F is chosen (b) Point G is chosen

Figure 1.1. The Complete Quadrangle Theorem set up

straight lines cross FB and FC at H and K respectively, join HK as in Figure 1.2. If this straight line crosses the original straight line AB at L, then L depends only on the original five points, not on F or G. This is a consequence of Lemma IV in our text of Pappus. Let us call this result the **Complete Quadrangle Theorem.** It is about how the straight line AB crosses the six straight lines that pass through pairs of the four points F, G, H, and K. Any such collection of four points, no three of which are collinear, together with the six straight lines that they determine, is called a **complete quadrangle** (*tam dörtgen*).[3] See also Figure 2.5 on page 34.

In Figure 1.2, if it should turn out that $HK \parallel AB$, this too happens independently of the choice of F or G.[4] We shall say in this case that the straight lines HK and AB *intersect*

[3]I don't think I actually defined this term in class today; but I would do so in the following week, as recorded on page 33.

[4]This result is basically Lemmas I and II of Pappus. By Lemma V, under the assumption that C and D coincide, if L and A should coincide, this always happens. Lemma VI and VII concern cases where $GK \parallel AB$, that is, E is at infinity. I did not say this in class today.

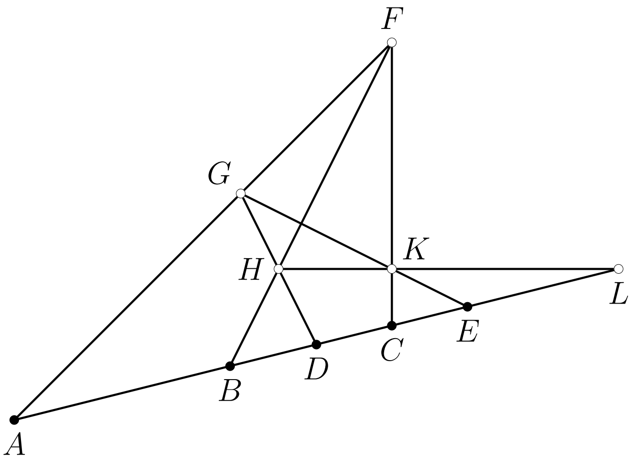


Figure 1.2. The Complete Quadrangle Theorem

at infinity. We define the points of the **projective plane** (*projektif düzlem*) to be:

- the points of the Euclidean plane, and
- **points at infinity** (*sonsuzdaki noktalar*), one for each class of parallel straight lines.

The points at infinity compose the **line at infinity** (*sonsuzdaki doğru*). Thus the projective plane respects two axioms:

1) any two straight lines intersect at exactly one point,
2) any two points lie on exactly one straight line.

The Complete Quadrangle Theorem can be understood as a theorem about the projective plane.

Later, through the work of Lobachevski, we shall study the **hyperbolic plane** (*hiperbolik düzlem*), where, through a given point, more than one straight line will pass that never intersects a given straight line, as in Figure 1.3.

Meanwhile, after Pappus's proof of the Complete Quadrangle Theorem, we are going to work through another proof,

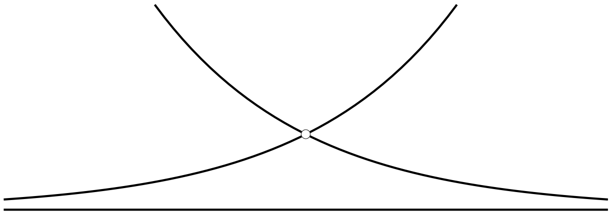


Figure 1.3. Straight lines in the hyperbolic plane

using **Desargues's Theorem.** Girard Desargues was a contemporary of Descartes. The theorem named for him is that if, as in Figure 1.4, the straight lines through corresponding vertices of triangles ΑΒΓ and ΔΕΖ meet at one point (namely Η), then the points Θ, Κ, and Λ, where corresponding sides of the triangles intersect, are on a straight line.[5] This is true in the projective plane: that is, some of the points Η, Θ, Κ, and Λ can be at infinity.

We shall prove Desargues's Theorem by using **Pappus's Hexagon Theorem.** This theorem is that if the vertices of a hexagon lie alternately on two straight lines, then the points of intersection of the three pairs of opposite sides of the hexagon also lie on a straight line. Thus in Figure 1.5, which shows a hexagon $ABCDEF$, if ACE and BDF are straight, then GHK is straight.

Pappus's Theorem is Lemmas VIII, XII and XIII in our text. Strictly, these cover only three of six cases of the theorem, since the two straight lines on which the vertices of the hexagon lie

[5] I vacillate between using Latin and Greek letters for points in diagrams. In the Pappus translation, I have retained Pappus's Greek letters; but students should not feel that using Greek letters is obligatory in their own work. A possible modern practice is to use capital Latin letters for points, Latin minuscules for straight lines, and capital Greek letters for planes; but I am not following this practice.

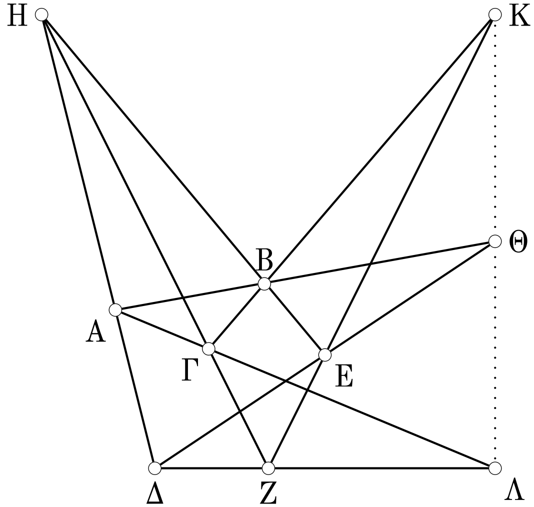


Figure 1.4. Desargues's Theorem

may be parallel or not, and none, one, or all three of the pairs of opposite sides of the hexagon may be parallel. The possibilities are tabulated in Figure 1.13 on page 27.

Lemma VIII is the case where the points of intersection of two pairs of opposite sides of the hexagon are at infinity, that is, the pairs are parallel, as in Figures 2.1 and 4.2 (pages 29 and 38). The conclusion is that the third pair of opposite sides are parallel. Pappus proves only the case where the two straight lines on which the vertices of the hexagon lie alternately are not parallel; the other case is easier. Pappus's proof is based on Propositions 37 and 39 of Book I of Euclid's *Elements*: namely, since two triangles ABC and ABD have a common base AB as in Figure 1.6, the triangles are equal if and only if $CD \parallel AB$.

Pappus's Lemmas XII and XIII are the case of the Hexagon Theorem where two pairs of opposite sides intersect, as in Fig-

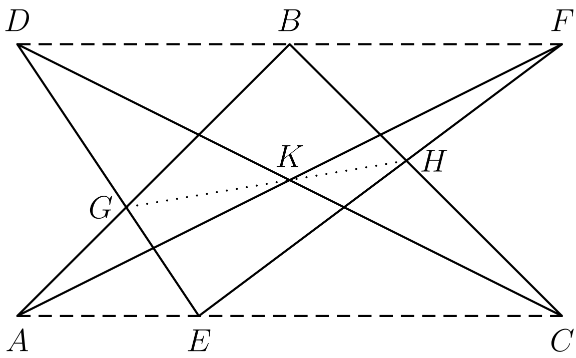


Figure 1.5. Pappus's Hexagon Theorem

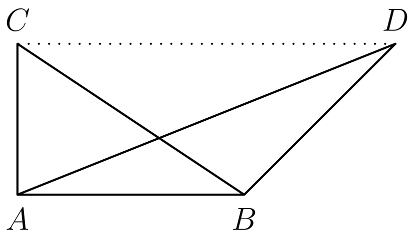


Figure 1.6. Triangles on the same base

ure 1.5, or in Figure 5.9 on page 53. The case where only one pair of opposite sides are parallel, as in Figure 4.3 on page 39, is apparently not treated.

One may identify even more cases of Pappus's Theorem, if one considers different orderings of the vertices of the hexagon on the two straight lines; but these orderings should not affect the proofs.

Pappus's proofs rely on Lemmas III, X, and XI. These in turn require a **theory of proportion.** The key element of this theory is Proposition 2 of Book VI of Euclid's *Elements*: if the straight line DE cuts the sides of triangle ABC as in

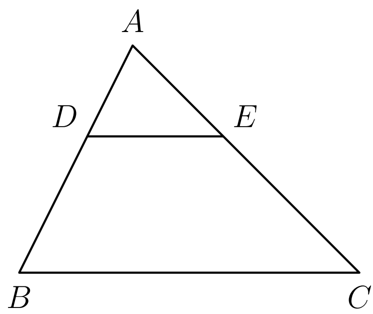


Figure 1.7. The Fundamental Theorem of Proportion

Figure 1.7, then

$$DE \parallel BC \iff AD : DB :: AE : EC. \qquad (1.1)$$

I propose to call this result the **Fundamental Theorem of Proportion.**[6] Here the expression $AD : DB$ denotes a **ratio** (*oran*), and $AD : DB :: AE : EC$ denotes a **proportion** (*orantı*). One may write the ratio in the modern form of AD/DB, and the proportion as the equation $AD/DB = AE/EC$. Familiar algebraic rules for manipulating such equations will apply. However, Euclid and Pappus never describe two ratios as being *equal,* but only as *the same.*

We have not actually defined ratios and proportions. Can we take the Fundamental Theorem, formulated in (1.1), as a definition? To do this, we need to know that, in Figure 1.7, if the lengths AD, DB, AE, and EC are fixed, but the angle

[6] I did not use this term in class that day. Apparently the theorem is called Thales's Theorem in Turkish [8] and some other languages, as one can tell from *Wikipedia*; but the name dates only from the 19th century [17]. The only historical justification for the name seems to be Plutarch's fanciful *Dinner of the Seven Wise Men* [20].

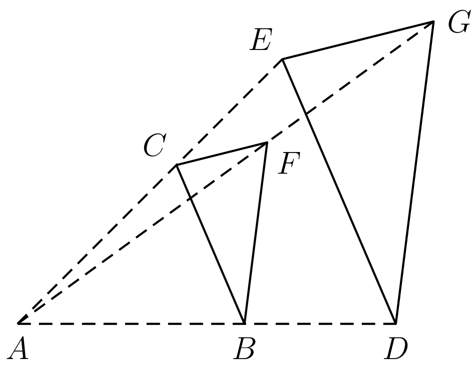


Figure 1.8. Proportion and Desargues

BAC changes, then the parallelism (or lack of it) of BC and DE will not change.

In Figure 1.8, suppose we know $BC \parallel DE$ and $BF \parallel DG$. The Fundamental Theorem gives us

$$AB : BD :: AC : CE, \qquad AB : BD :: AF : FG. \tag{1.2}$$

By noting that the ratio $AB : BD$ is common to each of these proportions, if we can conclude

$$AC : CE :: AF : FG, \tag{1.3}$$

then the Fundamental Theorem gives us $CF \parallel EG$. Euclid's definition of proportion does entail that (1.2) implies (1.3). But the Fundamental Theorem by itself does not: the theorem alone (treated as a definition of proportion) does not entail the property of *transitivity* that "sameness of ratio" ought to have. However, if we know

$$BC \parallel DE \ \& \ BF \parallel DG \implies CF \parallel EG, \tag{1.4}$$

then transitivity of sameness of ratio follows from the Fundamental Theorem, even when used as a definition.

The implication (1.4) is a special case of Desargues's Theorem. But we are going to prove this theorem by means of Pappus's Hexagon Theorem, which in turn will rely on the theory of proportion embodied in the Fundamental Theorem. Thus Pappus's Theorem and the Fundamental Theorem of Proportion are somehow equivalent.[7]

The theory of proportion found in Books V and VI of Euclid's *Elements* relies on the so-called Archimedean Axiom. This is that the difference between unequal finite straight lines can be multiplied so as to exceed either of these lines. In fact that axiom is not needed, but one can develop an adequate theory of proportion, solely on the basis of Book I of the *Elements.* I did this in the first-year course Analytic Geometry last semester, as follows.[8]

Let a minuscule Latin letter denote a **length,** that is, the class of finite straight lines that are equal to a given finite straight line. A product $a \cdot b$ can denote an **area,** namely the class of all plane figures that are equal to a rectangle of width a and height b. Then we can define

$$a : b :: c : d \iff a \cdot d = b \cdot c.$$

[7]Since in class today I did not give a name to the Fundamental Theorem, I just said Pappus's Theorem was equivalent to the theory of proportion. As noted earlier, the proof of Pappus's Theorem needs, in addition to proportion, a theory of *areas,* which requires the points on a straight line (without the point at infinity) to be *ordered,* so that one can say when two areas are being added rather than subtracted.

[8]I did not go into the remainder of this chapter in class today. From the Analytic Geometry class, I have extensive notes in Turkish. In that class, I had hoped my Euclidean approach to things would make sense to the students, who had just spent a semester reading Euclid. Probably most students preferred to use what they had learned about proportion in high school, however unjustified it was. I relied on this high-school knowledge myself, in the Geometries class.

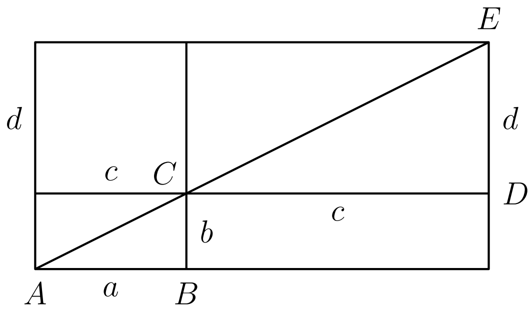


Figure 1.9. Proposition I.43 of Euclid's *Elements*

If these lengths are as in Figure 1.9, it follows from the *Elements* Proposition I.43 (and its converse, and I.28 and 29) that $a : b :: c : d$ if and only if, in the right triangles ABC and CDE, the angles at A and C (respectively) are equal.

We pass to a third dimension, defining the **volume** $a \cdot b \cdot c$ as the class of solid figures equal to a rectangular parallelepiped[9] having dimensions a, b, and c. We can now define

$$a \cdot b : c \cdot d :: e : f \iff a \cdot b \cdot f = c \cdot d \cdot e.$$

This allows us to derive from the proportion

$$a : b :: c : d$$

the additional proportions

$$a^2 : b^2 :: c^2 : d^2,$$
$$a^2 + b^2 : b^2 :: c^2 + d^2 : d^2.$$

If we know $a^2 + b^2 = e^2$ and $c^2 + d^2 = f^2$, as in Figure 1.10, then we can conclude

[9]See Appendix E.2 (page 148) for why this word should be pronounced with the stress on the antepenult.

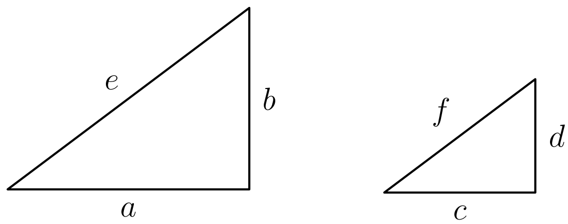


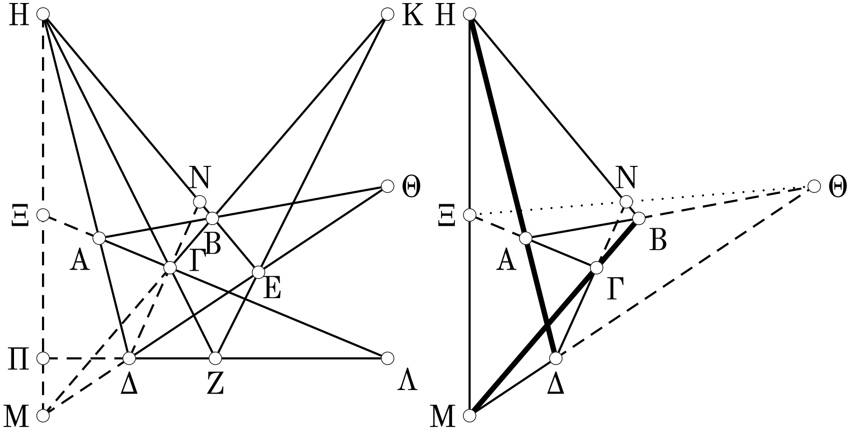
Figure 1.10. Similar right triangles

$$e : b :: f : d.$$

Ultimately we obtain the Fundamental Theorem.

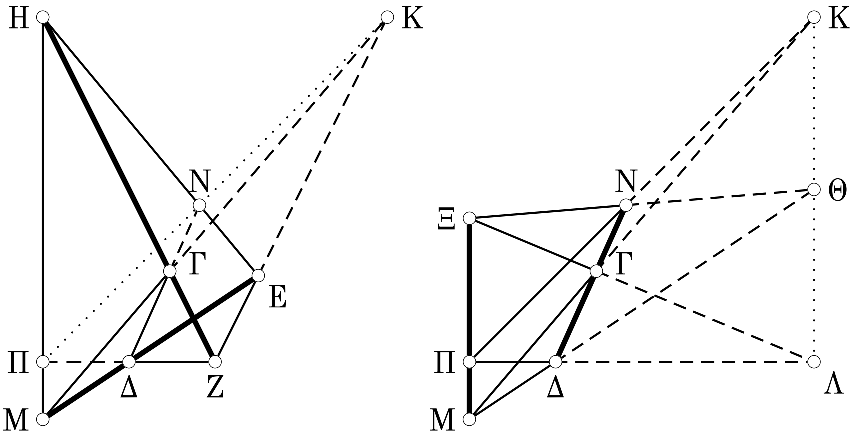
In present course, we shall also make use of a third dimension, but in a less algebraic, more geometric way. As noted, we shall prove Pappus's Lemma VIII using not proportions, but areas. By *projecting* from one plane onto another that is not parallel to it, we transform some parallel straight lines into intersecting straight lines, and *vice versa.* This will give us the remaining cases of the Hexagon Theorem.

With three applications of the Hexagon Theorem, we can prove Desargues's Theorem as sketched in Figures 1.11 and 1.12, using in turn the hexagons ΑΒΗΜΔΓ, ΓΔΖΕΗΜ, and ΓΜΔΠΝΞ. This gives us (1.4), as noted, and hence the Fundamental Theorem of Proportion.



(a) Construction (b) ΘΝΞ is straight

Figure 1.11. Proof of Desargues's Theorem: first steps



(a) ΚΝΠ is straight (b) ΘΚΛ is straight

Figure 1.12. Proof of Desargues's Theorem: last steps

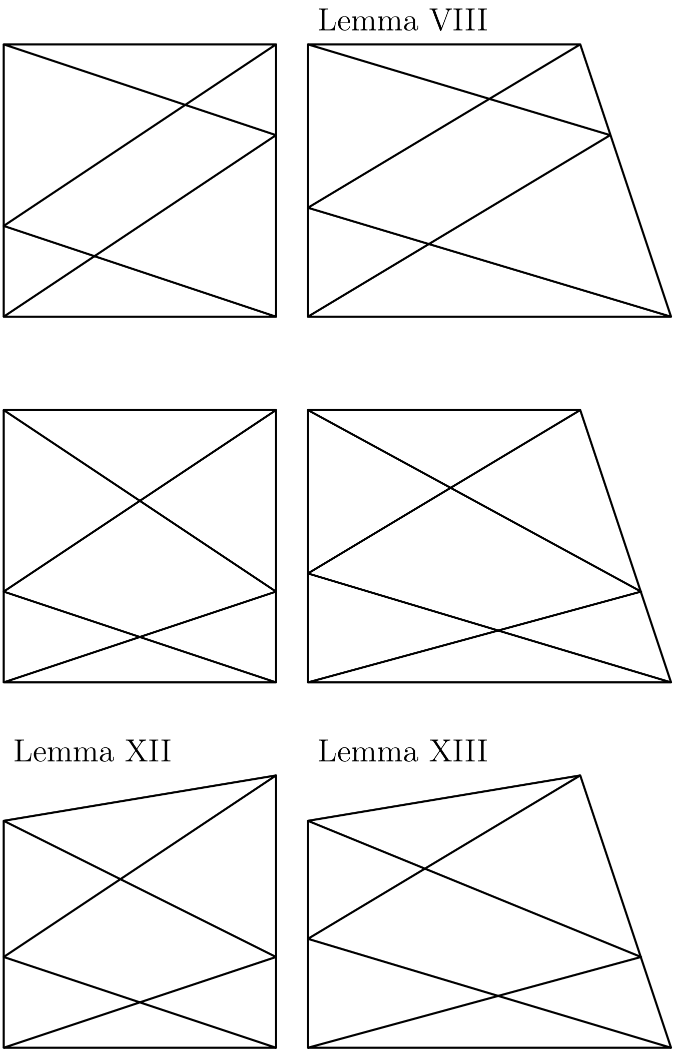


Figure 1.13. Cases of the Hexagon Theorem

2. October 6

Present are Verity, Eve, Sparkle, Charity, and Lucky, though some of them are late. Verity is at the board as I enter, trying to get ready to present **Lemma VIII.** She is confused about something.

As on page 19, Lemma VIII is that if the vertices of a hexagon are alternately on two intersecting straight lines, and two pairs of opposite sides are parallel, then the third pair are parallel.

Verity uses a diagram much as in Figure 2.1a, which is from the text. We are given the hexagon ΒΓΗΕΔΖ, whose vertices lie alternately on straight lines intersecting at Α. It is assumed that

$$\mathrm{B\Gamma} \parallel \mathrm{\Delta E}, \qquad \mathrm{BZ} \parallel \mathrm{EH}. \tag{2.1}$$

We are to prove

$$\mathrm{\Gamma H} \parallel \mathrm{\Delta Z}.$$

Verity observes that triangle ΒΖΕ cannot be equal to ΑΒΖ as the text says (because of my transcription mistake). She proposes ΒΖΕ = ΒΗΕ. She tries to justify this by noting that ΒΖ ∥ ΕΗ. She has not remembered properly Proposition I.37 from Euclid mentioned on page 19, perhaps because the parallel straight lines are vertical in her figure, not horizontal as in Euclid's. (I do not use the proposition number in class; I do not even remember it.)

Eve supports Verity's mistake. I go to the board to draw a diagram as in Figure 2.2, to bring out the error. Drawing

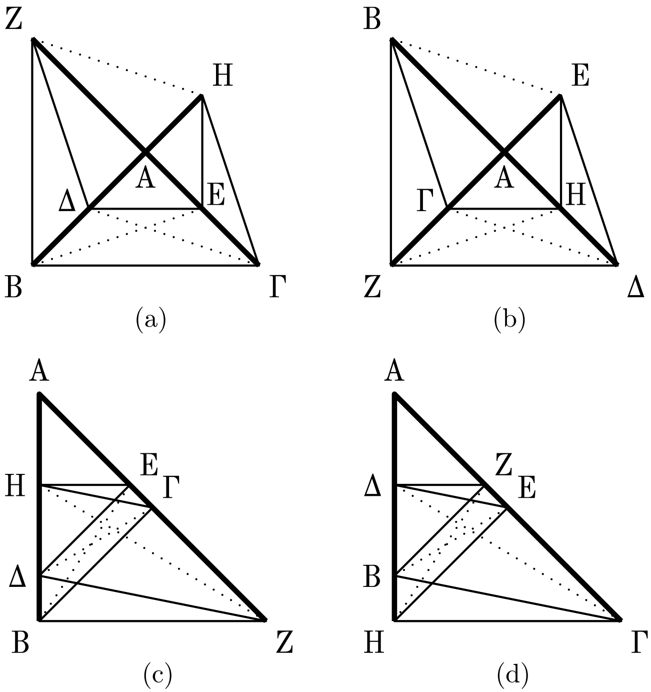


Figure 2.1. Lemma VIII

the parallels vertically, rather than horizontally as in Euclid's diagram, has caused confusion. Eventually the matter is understood.

Lemma VIII uses also the converse of I.37, which is I.39. I observe in effect that we use I.37 twice, and I.39 once. For the argument in short is as follows (where all three-letter combinations are triangles, not angles):

1) $\Delta\mathrm{BE} = \Delta\Gamma\mathrm{E}$, [*Elements* I.37, since $\mathrm{B}\Gamma \parallel \Delta\mathrm{E}$]
2) $\mathrm{ABE} = \Gamma\Delta\mathrm{A}$, [add $\Delta\mathrm{AE}$]

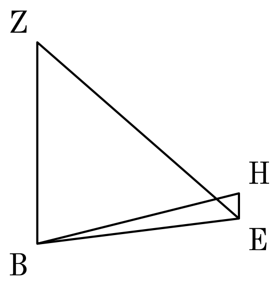


Figure 2.2. *Elements* I.37, misremembered

3)	BZE = BZH,	[*Elements* I.37, since BZ ∥ EH]
4)	ABE = AHZ,	[subtract ABZ]
5)	AΓΔ = AHZ,	[steps 2 and 4]
6)	ΓΔH = ΓZH,	[add AΓH]
7)	ΓH ∥ ΔZ.	[*Elements* I.39]

I indicate that there are other cases of the theorem, depending on the relative positions of the points on the two straight lines BAH and ZAΓ. I sketch some of the possibilities, though I do not make a systematic presentation in class. However, every possible configuration can be labelled as in one of the four diagrams of Figure 2.1. That is, Lemma VIII is really four theorems about the hexagon BΓHEΔZ, whose vertices lie alternately on straight lines intersecting at A, and where (2.1) holds; or else the lemma is one theorem whose proof has four parts, corresponding respectively to the following situations:

a) A lies between exactly one of the pairs of parallels,
b) A lies between both pairs of parallels,
c) the hexagon BΓHEΔZ lies between two of the opposite sides that are given as parallel,
d) the hexagon BΓHEΔZ lies between the opposite sides that

are not given as parallel.

For all four diagrams, the equations in the proofs are the same, but the ways that they are obtained differ, as follows:

case (a)	case (b)	case (c)	case (d)
add	subtract	add	add
subtract	subtract	subtract from	add
add	subtract	subtract	subtract from

To obtain different patterns, one can respectively apply the permutations

a) (Β Δ)(Γ Ε),

b) (Γ Ζ)(Δ Η),

c) (Β Η)(Ε Ζ), (Γ Ζ)(Δ Η), (Γ Η)(Δ Ζ),

d) (Γ Ζ)(Δ Η).

However, if one wants to keep Pappus's triangles exactly, one cannot avoid "subtracting from."

Lucky is to present **Lemma IV** (discussed on page 16), which is that if, in Figure 2.3,

$$\mathrm{AZ}\cdot\mathrm{B}\Gamma : \mathrm{AB}\cdot\Gamma\mathrm{Z} :: \mathrm{AZ}\cdot\Delta\mathrm{E} : \mathrm{A}\Delta\cdot\mathrm{EZ}, \tag{2.2}$$

then Θ, Η, and Ζ are in a straight line. Lucky generally has the appearance of an ambitious student. He is organizing seminars for the student mathematics club in our department. However, today, he writes on the board what is in the text, without understanding it. He does not understand that some of Pappus's manipulations are entirely formal, with no need to refer to the diagram; but Lucky keeps looking up at, and referring to, his diagram. Again with my intervention, we get things straightened out. By alternation of (2.2), we obtain

$$\mathrm{AZ}\cdot\mathrm{B}\Gamma : \mathrm{AZ}\cdot\Delta\mathrm{E} :: \mathrm{AB}\cdot\Gamma\mathrm{Z} : \mathrm{A}\Delta\cdot\mathrm{EZ}.$$

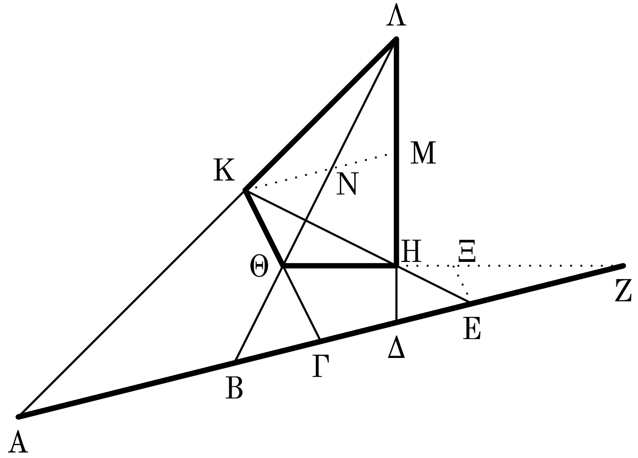


Figure 2.3. Lemma IV

For the left-hand member we have

$$\mathrm{AZ \cdot B\Gamma : AZ \cdot \Delta E :: B\Gamma : \Delta E}$$
$$\mathrm{:: B\Gamma : KN \;\;\&\;\; KN : KM \;\;\&\;\; KM : \Delta E;}$$

and for the right,

$$\mathrm{AB \cdot \Gamma Z : A\Delta \cdot EZ :: BA : A\Delta \;\;\&\;\; \Gamma Z : ZE.}$$

Assuming KM is drawn parallel to AZ, we have

$$\mathrm{NK : KM :: BA : A\Delta.}$$

Eliminating this common ratio from either side of the original proportion, and reversing the order of the members, we obtain

$$\mathrm{\Gamma Z : ZE :: B\Gamma : KN \;\;\&\;\; KM : \Delta E,}$$

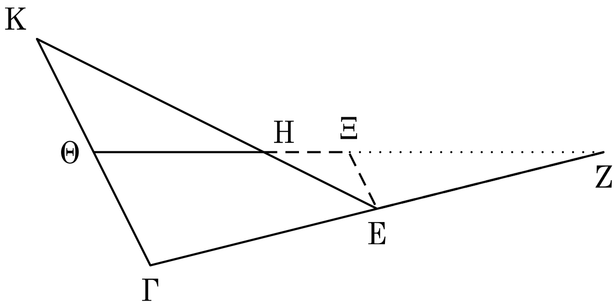


Figure 2.4. Lemma from Lemma IV

and therefore, by the Fundamental Theorem of Proportion applied to each ratio in the compound,

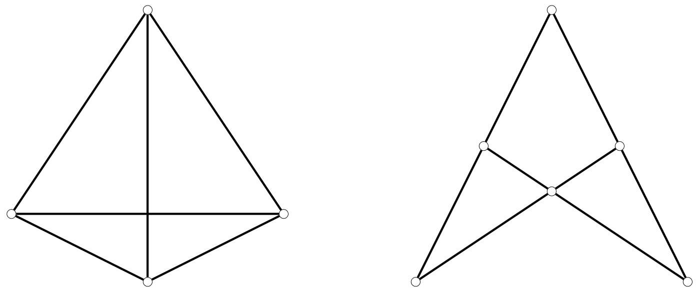
$$\Gamma\mathrm{Z} : \mathrm{ZE} :: \Theta\Gamma : \mathrm{K}\Theta \ \ \& \ \ \mathrm{KH} : \mathrm{HE}. \tag{2.3}$$

Pappus says now that ΘHZ is indeed straight, which seems premature, since he is going to have to argue this out. He may be alluding to the lemma whose diagram is in Figure 2.4. Here $\mathrm{E}\Xi \parallel \Theta\Gamma$, and ΘH is extended to Ξ. Then from (2.3) we have

$$\begin{aligned}\Gamma\mathrm{Z} : \mathrm{ZE} &:: \Theta\Gamma : \mathrm{K}\Theta \ \ \& \ \ \mathrm{K}\Theta : \mathrm{E}\Xi \\ &:: \Theta\Gamma : \mathrm{E}\Xi.\end{aligned}$$

By the Fundamental Theorem again, the points Θ, Ξ, and Z must be collinear, and therefore the same is true for Θ, H, and Z.

I shall later give the converses of this lemma and Lemma IV itself as exercises; see page 58. Meanwhile, in class today, I explain the lemma in terms of the **complete quadrangle,** as described on page 16 and as shown in Figure 2.5a. I used the term *tam dörtkenarlı* in the introduction to the translation



(a) A complete quadrangle (b) A complete quadrilateral

Figure 2.5. "Complete" figures

of Pappus; but the term should be *tam dörtgen.* The former term should be used for the **complete quadrilateral,** which consists of four straight lines, no three concurrent, together with the six points in which pairs of the straight lines intersect, as in Figure 2.5b. But then a complete quadrilateral is just what we turned out to be considering in Figure 2.4.

Lemma IV is that, if a straight line cuts the sides

$$\mathrm{\Lambda K}, \qquad \mathrm{\Lambda\Theta}, \qquad \mathrm{K\Theta}, \qquad \mathrm{\Lambda H}, \qquad \mathrm{KH}, \qquad \mathrm{\Theta H}$$

of a complete quadrangle at the points A, B, Γ, Δ, E, Z respectively so that (2.2) holds, and if, as in Figure 2.6, the same straight line cuts the sides

$$\mathrm{T\Sigma}, \qquad \mathrm{TP}, \qquad \mathrm{\Sigma P}, \qquad \mathrm{T\Pi}, \qquad \mathrm{\Sigma\Pi}$$

of another complete quadrangle also at A, B, Γ, Δ, E, then it must cut ΣT at Z. By reversing the steps of the argument, one shows that (2.2) must hold anyway. Thus we obtain the Complete Quadrangle Theorem.

In the break I made a printout of the text for the newcomers. At the end of class, I make the assignments:

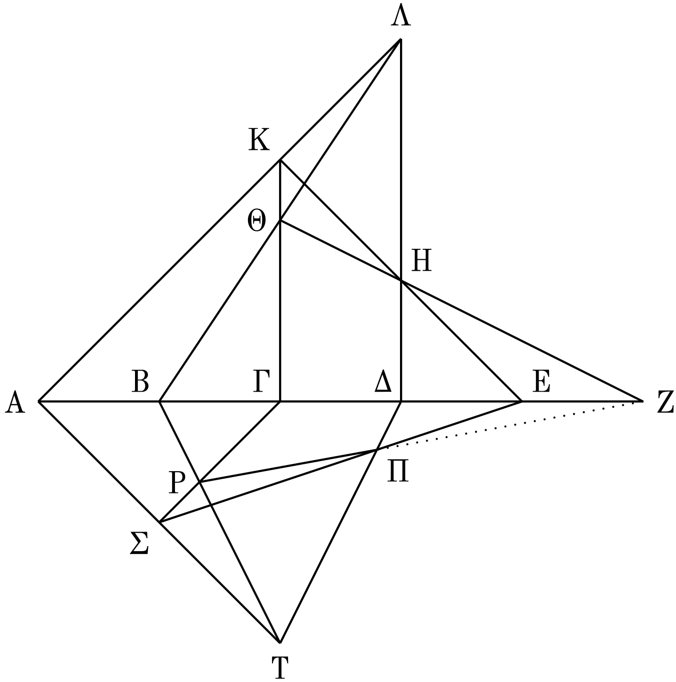


Figure 2.6. The Complete Quadrangle Theorem

Lemma III: Sparkle;
Lemma X: Eve;
Lemma XI: Charity.

3. October 13

In response to the October 10 Ankara Bombing, labor unions called a general strike for the following three days, the last of these being today. Students and teachers have joined the strike. Students of our university have arranged to hold a forum in our building in Bomonti at 10 a.m. It is suspected that, before this, they will make noise and otherwise ensure that no classes can be held.

I go to the classroom anyway. The disgusting crime in Ankara would seem to mean that *more* education is needed, not less. I mean liberal education though, not technical education. In any case, spending time doing mathematics—or just learning *anything*—may aid a person who is in mourning.

Eve wrote me yesterday, to ask if class would be held, since she had heard that many classes were being cancelled. I wrote back that I would be in class, and I thought everybody should go *somewhere,* be it to class, to the forum, or to some other demonstration. The slaughter in Ankara should not be treated as an opportunity for a holiday. When Sparkle wrote me a similar question, I forwarded to her my reply to Eve.

I hope that the remaining three students understand that, in a real strike, one does not ask permission from one's employer. Of course we teachers are not our students' employers; neither are the students *our* employers. The Turkish state employs us.

4. October 20

Only Sparkle is present at the start of class, and she asks me about one step in the proof of **Lemma III,** namely

$$\mathrm{EZ} : \mathrm{ZH} :: \mathrm{E\Theta} : \mathrm{\Theta M} \implies \mathrm{\Theta E} \cdot \mathrm{HZ} = \mathrm{EZ} \cdot \mathrm{\Theta M}, \tag{4.1}$$

that is, the step of deducing the equation from the proportion. I sketch a diagram as in Figure 4.1, where $a : b :: c : d$ and $ad = bc$. (See also page 24.)

Meanwhile, everybody else but Eve shows up; she will come at about 9:20. I recall Lemma VIII, but draw its diagram as in Figure 4.2 (not Figure 2.1a). We state the theorem as

$$\mathrm{AB} \parallel \mathrm{EZ} \ \& \ \mathrm{B\Gamma} \parallel \mathrm{\Delta E} \implies \mathrm{A\Delta} \parallel \mathrm{\Gamma Z},$$

assuming ΑΓΕ and ΒΔΖ are straight. I observe that the statement does not involve an intersection point of the two straight lines. I leave it as an **exercise** to prove the theorem in case ΑΓΕ and ΒΔΖ are parallel. (See page 58.)

What if ΒΓ and ΔΕ meet at a point Θ, as in Figure 4.3? After some thought, students agree that, as a result of Lemma VIII, ΑΔ and ΓΖ must intersect at a point Κ. I say that in this case

$$\mathrm{\Theta K} \parallel \mathrm{AB}.$$

Sparkle seems to think this is clear. I say we are going to prove it using **perspective.** In Turkish this is *perspektif* or *görünge.* The students have not seen the latter word, though it is in Püsküllüoğlu's Turkish dictionary [21]. I hold two pens

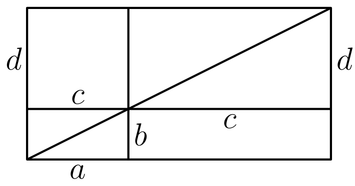


Figure 4.1. Cross multiplication

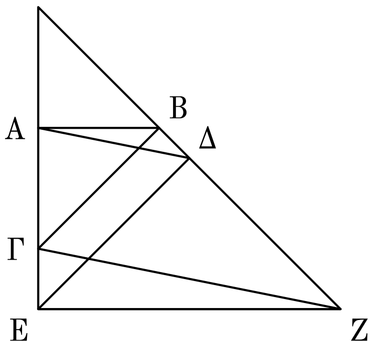


Figure 4.2. Lemma VIII again

parallel, observing that they do not look parallel when pointed towards my eye.

Now I invite Sparkle to present **Lemma III.** She draws the diagram complete at the beginning, as in Figure 4.4, then presents the argument in numbered steps. She draws separate diagrams to explain the first proportions, namely

$$\begin{aligned} \mathrm{EZ} : \mathrm{ZA} &:: \mathrm{E\Theta} : \Theta\Lambda, \\ \mathrm{AZ} : \mathrm{ZH} &:: \Theta\Lambda : \Theta\mathrm{M}; \end{aligned} \tag{4.2}$$

and she writes out Pappus's explanation, "because the two are the same as ΘK : ΘH." But she does not seem to understand the explanation. The straight lines ΘK and ΘH are not in her

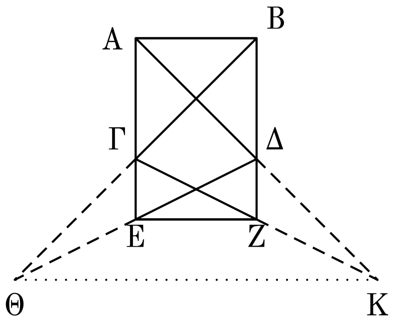


Figure 4.3. Another case of Pappus's Theorem

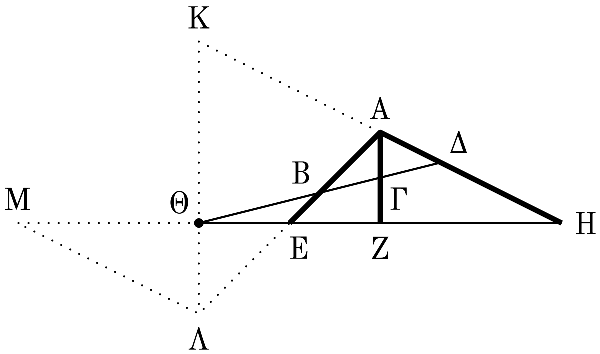


Figure 4.4. Lemma III

supplementary diagrams until I invite her to add them. The diagram ought to be as in Figure 4.5; but when Sparkle draws ΘK, she does not make it tall enough so that HA extends to K.

Pappus's argument is as follows. The straight lines ΘE and ΘΔ cut the straight lines AB, ΓA, and ΔA as in Figure 4.4. The diagram is completed by making

$$\mathrm{K\Lambda} \parallel \mathrm{Z\Gamma A}, \qquad \mathrm{\Lambda M} \parallel \mathrm{\Delta A}.$$

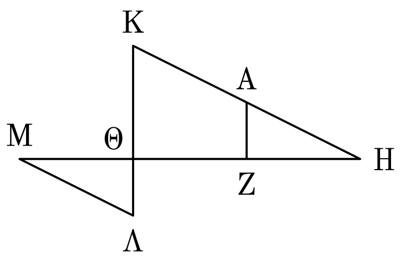


Figure 4.5. Lemma III, auxiliary diagram

Then the proportions (4.2) hold, so *ex aequali,* and by (4.1),

$$\mathrm{EZ} : \mathrm{ZH} :: \mathrm{E\Theta} : \Theta\mathrm{M},$$
$$\Theta\mathrm{E} \cdot \mathrm{HZ} = \mathrm{EZ} \cdot \Theta\mathrm{M}.$$

Being equal, these areas have the same ratio to $\mathrm{EZ} \cdot \Theta\mathrm{H}$, and so

$$\begin{aligned} \Theta\mathrm{E} \cdot \mathrm{HZ} : \mathrm{EZ} \cdot \Theta\mathrm{H} &:: \mathrm{EZ} \cdot \Theta\mathrm{M} : \mathrm{EZ} \cdot \Theta\mathrm{H} \\ &:: \Theta\mathrm{M} : \Theta\mathrm{H} \\ &:: \Lambda\Theta : \Theta\mathrm{K}. \end{aligned} \tag{4.3}$$

When Sparkle reaches this point, I try so suggest that the proof is really over: for the ratio $\Lambda\Theta : \Theta\mathrm{K}$ is independent of the choice of the straight line through Θ that cuts the three straight lines that pass through A. In particular, we can conclude immediately

$$\Theta\mathrm{E} \cdot \mathrm{HZ} : \mathrm{EZ} \cdot \Theta\mathrm{H} :: \Theta\mathrm{B} \cdot \Delta\Gamma : \mathrm{B}\Gamma \cdot \Theta\Delta,$$

which is the theorem.

However, Pappus does not conclude immediately, but proceeds strangely, and Sparkle wants to follow him, so I let her.

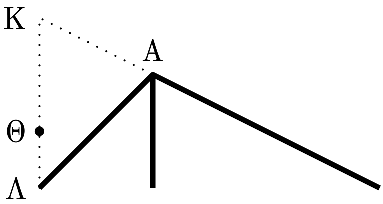


Figure 4.6. Lemma III, simplified

Then I draw a separate diagram, as in Figure 4.6, to make my point that the extra work is not needed. I say that the ratio

$$\Theta\mathrm{E} \cdot \mathrm{HZ} : \mathrm{EZ} \cdot \Theta\mathrm{H}$$

can be called the **cross ratio** (*çapraz oran*) of the points Θ, E, Z, and H. In stating Lemma III, Sparkle seemed to have got the idea that the ratio was something special: at any rate, she had learned it as the ratio of the product of the two outer segments to the product of the whole with the inner segment.

Eve presents **Lemma X** after the break. First I ask her if she has seen the error in the text: in the proportion

$$\Delta\Theta \cdot \mathrm{B}\Gamma : \Delta\underline{\Theta} \cdot \mathrm{B}\Theta :: \Gamma\Delta \cdot \Theta\mathrm{N} : \Delta\underline{\Theta} \cdot \mathrm{B}\Theta,$$

the two underlined Θ should be Γ. She checks her handwritten notes: her proportion is correct. But the proportion turns out to be correct in the text that the students have. My own copy is a printout of an earlier version, and I have not noted there that I made the correction.

Meanwhile I ask Eve if she recognizes that her proposition is the converse of Sparkle's. She seems to do so, but I am not sure how well, since the positioning of the lines is different, as in Figure 4.7. (On the other hand, I have forgotten that, at the end of the proof, Pappus mentions its being a

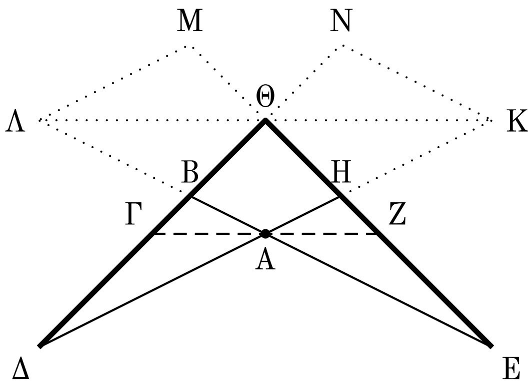


Figure 4.7. Lemma X

converse to something earlier, and in the translation I have named that something earlier as Lemma III.) In Lemma III, the two straight lines cut the three straight lines on the same side of their intersection point; in Lemma X, on opposite sides. As Eve writes on the board, I try to sketch an alternative diagram, as in Figure 4.8, next to hers, though this turns out to be a distraction. I talk about the diagram after the demonstration, though I do not fill in all of the auxiliary straight lines.

In Lemma X, the hypothesis is

$$\Theta\Delta \cdot \mathrm{B}\Gamma : \Delta\Gamma \cdot \mathrm{B}\Theta :: \Theta\mathrm{H} \cdot \mathrm{ZE} : \Theta\mathrm{E} \cdot \mathrm{ZH}. \tag{4.4}$$

Pappus makes KΛ parallel to ΓA, and then extends AB and AΔ to meet KΛ at two points, which he writes as K and Λ, though today we might say this is the wrong order. Then we ensure ΛM ∥ AΔ and KN ∥ AB, with EΘ extended to M, and ΔΘ to N. Now we repeat part of the proof of in Lemma III, at least in the configuration of Figure 4.9, with two of the three concurrent straight lines interchanged. That is, using only the

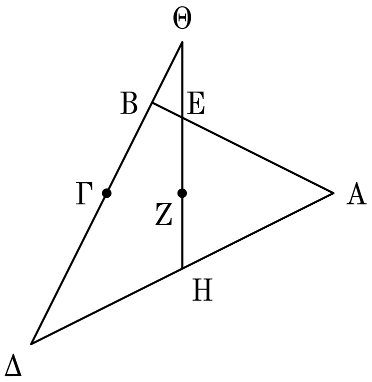


Figure 4.8. Lemma X: an alternative configuration

part of the diagram shown in Figure 4.10a, we have

$$\begin{aligned}\Delta\Theta : \Theta\mathrm{N} &:: \Delta\Gamma : \Gamma\mathrm{B},\\ \Delta\Theta \cdot \Gamma\mathrm{B} &= \Delta\Gamma \cdot \Theta\mathrm{N},\\ \Delta\Theta \cdot \mathrm{B}\Gamma : \Delta\Gamma \cdot \mathrm{B}\Theta &:: \Gamma\Delta \cdot \Theta\mathrm{N} : \Delta\Gamma \cdot \mathrm{B}\Theta\\ &:: \Theta\mathrm{N} : \Theta\mathrm{B}\\ &:: \mathrm{K}\Theta : \Theta\Lambda.\end{aligned}$$

Now we move to the part of the diagram shown in Figure 4.10b, where we have proportions corresponding to the last two:

$$\begin{aligned}\mathrm{K}\Theta : \Theta\Lambda &:: \mathrm{H}\Theta : \Theta\mathrm{M}\\ &:: \Theta\mathrm{H} \cdot \mathrm{ZE} : \Theta\mathrm{M} \cdot \mathrm{ZE}.\end{aligned}$$

In sum, we have shown

$$\Delta\Theta \cdot \mathrm{B}\Gamma : \Delta\Gamma \cdot \mathrm{B}\Theta :: \Theta\mathrm{H} \cdot \mathrm{ZE} : \Theta\mathrm{M} \cdot \mathrm{ZE}.$$

But the left member already appears in (4.4). Hence the right members of the two proportions are the same, that is,

$$\Theta\mathrm{H} \cdot \mathrm{ZE} : \Theta\mathrm{E} \cdot \mathrm{ZH} :: \Theta\mathrm{H} \cdot \mathrm{ZE} : \Theta\mathrm{M} \cdot \mathrm{ZE},$$

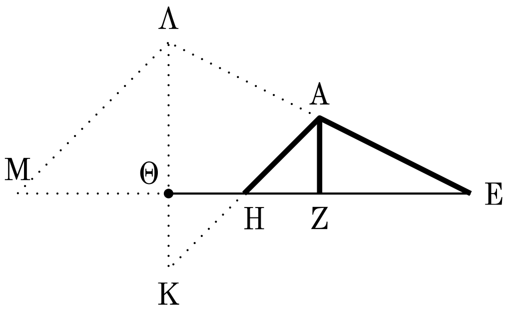


Figure 4.9. Lemma III reconfigured

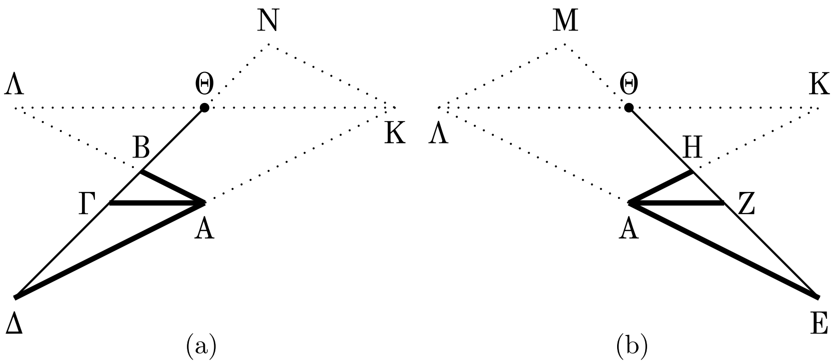


Figure 4.10. Lemma X: two halves of the proof

$$\Theta\mathrm{E} \cdot \mathrm{ZH} = \Theta\mathrm{M} \cdot \mathrm{ZE},$$
$$\Theta\mathrm{M} : \Theta\mathrm{E} :: \mathrm{HZ} : \mathrm{ZE}.$$

Thus what we did in Figure 4.10a, we have done in reverse in Figure 4.10b. It remains to draw the conclusion $\mathrm{AZ} \parallel \mathrm{K\Lambda}$, corresponding to the hypothesis $\mathrm{A\Gamma} \parallel \mathrm{K\Lambda}$. Pappus argues as follows (the bracketed proportions replaced with a reference to addition and alternation):

$$\left[\begin{array}{c}\Theta\mathrm{M} + \Theta\mathrm{E} : \Theta\mathrm{E} :: \mathrm{HZ} + \mathrm{ZE} : \mathrm{ZE}, \\ \mathrm{ME} : \Theta\mathrm{E} :: \mathrm{HE} : \mathrm{ZE},\end{array}\right]$$

$$\mathrm{ME} : \mathrm{EH} :: \Theta\mathrm{E} : \mathrm{EZ},$$

$$\Lambda\mathrm{E} : \mathrm{EA} :: \Theta\mathrm{E} : \mathrm{EZ},$$

and so $\mathrm{AZ} \parallel \mathrm{K}\Lambda$, which means $\Gamma\mathrm{AZ}$ must be straight.

Charity will proceed next week with Lemma XI, as planned. Following this, the assignments are
Lemma XII: Verity;
Lemma XIII: Lucky.

5. October 27

By email at 8 a.m., Lucky says he cannot come to class. At the beginning of class, only Charity is present. I figure it is better for me to present things to her—things that she might later present to the others—than for her to present her proposition to me.

So I start talking to Charity about perspective. But first I assign, as an **exercise,** to show

$$\frac{AC \cdot BD}{AD \cdot BC} = \frac{EG \cdot FH}{EH \cdot FG} \iff \frac{AB \cdot CD}{AD \cdot BC} = \frac{EF \cdot GH}{EH \cdot FG}, \tag{5.1}$$

given that $ABCD$ and $EFGH$ are straight, as in Figure 5.1. The others can copy this from the board when they come in. (See page 58.)

I proceed to draw something like Figures 4.2 and 4.3 again, though using Latin letters, and without committing to whether the bounding lines are parallel. Referring to Figure 4.3, I say that if ΑΒ ∥ ΕΖ, but ΒΓ and ΕΖ meet at Θ, then ΑΔ and ΓΖ must meet at a point Κ, and moreover ΘΚ ∥ ΑΒ.

To show why this follows, first I draw something like Figure 5.2. In the process, Verity and then Sparkle come in. The **projection** (*izdüşümü*) of A onto the horizontal plane is B; but C has no projection, or else the projection is the point at infinity. Thus if two straight lines in the (approximately) vertical plane meet at C, then their projections in horizontal plane must be parallel. I sketch this "in perspective," roughly as in Figure 5.3. (For the figure here, I use the `pst-3dplot`

Figure 5.1. Cross ratios

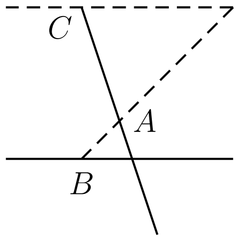


Figure 5.2. Perspective

extension to `pstricks`; but it does not provide the facility of showing automatically when one surface is behind another.)

So now in Figure 5.4, if AB and ED are parallel to the **horizon** (*ufuk*), and we place G *on* the horizon, then the projections of AB and ED onto the ground will be parallel, as will those of BC and FE. Then the projections of AF and CD must be parallel, by Lemma VIII, and so H must be on the horizon: that is, in the original diagram, $GH \parallel AB$.

People seem to agree with this argument, though Charity says it was not quite a proof. Indeed, it was not polished. I had not planned to give it.

The time is about 9:40. Eve has not shown up (and will not show up). Charity suggests that we should take a break before she presents **Lemma XI,** but others suggest that we just continue. Charity is very excited. She first gives a four-part outline. She will (1) state the theorem, (2) draw the diagram, (3) give the proof, then (4) consider the other case (which Pappus refers to at the end of his own proof; I *think* this was her outline). Charity speaks while she writes, and

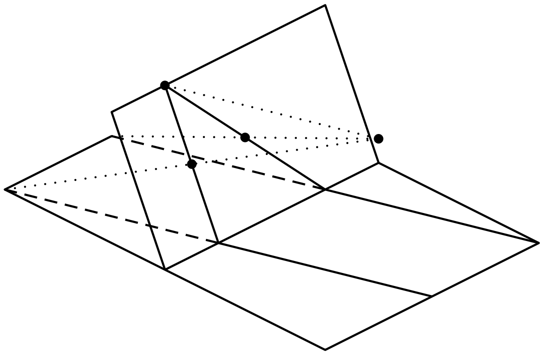


Figure 5.3. Parallel lines in perspective

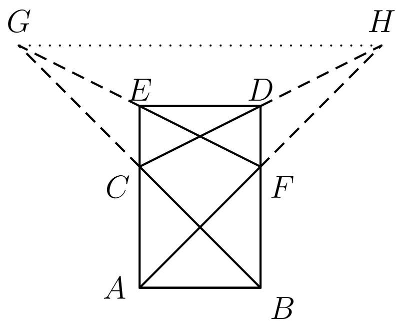
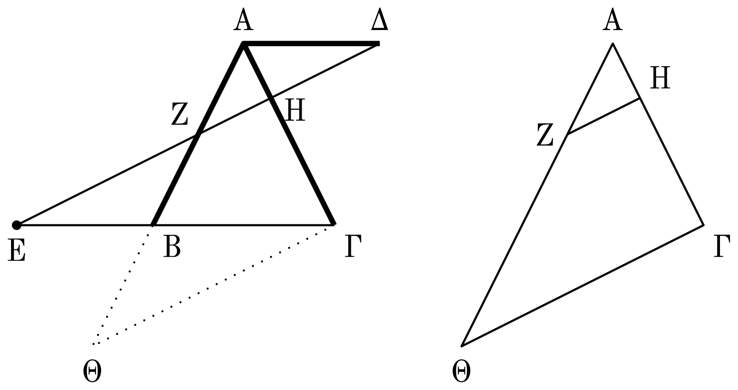


Figure 5.4. Pappus's Theorem in perspective

she looks at her classmates, demanding their attention and agreement. She tells us that she will replace Γ with C, though in the event she does not also replace Δ with D. She writes proportions as such, but also as fractions; and she *says* she is going to do this. Charity's is the most polished student presentation that I remember seeing so far.

Charity writes what she will prove as

$$\Delta\mathrm{E}\cdot\mathrm{ZH}:\mathrm{EZ}\cdot\mathrm{H}\Delta::\Gamma\mathrm{B}:\mathrm{BE}, \tag{5.2}$$



(a) Full diagram (b) First step

Figure 5.5. Lemma XI

as in the text, and then as something like

$$\frac{\Gamma B}{BE} = \frac{\Delta E \cdot ZH}{EZ \cdot H\Delta} = \frac{\Delta E}{EZ} \cdot \frac{ZH}{H\Delta}$$

The diagram is in Figure 5.5a. In establishing

$$\Gamma A : AH :: \Gamma\Theta : ZH, \tag{5.3}$$

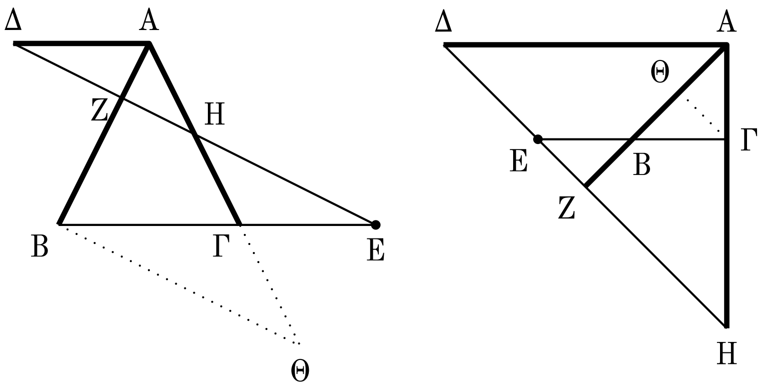
Charity draws a separate diagram, as in Figure 5.5b. She proceeds as Pappus does, though using fractional notation. Thus (5.3) becomes

$$\frac{\Gamma A}{AH} = \frac{\Gamma\Theta}{ZH}.$$

The next step is

$$\frac{\Gamma A}{AH} = \frac{E\Delta}{\Delta H},$$

which is explained with reference to the "butterfly" (*kelebek*), presumably the one whose wings are the triangles AΔH and



(a) Original case relabelled (b) Genuinely new case

Figure 5.6. Lemma XI variations

ΓEH. The two equations yield

$$\frac{\Gamma\Theta}{\mathrm{ZH}} = \frac{\mathrm{E}\Delta}{\Delta\mathrm{H}}, \qquad \Gamma\Theta \cdot \Delta\mathrm{H} = \mathrm{E}\Delta \cdot \mathrm{ZH},$$

and then

$$\frac{\Delta\mathrm{E} \cdot \mathrm{ZH}}{\Delta\mathrm{H} \cdot \mathrm{EZ}} = \frac{\Gamma\Theta \cdot \Delta\mathrm{H}}{\Delta\mathrm{H} \cdot \mathrm{EZ}} = \frac{\Gamma\Theta}{\mathrm{EZ}} = \frac{\Gamma\mathrm{B}}{\mathrm{BE}},$$

which is the desired result. Thus written, the reduction of the ratio of products becomes transparent, at least to the modern student. Charity gets Verity's confirmation of the reduction.

Pappus observes that AΔ can be drawn on the other side. Charity thinks the diagram is as in Figure 5.6a in this case. The correct proportion in this case would be

$$\Delta\mathrm{E} \cdot \mathrm{ZH} : \Delta\mathrm{Z} \cdot \mathrm{HE} :: \Gamma\mathrm{B} : \Gamma\mathrm{E}, \tag{5.4}$$

or else

$$\Delta\mathrm{E} \cdot \mathrm{ZH} : \Delta\mathrm{H} \cdot \mathrm{ZE} :: \Gamma\mathrm{B} : \mathrm{BE},$$

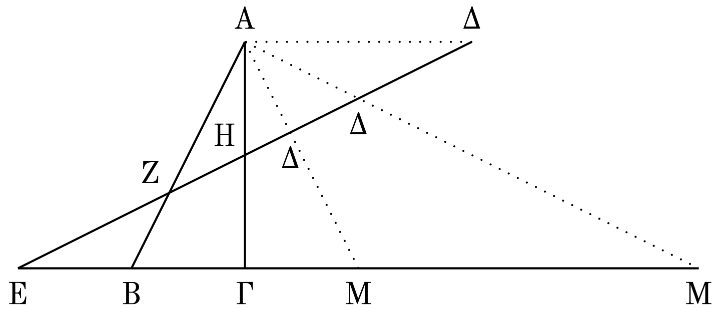


Figure 5.7. Lemma XI as limiting case of Lemma III

but I am not sure Charity gets this, though she discusses it with the others (mainly Verity, I think). Charity does recognize that her proposition is somehow a special case of Lemma III, proved by Sparkle; but I think this needs to be made clearer, so I go to the board.

First I observe that Charity's alternative diagram is just the reverse of the original diagram; so (5.4) holds immediately. But perhaps what is meant is a diagram as in Figure 5.6b, where Δ has moved to the other side, *but* E *has not.* This diagram is just as Figure 5.5a, as regards parallelism and straightness. The order of points on given straight lines may have changed, but the original proof never relied on this: it did not use addition or subtraction. Thus (5.2) should still hold.

To see Lemma XI as a special case (or a variant) of Lemma III, we may draw the diagram as in Figure 5.7. From Lemma III we know

$$\frac{\mathrm{EZ}}{\mathrm{ZH}} \cdot \frac{\mathrm{H\Delta}}{\mathrm{E\Delta}} = \frac{\mathrm{EB}}{\mathrm{B\Gamma}} \cdot \frac{\mathrm{\Gamma M}}{\mathrm{EM}}.$$

In the limit as M goes to infinity, ΓM/EM goes to unity, since

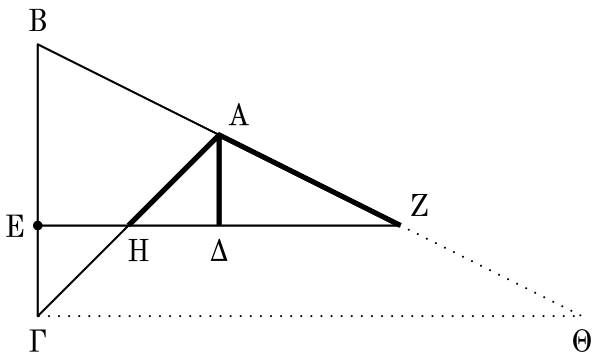


Figure 5.8. Lemma XI, third case

this ratio is $\Gamma\mathrm{M}/(\mathrm{E}\Gamma + \Gamma\mathrm{M})$, and

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Thus we obtain the result of Lemma XI. Charity says this is a real proof. I suggest that in analysis such proofs are real, because there is a precise definition of limit.

One may also observe, although I do not do it, that we nearly proved Lemma XI by establishing (4.3) on page 40 in proving Lemma III. More precisely, this gives us a third case of Lemma XI, as in Figure 5.8. Rewritten for this figure, (4.3) becomes

$$\Gamma\mathrm{E} : \mathrm{EB} :: \mathrm{HE} \cdot \mathrm{Z}\Delta : \mathrm{H}\Delta \cdot \mathrm{ZE},$$
$$\Gamma\mathrm{B} : \mathrm{EB} :: \mathrm{HE} \cdot \mathrm{Z}\Delta + \mathrm{H}\Delta \cdot \mathrm{ZE} : \mathrm{H}\Delta \cdot \mathrm{ZE};$$

and the sum reduces as follows:

$$\begin{aligned}&\mathrm{HE} \cdot \mathrm{Z}\Delta + \mathrm{H}\Delta \cdot \mathrm{ZE}\\ &= \mathrm{HE} \cdot \mathrm{Z}\Delta + \mathrm{H}\Delta \cdot \mathrm{EH} + \mathrm{H}\Delta \cdot \mathrm{H}\Delta + \mathrm{H}\Delta \cdot \Delta\mathrm{Z}\end{aligned}$$

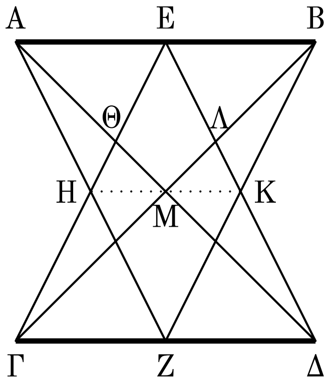


Figure 5.9. Lemma XII

$$= (\mathrm{EH} + \mathrm{H\Delta}) \cdot \mathrm{H\Delta} + (\mathrm{EH} + \mathrm{H\Delta}) \cdot \mathrm{\Delta Z}$$
$$= \mathrm{E\Delta} \cdot \mathrm{HZ}.$$

But the auxiliary triangle used for the proof of Lemma XI is different than for Lemma III.

I suggest taking a break now. It is about 10:20. Verity wants to start presenting **Lemma XII** though, so she can finish for her 11:00 class. But she wants to write things on the board before talking. I suggest she do this while the others take a break. She writes the enunciation only, along with the diagram, as in Figure 5.9. She writes "Lemma XII" in red, the enunciation in black, then "Proof" (*Kanıt*) in red again.

While waiting for the others, I confirm with Verity that she will use Charity's proposition. She does not seem to recognize that she will also use Lemma X. When she writes out the proof, indeed she makes the application of Lemma X as Pappus does, but without justifying it. Disappointingly, she seems to think it justifies itself, because it is in the text. She agrees that it is an application of Lemma X when I point this out.

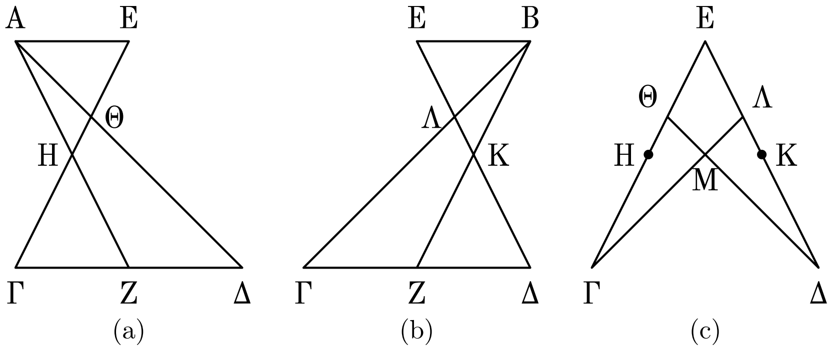


Figure 5.10. Steps of Lemma XII

This application is the third step of the proof, the first two steps each being an application of Lemma XI. When Verity recapitulates, she tries to clarify which triangles are used in each of the first two steps; but then she is confused about the cutting straight lines, though she has written the results correctly before. I draw supplementary diagrams next to her steps, as in Figure 5.10. Thus Pappus's argument is:

a) By Lemma XI,

$$\mathrm{\Delta Z : Z\Gamma :: \Gamma E \cdot H\Theta : \Gamma H \cdot \Theta E.} \tag{5.5}$$

b) By Lemma XI again, inversion, and (5.5),

$$\begin{aligned} \mathrm{\Gamma Z : Z\Delta} &:: \mathrm{\Delta E \cdot \Lambda K : \Delta K \cdot \Lambda E,} \\ \mathrm{\Delta Z : Z\Gamma} &:: \mathrm{\Delta K \cdot \Lambda E : \Delta E \cdot \Lambda K,} \\ \mathrm{\Gamma E \cdot H\Theta : \Gamma H \cdot \Theta E} &:: \mathrm{\Delta K \cdot \Lambda E : \Delta E \cdot K\Lambda.} \end{aligned}$$

c) By Lemma X then, HMK is straight.

At the end of class, I note that Lucky (if he comes next week) will prove Lemma XIII, which is Lemma XII in case

AB and ΓΔ intersect. I observe that we can also prove this by using perspective, and I ask Sparkle to do this for next week. She expressed concern that only one new proposition was expected; now she will present a second. She makes an appointment to meet me next Monday at 12:30. Now I learn that the students do not recognize *yarım,* "half," as an expression for half past twelve. Older people use it this way, and dictionaries give this meaning; but the students think I am saying *yarın,* "tomorrow."

I say that everybody should think about how to prove both lemmas, XII and XIII, using perspective.

6. November 3

Yesterday Sparkle visited my office as planned. But she did not know what she was supposed to prepare for class. She thought it had something to do with Lemma III, which she had already presented, and not Lemma XIII. She noticed, and became interested in, my cardboard parabola, situated with the corresponding axial triangle and base of a cone, as in Figure 6.1; so I tried to explain it:

The base of the axial triangle is a diameter of the base of the cone. A plane cuts the cone at right angles to that diameter, but parallel to a side of the axial triangle. Then the resulting chord of the base of the cone is bisected by the diameter. The square on one of the halves of the chord is equal to the product of the segments of the diameter (Sparkle seems to accept this readily). But one of those segments remains unchanged if we cut the cone by a new plane parallel to the old base. Thus if the segments of the diameter are a and y, while the half of the chord is x, then

$$x^2 = ay.$$

When I said I had taught this in Analytic Geometry (*Analitik Geometri*, MAT 104) last spring, Sparkle said she had not got much out of the course when she had taken it, because of the teacher that year (who is no longer in the department).

I had another cardboard model, as in Figure 6.2: a cardboard rectangle, bisected, scored, and folded along a straight line parallel to two sides, so that, after the folding, those sides remain parallel, but the halves of one of the other sides be-

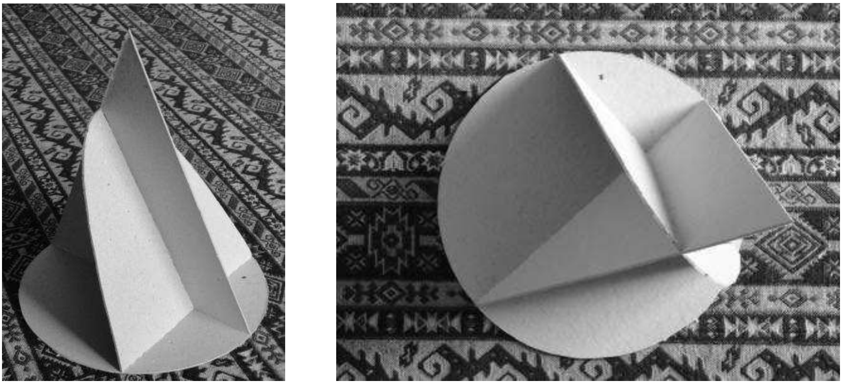


Figure 6.1. Parabola in cardboard

come distinct intersecting straight lines. Between those two halves, the figure of Lemma XIII is formed with thread, while between the parallel sides, a corresponding figure of Lemma XII is formed. From a point along the extension of the scored line, the two figures appear as one. The unfolded cardboard is as in Figure 6.3, where the path of the thread is as in Figure 6.4. I had not decided whether to show this to Sparkle and the rest of the class before Sparkle herself explained the depicted result. But since she came to me, not having understood what to do in the first place, I discussed the model with her. I was not sure she felt this left her with much to say in class. I also talked about how Lemma III had already proved a form of Lemma XI.

Sparkle does not show up for class. Maybe she is sick. Lucky has written an email to explain how he is still sick, taking antibiotics, and so on. As last week, so today, Eve does not make an appearance.

Charity and Verity do come. However, nobody is in class at 9 a.m. On the board, I start writing the **exercises** that I

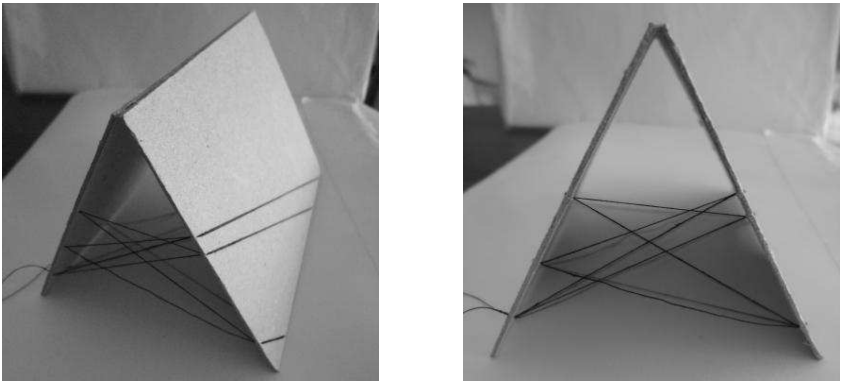


Figure 6.2. Pappus's Theorem in thread and cardboard

want to assign:

1. The remaining three cases of Lemma VIII, as in Figure 2.1 on page 29.
2. The fifth case of this lemma, where HB ∥ EZ (I assigned this before: see page 37).
3. The proof of the equivalence (5.1) on page 46.
4. The converse of the lemma embedded in the proof of Lemma IV, namely that in Figure 2.4 on page 33, if (2.3) on page 33 holds, then ΘHZ is straight.
5. The converse of Lemma IV itself.

Charity shows up at about 9:07. I have brought the cardboard model that I showed Sparkle, so I talk about this. I start proving a simple application of Pappus's Theorem (that is, Lemmas VIII, XII, and XIII) to the figure of Lemma IV. The result is in Coxeter [5, pp. 240–1]. First, referring to Figure 2.3 on page 32, now adapted as Figure 6.5, assuming that the solid lines that look straight *are* straight, I observe that what

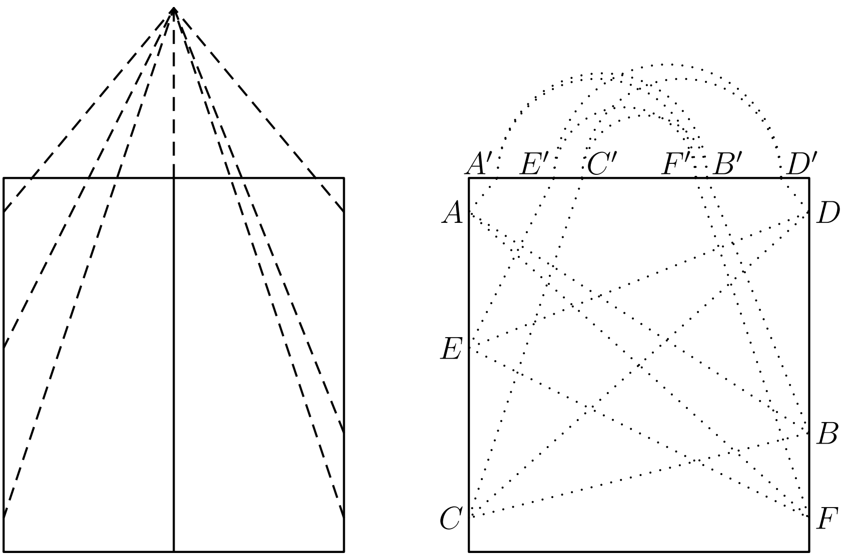


Figure 6.3. Plan for the cardboard Pappus's Theorem

we know from the converse of Lemma IV is

$$\frac{\mathrm{AZ}\cdot\mathrm{B\Gamma}}{\mathrm{AB}\cdot\mathrm{\Gamma Z}}=\frac{\mathrm{AZ}\cdot\mathrm{\Delta E}}{\mathrm{A\Delta}\cdot\mathrm{EZ}} \tag{6.1}$$

(which is (2.2) on page 31), and this is the sameness of the cross ratios of $(\mathrm{A},\mathrm{B},\Gamma,\mathrm{Z})$ and $(\mathrm{A},\Delta,\mathrm{E},\mathrm{Z})$. Here B, Γ, and Z are where three straight lines through Θ are cut; and Δ, E, and Z are where three straight lines through H are cut. This gives us five straight lines in all, since the straight line through Θ and H passes through Z. The other four straight lines, in different pairs, meet at K and Λ, and the straight line through these passes through A.

Verity shows up at some point during this.

For the application of Pappus's Theorem, we observe that, in Figure 6.5, HΘKΛ is a complete quadrangle, whose six sides

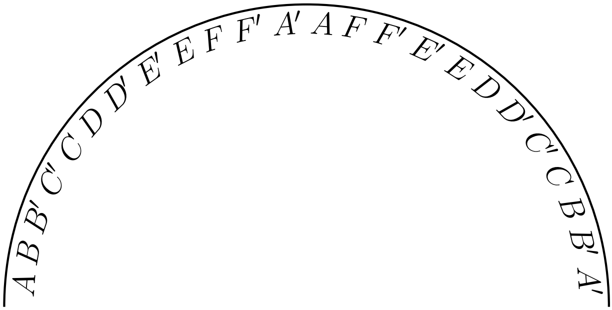


Figure 6.4. Path of thread for Pappus's Theorem

are cut by a straight line at Α, Β, Γ, Δ, Ε, and Ζ. These points then compose a **quadrangular set** (the term is in Coxeter). We shall obtain another complete quadrangle yielding the same quadrangular set. Let ΕΘ cut ΑΛ at Μ, and let ΖΘ cut ΚΒ at Ν. Consider the hexagon ΗΛΚΒΕΘ, whose vertices lie alternately on ΗΚ and ΛΘ, as in Figure 6.6. The pairs of opposite sides intersect at Δ, Μ, and Ν respectively, and so these lie on a straight line. The new complete quadrangle is thus ΘΚΜΝ.

I go on to derive Lemma III without auxiliary triangles. In Figure 6.7, where three straight lines through A are cut by a straight line through B at C, D, and E, and the straight line through B that is parallel to AD cuts AC and AE at F and G respectively,

$$\frac{BF}{BG} = \frac{BF}{BC} \cdot \frac{BC}{BG} = \frac{DA}{DC} \cdot \frac{BC}{BG} = \frac{DA}{BG} \cdot \frac{BC}{DC} = \frac{DE \cdot BC}{BE \cdot DC}.$$

I state Desargues's Theorem (see page 18), indicating that we shall use Pappus's Theorem to prove it. Then Desargues's Theorem will give us the Complete Quadrangle Theorem. It will also allow us to make the Fundamental Theorem of Pro-

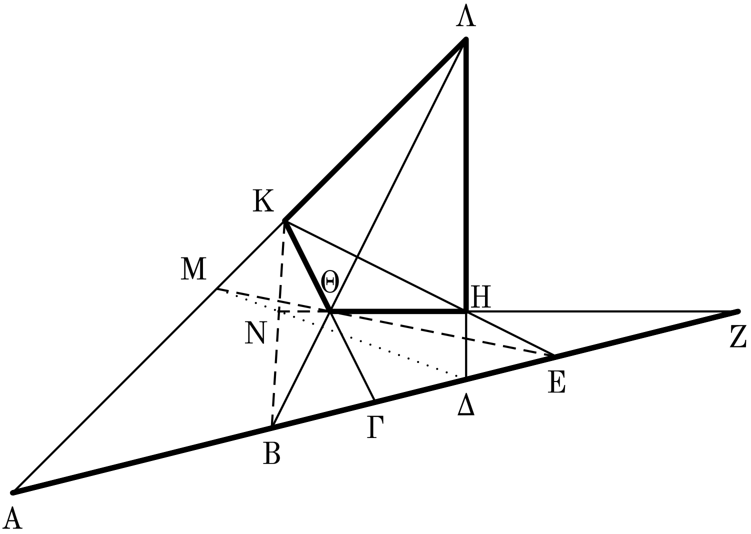


Figure 6.5. Pappus's Theorem applied to Lemma IV

portion (page 1) true by definition.

Desargues was a contemporary of Descartes, but his geometry did not catch on as well as Descartes's algebra. It is possible to make projective geometry algebraic. The Euclidean plane can be modelled by $\mathbb{R} \times \mathbb{R}$, so that points correspond to ordered pairs (x, y). How do we add the points at infinity?

There is one point at infinity for each family of parallel lines. Thus there is a point at infinity for each *direction* (*yön*) in the plane. Also the ordinary points of the plane can correspond to directions: if we take a point outside the plane, then a point in the plane corresponds to the direction of the straight line through that point and the point outside. Thus the points of the projective plane can be understood as straight lines through a point in space.

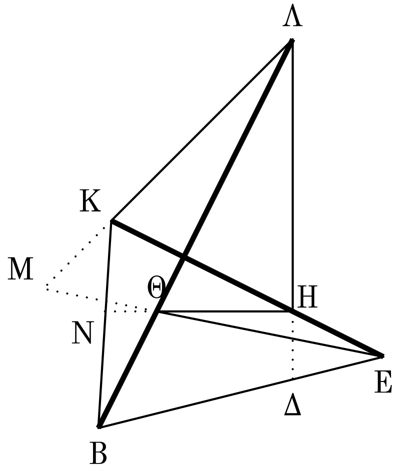


Figure 6.6. Pappus's Theorem applied to Lemma IV

If we take a sphere with that point as center, then each point of the projective plane becomes a pair of antipodal points on the sphere. If we take just half of the sphere, then the ordinary points of the projective plane correspond to points of the interior of the hemisphere; but points at infinity correspond to pairs of opposite points on the bounding circle of the hemisphere.

I have presented all of this informally. That is, it was unprepared. Evidently I was counting on Lucky and Sparkle to show up.

I have brought a printout of Lobachevski (the 1891 translation of Halsted appended to the Bonola book [3]). I want to see how well the students can handle the English. Charity seems more confident than Verity; but they both agree with my suggestion that I can discuss propositions ahead of time with the students who will present them.

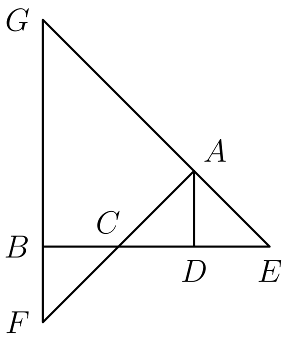


Figure 6.7. Lemma III with no new triangle

They have asked about examinations. I say I do not want to give a midterm exam, though I can do it. If I am not going to give an exam, students must come to class and give presentations. This is the preference of Verity and Charity. They suggest that the students who do not show up should just fail.

I make assignments from the exercises above (page 58): Charity is to do 1; Verity, 3.

7. November 10

I wake up early this morning as usual (around three or four o'clock) and prepare three pages of notes, covering respectively

(1) Desargues's Theorem,
(2) the dual of Pappus's Theorem,
(3) the Complete Quadrilateral Theorem.

But I feel increasingly ill, and eventually I go back to bed. I ask Ayşe to photograph the first page of my notes and send it to the students.[1] (First I consider sending all three pages.) The students are asked to work through the proof of Desargues's Theorem and report back to me. I stay home.

My diagram is labelled as in Figure 7.1 (which is just Figure 1.11a from page 26, relabelled); but the relative slopes of some of the original straight lines are different, and so the constructed straight line $GPQM$ in my notes lies at the top, not at the left. The text of my notes reads as follows (translated into English; the original is in Figure 7.4).

Geometries, 2015.11.10
Pappus, ~300
Desargues (Girard) 1591–1661
Descartes (Rene) 1596–1650

[1] Ayşe's mobile seems the most convenient device with the desired capacity. It may be possible to send photographs with my own mobile, but I have not figured out how to do it. I can however tether the mobile to our laptop computer, and in this way I can in principle send photographs. We currently have no other internet connection at home, besides our mobiles.

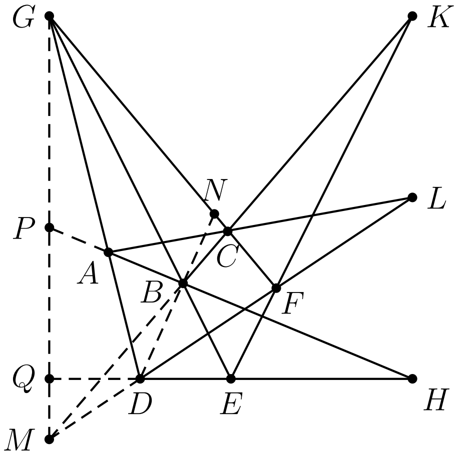


Figure 7.1. Desargues's Theorem, 2015.11.10

Desargues's Theorem.
Let AD, BE, and CF intersect at G.
Let intersect

- AB and DE at H,
- BC and EF at K,
- AC and DF at L.

We shall show that HKL is straight.
Let BC and DF intersect at M.
In the hexagon $ACGMDB$, AGD ve CMB are straight, and

- AC and MD intersect at L.
- Let CG and DB intersect at N;
- let GM and BA intersect at P.

Then by Pappus's Theorem, LNP is straight.
In the hexagon $BDEFGM$, BEG ve DFM are straight, and

- BD and FG intersect at N.
- Let DE and GM intersect at Q.
- EF and MB intersect at K.

Then NQK is straight.

> In the hexagon $BMDQNP$, BDN and MQP are straight. Because BM and QN intersect at K (because NQK is straight); MD and NP, at L (because LNP is straight); DQ and PB, at H: KLH is straight.

Lucky sends me a report with four photographs of writing on the whiteboards, along with a list of the four students in attendence, each of whom, he says, has done some of the writing: Verity, Sparkle, Charity, and Lucky. The photographs, cropped by me, are in Figures 7.2 and 7.3. Apparently the students have first copied out my notes exactly, even down to the sloppy diagram (but not including the names and dates of Pappus, Desargues, and Descartes). But then the students have separated out the three hexagons, sketching each one separately, and emphasizing the straight lines (which I had not drawn) through the points of intersection of the pairs of opposite sides.

Eve sends me an email saying she was in class too. I write to Lucky that he has forgot Eve, according to her. He replies that he listed everybody in attendence.

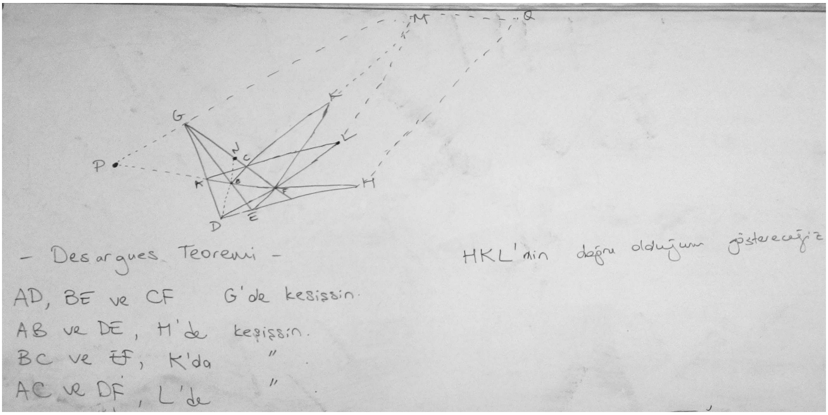


Figure 7.2. Students' notes: first board

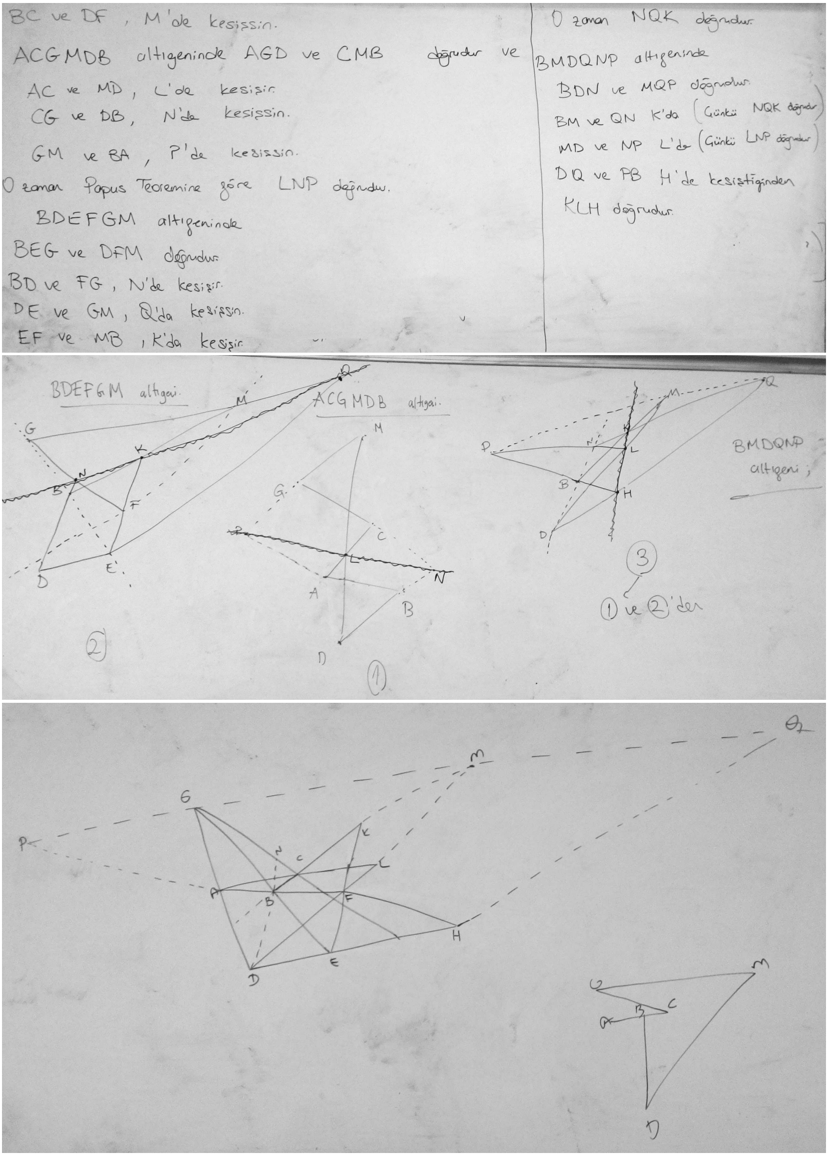


Figure 7.3. Students' notes: second, third, and fourth boards

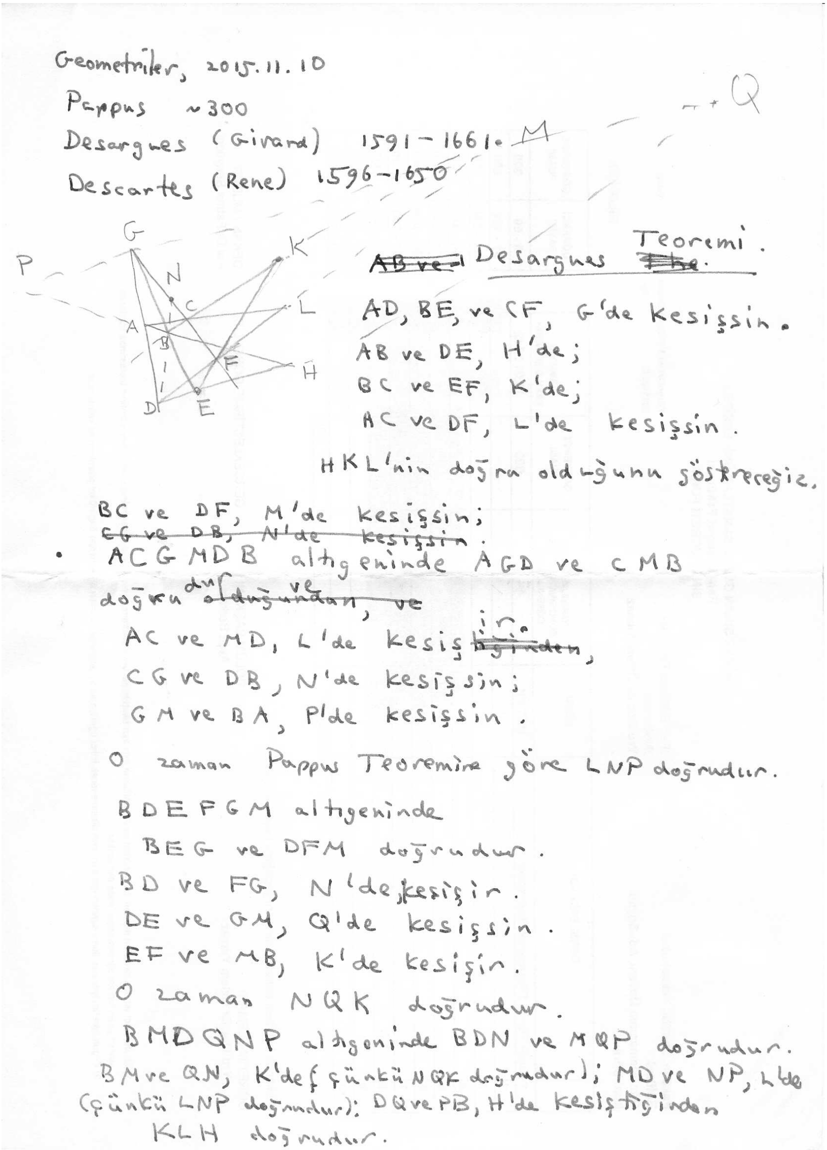


Figure 7.4. My lecture notes

8. November 17

Last Friday, a student found me who was registered for the course, but who had never come to class. I shall call him Hapless. He wanted to be given some special work to do so that he could pass the course. I pointed out that the semester was half over. I could not say that it was impossible to pass the course at this point. A diligent student could do it. However, having known Hapless from Euclid class, I was pretty sure he was not that student.

Where had Hapless been all semester? I did not find out; but students do sign up for courses that they do not intend to work on, because there is no penalty for failing. They might somehow be able to pass, and so they take the chance. Apparently Hapless hoped this would be the case for my course. He could not tell me what we were reading. He had not read the course webpage. I told him to do this, and then talk to me.

I have not seen him since. He does not come to class today. (Nor will he ever.)

Nobody is present at the beginning of class today. Charity comes at maybe seven minutes past nine. Others trickle in over the next half-hour. Eve does not come though: Sparkle says she is seeing the dentist.

The students agree that Eve did not actually come last week. Later, Eve will send me an apologetic email for the "misunderstanding."

I draw the figure for Desargues's Theorem and try to show how its converse is its dual. In Figure 8.1, If the minuscule let-

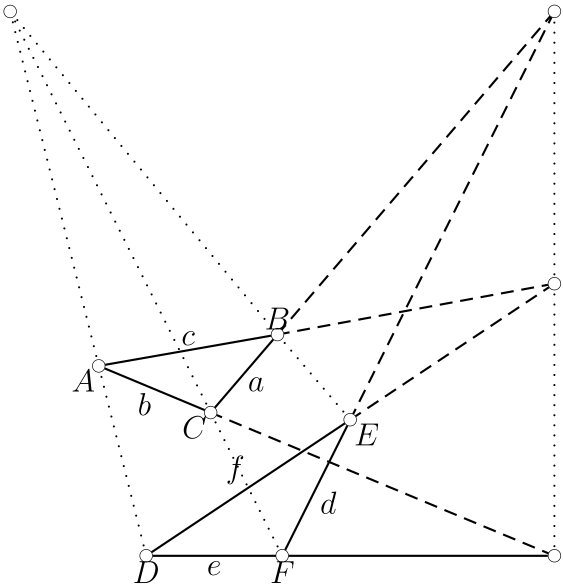


Figure 8.1. Desargues's Theorem and its converse

ters denote straight lines, and two such letters together denote the intersection point of those straight lines, then Desargues's Theorem is that if AD, BE, and CF meet a common point, then ad, be, and cf meet a common straight line. We get the converse by interchanging points and straight lines. This means the converse is the **dual** (*dual*) of the original theorem.

We have been using two axioms:

1. Any two straight lines meet exactly one common point.
2. Any two points meet exactly one common straight line.

Each of these is dual to the other. In class I do not actually come up with a verb like "meet" here to describe both what a straight line can do to a point and a point to a straight line. I mostly talk out loud about duals. The students seem to get

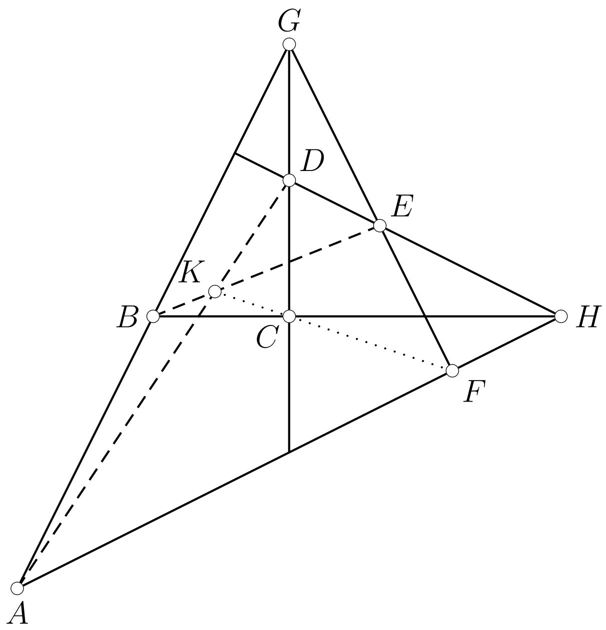


Figure 8.2. The dual of Pappus's Theorem

the idea.

I prove the dual of Pappus's Theorem. The Theorem itself is that if the vertices of a hexagon alternately meet two straight lines, then the points met by the pairs of opposite sides meet a common straight line. The dual then is that if the sides of a hexagon alternately meet two points, then the straight lines met by pairs of opposite vertices meet a common point. So, in the hexagon $ABCDEF$, let AB, CD, and EF intersect at G, and let BC, DE, and FA intersect at H, as in Figure 8.2. If the diagonals AD and BE meet at K, then the diagonal CF also passes through K. For we can apply Pappus's Theorem itself to the hexagon $ADGEBH$, since AGB and DEH are straight. Since AD and EB intersect at K, and DG and BH

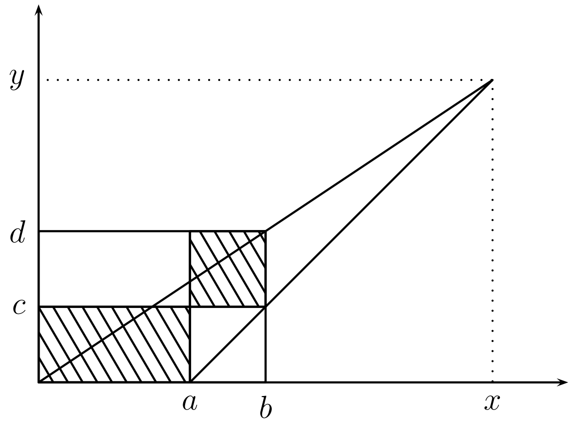


Figure 8.3. The dual of Pappus's Theorem, analytically

at C, and GE and HA at F, it follows that KCF is straight.

In case the points G and H are at infinity, we can work out the dual of Pappus's Theorem analytically. In Figure 8.3, the hexagon is hatched, though perhaps not so in class. Given that $(0,0)$, (b,d) and (x,y) are collinear, and $(a,0)$, (b,c), and (x,y) are collinear, we want to show $(0,c)$, (a,d), and (x,y) are collinear. That is, we want to show

$$\frac{d}{b}=\frac{y}{x}\ \&\ \frac{c}{b-a}=\frac{y}{x-a} \implies \frac{d-c}{a}=\frac{y-c}{x}.$$

I leave this as an **exercise.** (Maybe I do all of this at the end of class.)

I argue that the converse of Desargues's Theorem must now by true, since it follows from our two axioms and Pappus's Theorem, and the duals of these are also true, and the dual of Desargues's Theorem is its converse.

I now use that Theorem and its converse to prove the Complete Quadrilateral Theorem. In Figure 8.4, by the converse

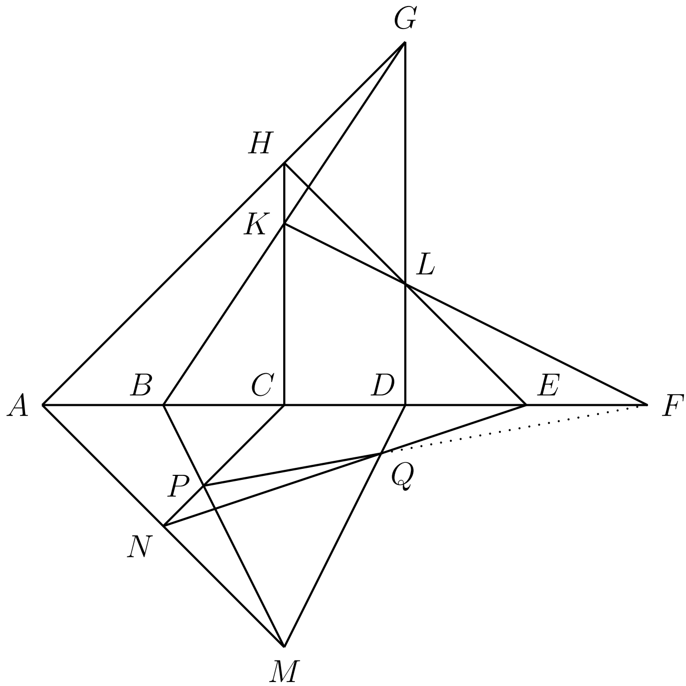


Figure 8.4. The Complete Quadrangle Theorem

of Desargues's Theorem applied to triangles GHL and MNQ, since

- GH and MN meet at A,
- GL and MQ meet at D, and
- HL and NQ meet at E,

and ADE is straight, it follows that GM, HN, and LQ intersect at a common point R (not drawn). Likewise, in triangles GHK and MNP, since

- GH and MN meet at A,
- GK and MP meet at B, and
- HK and NP meet at C,

and ABC is straight, it follows that KP passes through the intersection point of GM and HN, which is R. So now we know that HN, KP, and LQ intersect at R. Therefore, by Desargues's Theorem, the respective sides of triangles HKL and NPQ intersect along a straight line. But HK and NP intersect at C, and HL and NQ intersect at E; and KL intersects CE at F; therefore PQ must also intersect CE at F.

I use for the argument the third page of my notes prepared for last week, but it turns out to be in error. It starts by considering GKL and MPQ, but this does not work. Either I worked through the argument then by just following the letters, and not the diagram; or else I was copying from Coxeter, and trying (but failing) to change his letters to mine.

The break occurs at some point. During the break, I print out four copies of the Lobachevski. The original text is 47 pages; these are printed two to a side on 12 sheets. I show the students how I drill three holes through the sheets, thread the holes, tie the thread, then fold the sheets to make a booklet. I have forgotten to bring padding to put below the sheets being drilled. I just use my own booklet, and am careful not to drill into it, at least not too far. I have a little battery-powered drill, with the thinnest bit I could find.

Three booklets remain to be made. Lucky takes up the drill; Verity, the needle and thread. Lucky holds sheets across the gap between the two tables on the dais. This method does not appear to work very well, but I do not interfere. Lucky also folds the sheets *before* drilling, whereas I folded only *one* sheet, as a guide, before drilling. I think it is better do drill and sew *before* folding all of the sheets; but I leave the students to find their own way. Verity starts sewing one booklet from the wrong side, and she thinks she has to cut the thread and start over. Here perhaps I do interfere, but Verity has asked my advice. I pull the thread free, so that she can use it again.

When I entered graduate school in 1989 and became one of three teaching assistants for a calculus lecture, I was surprised by the physical work that we had to do. Exam solutions were written on four sheets, which we assistants had to clip together. After the exam, we had to separate the pages, so that the lecturer and assistants could each take home and grade the same page from all of the students in the lecture. Finally, we reassembled the sheets for return to the students.

In the previous year, working at a farm, I had been doing repetitive physical labor, like pulling weeds or picking cucumbers. Now, in graduate school in mathematics, I still had to contend with repetitive physical labor. It was something of a shock, until I accepted that there was no work that used the mind alone. So I am pleased that in my Geometries class, I have given the students some small experience of the purely bodily effort that goes into what they read.

Lobachevski begins his treatise with 15 propositions, stated without proof. In class, we read some of these together. The students seem to handle the English fine, especially Sparkle; the others, at least, do not complain. I make assignments:

17 Sparkle,
18 Verity,
19 Lucky,
20 Charity.

(I think this is right; but next week Sparkle will have worked on 18, not 17.) Charity seems relieved to think that we shall not get to her proposition next week, because she has other work to do.

After class, Lucky points out that he never got to present Lemma XIII from Pappus, and he offers to present more of the Lobachevski instead. I suggest that he explain Theorem 16.

Part II.

Hyperbolic Geometry

9. November 24

Sparkle shows up a few minutes past nine. A little later, Charity comes. Apparently Lucky sends me an email around this time, but I do not see it till later: there is an accident on the road, and his bus is stuck in traffic. When I see him the next day, he will say he came about half an hour late to the department, and he did not want to interrupt the class. This makes no sense: neither he nor anybody else has ever been reluctant to enter late before.

In class, Sparkle and Charity teach me a new word: *suistimal.* This means abuse,[1] and they say the other students are abusing my good nature. I point out that I have wanted to use class participation in lieu of exams; but it seems this will not work.

Meanwhile, Sparkle has studied **Theorem 18,** which Lobachevski enunciates as,

Two lines are always mutually parallel.

Sparkle has written out the English text with space between the lines to make a Turkish translation. She makes remarks about *devrik cümle.* This means "inverted sentence," and I know the term from Geoffrey Lewis. In his *Turkish Grammar* [9, XV 2, p. 239], he describes the *devrik cümle* school of Turkish writers, who feel free to play around with Turkish word order, given that peasants (from the point of view of the urban elite) do the same thing, and inflections still make the

[1] See Appendix E.3, page 149.

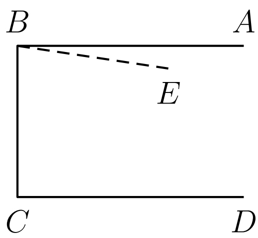


Figure 9.1. Parallel lines without Euclid's Fifth Postulate

syntax clear. From the point of view of Turkish, English sentences are inverted, with qualifying phrases coming *after* the words qualified, and not before. It has been Sparkle's challenge to come to terms with this feature of the long sentences of Lobachevski (in English translation).

Sparkle has not understood the *point* of her theorem, because she has not understood that parallelism has a new meaning. She has not understood that we are doing a new geometry now. Well, perhaps I have not made this crystal clear. I left the students to read the Lobachevsky; but this may be harder for them than for me to read Turkish.

I give the account of **Theorem 16** that Lucky was supposed to give. In Figure 9.1, if the angles ABC and BCD are right, then BA and CD do not meet, because if they did, a triangle would be formed in which two angles are together equal to two right angles. This is impossible, by Euclid's Proposition I.17. This proposition follows from I.16, that an exterior angle of a triangle is greater than either of the opposite interior angles. I repeat the proof, since the students do not well remember Euclid from three years ago. I do note Lobachevski's Theorem 7: "Two straight lines cannot intersect, if a third cuts them at the same angle." (Lobachevski's propositions are labelled only with numbers; in class I generally call them propositions, *önermeler*; but apparently he or his translator refers to them as theorems.)

I repeat the Fifth Postulate: that if the angles ABC and BCD were together less than two right angles, then BA and CD would intersect when extended. We are now assuming that this fails, so that some lines like BE also do not meet CD when extended. There is a boundary line between the lines through B that meet CD on the side of D and those that do not. It is the boundary line that is called **parallel** to CD.

Lobachevski just assumes that such a boundary line exists. The students accept that it exists, and I do not question this; it is not the most important issue now. Right now, we have to observe that the definition of parallelism is not symmetric. If BE is parallel to CD, it is not clear whether CD is parallel to BE.

Sparkle goes to the board to present her proposition, but she cannot present it cleanly. I need to help with a lot of the translation. At the end I point out that she was supposed to get my help *before* class. If there had been four other students in class, what were they going to do during her presentation? As it is, Charity does get involved in the work of understanding the proposition.

In Figure 9.2, angle ACD is right, and through A, AB is drawn parallel to CD.

Sparkle is translating "line" as *çizgi,* which is correct for Euclid; indeed, it is better than "line" as a translation for Euclid's $\gamma\rho\alpha\mu\mu\acute{\eta}$, since this and *çizgi* both mean something scratched, while a line is something drawn or stretched. In Lobachevski, "line" means straight line: *doğru çizgi,* or simply *doğru.*

We draw any line CE in the right angle ACD, and we want to show that it meets AB. Drop to it the perpendicular AF. In

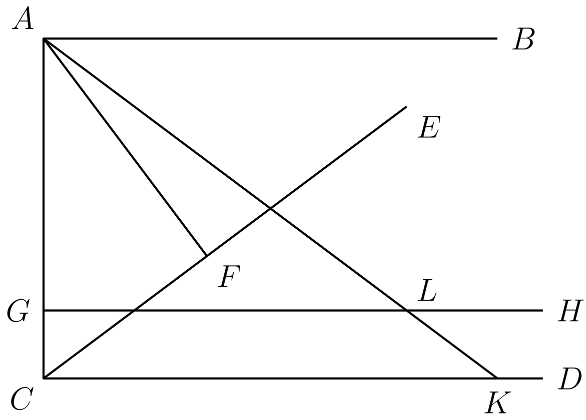


Figure 9.2. Theorem 18

the right triangle ACF, the angle ACF is acute, so $AF < AC$ by Euclid I.19, or Lobachevski's Theorem 9. So we can find on AF the point G such that $AG = AF$.

Now Lobachevski "slides" $EFAB$ so that it becomes $HGAK$. The point is that angle BAK is made equal to angle FAC, so AK may be assumed to cut CD at K, by Theorem 16, that is, the definition of parallelism. Also GH is perpendicular to AC, so it does not cut CD, by Theorem 4; and therefore it must cut GH at a point L, by Theorem 3.

As Lobachevski says now, AL must be the distance along AB from A where CE cuts AB. So, on the assumption that AB is parallel to CD, also CD must be parallel to AB.

Charity says she did not prepare 20, because she was studying for an exam.

I observe that I want to cover Lobachevski's Theorems

- 1–24,
- the part of 25 (straight lines parallel to a third are parallel to one another) taking place in one plane,

- 29–33,
- 36.

These are the propositions about the plane, not space. We could cover them all if the students were diligent, but I doubt they will be.

Sparkle and Charity have asked to leave early to register for “formation” (*formasyon*), the courses they need to take to qualify to be teachers. But as class nears the end, they prefer to sit and chat with me. They say this explicitly when I suggest that they can leave. I ask where they live, and how they get to the department. Sparkle is in Kağıthane and rides a single bus to come, but it takes an hour and a half. A private car would take twenty minutes, maybe half an hour. I suggest that walking might be an option: one can walk a long way in an hour and a half. Yes, but there are hills, it is pointed out.

Charity lives in a dormitory near by. Her family are from Trabzon, but they live in Zonguldak now.

We talk about some cultural attractions in Istanbul, such as Santralistanbul in Kağıthane. I mention having walked to Piyale Paşa Camii, which I think is in the direction of Kağıthane. It is not, but it was built by Mimar Sinan, and a tour of the Mimar Sinan creations throughout the city can be a worthwhile activity. The students have mentioned the high entrance fees of Ayasofya and Topkapı (or of one of these, at least); I point out that the mosques are free, as is Istanbul Modern, to us (at least it is free to Mimar Sinan *teachers*; I cannot affirm categorically that is free to students as well, though I shall learn later that it is). When I mention old churches, the students mention those along İstiklâl Caddesi. I explain that I mean *Byzantine* churches, like what is now Kalenderhane Camii, which you see when you exit the Vezneciler metro station.

10. December 1

Verity saw me on Thursday and made some excuse for not having been in class. Today, everybody will come to class, eventually. First it is only Charity as usual. I ask her if she knows anything about Alp Arslan, whose chivalrous treatment of Emperor Romanus IV Diogenes, after the Battle of Manzikert in 1071, I have been reading about in Michael Attaleiates [1]. Looking at the Attaleiates book, with the original Greek facing the English translation, Charity says her family used to know *Rumca,* which I understand to mean Greek as spoken in Turkey.

When Sparkle comes, she cannot explain what she proved last week. Neither she nor Charity can say what the new concept of parallelism is. So I go over it again. As I am doing this, Verity arrives. She cannot present Theorem 17. There seems to be confusion about what she is supposed to do. When Eve comes, she is eager to present Theorem 21. I am about to invite her to do so; but meanwhile, Lucky has come, and I suggest that he present 19. He says something about 16 too, but I think it is that he can present this *after* 19. This does not make any sense at the moment, so I tell him just to do **Theorem 19.**

Now it becomes clearer why he may have wanted to change the order of the propositions. He asks me if he should write the English of 19 on the board, or just the Turkish. I say just the Turkish. He proceeds to start translating the English. But he does not seem to understand what it means.

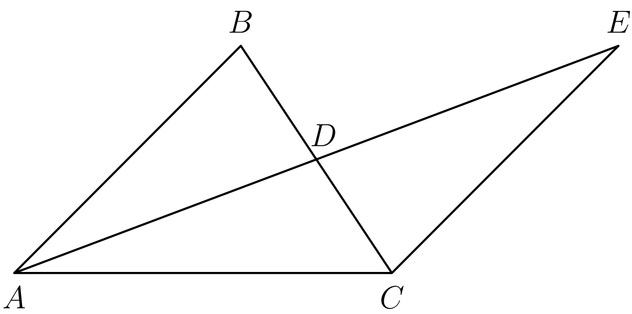


Figure 10.1. Theorem 19

The proposition is that the angles of a triangle add up to no more than two right angles. I ask the class: Don't we already know this from Euclid? Don't we know that the angles of a triangle are *equal* to two right angles? Yes, we do, they say. But I observe that Euclid's proofs require the Fifth Postulate. (Writing later, I shall not be able to remember whether I review the meaning of the Fifth Postulate now, or I already did so in talking about parallelism earlier.)

Lucky proceeds to write out the assumption that the angles of a triangle ABC add up to $\pi + \alpha$. He does not seem to understand that he is beginning a proof by contradiction. In fact he does not seem to understand anything at all. For, he asks me what "halve it [namely BC] in D" means, and likewise for "prolongation" and "congruent." Or perhaps he is only wondering how to say things in Turkish. ("To halve" is *ikiye bölmek.* "To prolong" is *uzatmak,* though I have no ready translation for the noun "prolongation." "Congruent" is *çakışan,* though in Euclid, for bounded straight lines and angles at least, it is simply *eşit,* equal.)

In Figure 10.1, we bisect BC at D, we prolong AD to E so that $DE = AD$, and we draw CE. The vertical angles ADB

and EDC are equal.

At some point I explain that a "vertex" can be *çokgenin köşesi* (the "corner" of a polygon) or *koninin tepesi* (the "peak" of a cone). In Latin it can mean *kafa* I say, knocking on the crown of my head; so "vertical angles" are *kafa kafaya* I say, knocking my fists together. Sparkle says she will never forget the meaning of "vertical" now. (Vertical angles in Turkish are *ters açılar,* "opposite" angles.)

Vertical angles are equal, by Theorem 6. The triangles ABD and ECD are now congruent, by Side Angle Side, which is part of Theorem 10. Thus

$$\angle BAD = \angle AEC, \qquad \angle ABC = \angle DCE,$$

and so the sum of the angles of triangle ACE is just the sum of the angles of triangle ABC. I think this is clear from the diagram; but Lucky is not trying to explain the claim in terms of the diagram. In the diagram at hand—Lucky's second diagram, the first being on the other board, now raised overhead—the straight line AC is not even drawn. Lucky follows Lobachevski in saying immediately that the sum of the angles of triangle ACE is $\pi + \alpha$; there is no recollection that this is only because $\pi + \alpha$ is the sum of the angles of triangle ABC.

I ask for clarification. Verity speaks, and I invite her to explain at the board. She gestures at and talks about whole triangles; but I say we are concerned with *angles.* Ultimately she makes a labelling as in Figure 10.2, and she writes something like

$$a + x + b + y = \pi + \alpha,$$
$$x + y + b + a = \pi + \alpha.$$

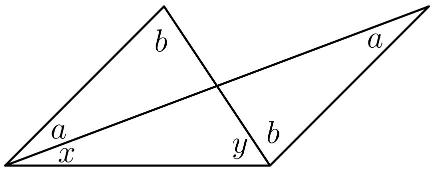


Figure 10.2. Theorem 19 with angles labelled

She writes α as a, and indeed the α in Lobachevski looks like our a; but Lobachevski's symbol is really an alpha, because his Latin letters are always roman, that is, upright, so that the first minuscule of the alphabet is not a, but a. My greater concern is that the second equation is correct only because its left member is equal to the left member of the first equation, but Verity has not made this clear.

Lucky says he thought we had to do everything in the style of Euclid, without equations like this. Verity too has said something like this, as being the reason she did not immediately write the equations. I say any method can be used, if it is correct.

Lucky does not understand how the proposition continues. I just go to the board to lecture on this. BC was chosen as the shortest side of triangle ABC, so the angle at A must be the smallest, by Theorem 9 (which Lucky has cited; but then so does Lobachevski himself). Call this angle β. Then the less—call it γ—of the angles EAC and AEC is no greater than half of β:

$$\gamma \leqslant \frac{\beta}{2}.$$

If we do to triangle ACE what we did to ABC, we get a triangle with an angle δ such that

$$\delta \leqslant \frac{\gamma}{2} \leqslant \frac{\beta}{4}.$$

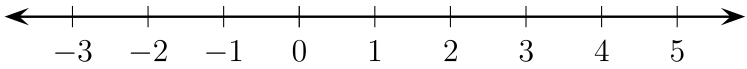


Figure 10.3. The real number line

At each step, we get a triangle whose angles add up to $\pi + \alpha$, but one of the angles has the upper bound $\beta/2^n$. If n is large enough, then

$$\frac{\beta}{2^n} < \alpha,$$

which means in the *next* triangle, *two* angles add up to less than α, and so one angle is greater than π, which is absurd.

Why can we make n large enough? Well, do the students know the Archimedean property of the real numbers? Nobody admits to it, even though the students confirm that they have taken all four semesters of analysis that we require. I explain. Every for every real number, there is a greater rational number, even a greater natural number. On the real number line as in Figure 10.3, the integers are unbounded. In logical jargon,

$$\forall\alpha\ \big(\alpha \in \mathbb{R} \Rightarrow \exists n\ (n \in \mathbb{N} \land \alpha < n)\big).$$

(I do not worry about taking absolute values.)

In Theorem 19, we want $2^n > \beta/\alpha$; we achieve this by letting $n > \beta/\alpha$. I do not talk about the assumption that angles (or rather their measurements) are real numbers.

The time is about 10:10. We take a break. Charity still cannot present Theorem 20, because she has *another* exam. Last week's exam was the ALES: *Akademik Personal ve Lisansüsü Eğitimi Giriş Sınavı* ("Academic Personel and Graduate Education Entrance Examination"), apparently a Turkish GRE. It is a big deal, Charity says. Her exam this week is for some other course in the department.

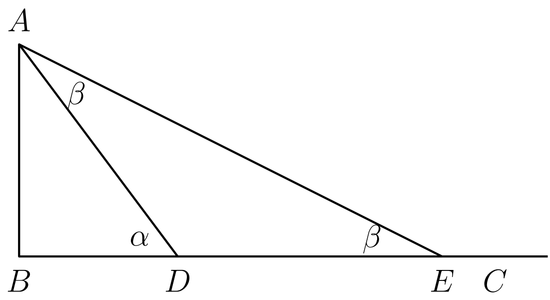


Figure 10.4. Theorem 21

Eve presents **Theorem 21** in a more polished style than Lucky's; but then the text is about half the length of that of 19. Also Eve may know English better than Lucky; on Facebook she claims to know French as well. In any case, she goes through Lobachevski's construction. In Figure 10.4, the angle at B is right, and $DE = AD$. Then the angles DEA and DAE are equal, and therefore each is either half of angle BDA, or less. But Eve cannot explain clearly why. She seems to know that, in the added labelling of Figure 10.4, $\alpha \geqslant 2\beta$; but this may be only because there is a proposition in Euclid that the exterior angle is equal to the sum of the opposite interior angles. Again, we no longer have all of Euclid, but as Verity explains, we have that angle ADE is $\pi - \alpha$, and so, by Theorem 19 (Lobachevski cites 20, as well as 8 for the equality of angles DAE and DEA),

$$\pi - \alpha + 2\beta \leqslant \pi,$$
$$\beta \leqslant \frac{\alpha}{2}.$$

What next? Again Eve seems to have missed the point. It is true that Lobachevski is imprecise. After finding that the angle AED is "either $\frac{1}{2}\alpha$ or less," he says,

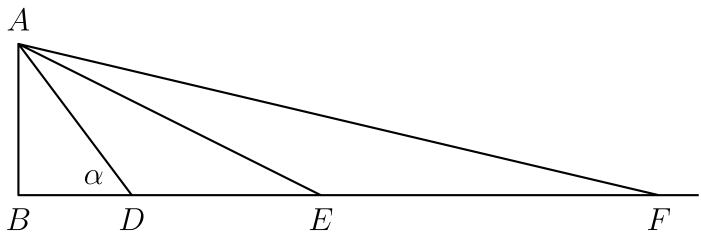


Figure 10.5. Theorem 21 continued

> Continuing thus we finally attain to such an angle, AEB, as is less than any given angle.

The point is that, at the beginning, there is some given angle, say γ. If $\alpha < \gamma$, we are done. Otherwise, we find β, and if $\beta < \gamma$, we are done. Otherwise, we proceed as in Figure 10.5, where $EF = AE$, so angle AFB is no greater than $\alpha/4$, and so on. As before, eventually we find a straight line passing through A that meets BC in an angle that is less than the angle given at the beginning.

Lobachevski does not draw an explicit conclusion from 21, and so I fail to observe it: If $\Pi(p) < \pi/2$ for some p, then there is a right triangle with a leg of p having positive defect:[1] for one of the acute angles will be less than $\Pi(p)$, while the other can be as small as we like. See page 101.

I list the propositions that we either have done or want to do. There are fifteen. Some students volunteer for particular propositions in each section of five; the rest have to take what is left. The list ends up as follows.

[1] Originally I said the defect would be at least $\pi/2 - \Pi(p)$.

17 Verity
18 Sparkle
19 Lucky
20 Charity
21 Eve

22 Sparkle
23 Eve
24 Charity
25 Lucky (only the plane part)
29 Verity

30 Eve
31 Lucky
32 Verity
33 Sparkle
36 Charity

I say that in the remaining three weeks, we ought to be able to cover this, if students will be properly prepared, meeting me *before* class to clarify any difficulties. Verity arranges to see me Friday afternoon.

11. December 8

Verity did not come on Friday. I saw yer yesterday, and she said she had had something else to do.

Only Charity and Verity come to class today. According to Verity, Sparkle is too tired from working for another class. She does not know about the others. In fact Sparkle will send me an email saying, “Yesterday I was very tired, and today I could not wake up.” Eve will write to say she is sick. I do not hear about Lucky.

Verity presents 17, and Charity 20, with some understanding. At least Verity can follow **Theorem 17** step by step, though without seeing the point. It is true that Lobachevski is not quite clear. He enunciates the proposition as,

> *A straight line maintains the characteristic of parallelism at all its points.*

By definition, it is a line *through a given point,* in a given direction, that is parallel to a given line that does not pass through the given point. Verity is to prove that the parallel is the parallel through *any* of its points.

Verity draws Lobachevski’s diagram, as in Figure 11.1, and proceeds with the argument. It would be better to construct the diagram as needed. Also, there really should be two diagrams. Perhaps Lobachevski economizes with one, to save printing costs. (This however will appear unlikely, since Theorem 20 will have three diagrams; see page 96.)

But Lobachevski could be clearer in words. After the enunciation quoted above, he says,

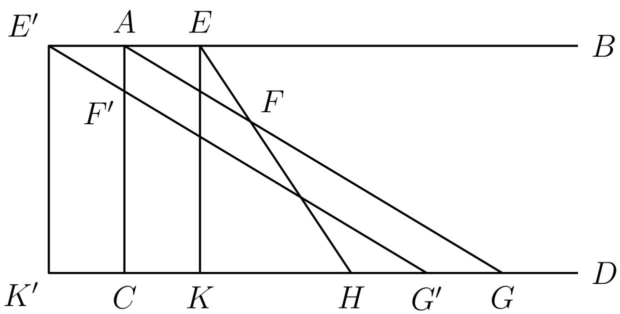


Figure 11.1. Theorem 17

> Given AB (Fig. [11.1]) parallel to CD, to which latter AC is perpendicular. We will consider two points taken at random on the line AB and its production beyond the perpendicular.

He does not emphasize that AB is the parallel *through* A to CD. Perhaps he does not see the need, since he understands AB not as the infinite straight line through A and B, but as the line with these endpoints. But in this case he might enunciate the proposition as something like, "Any segment of a parallel or the extension of a parallel is still parallel."

A further confusion arises from Lobachevski's failure to observe that the "two points taken at random" are to be taken on either side of the point A. The proof need not consider the two points at once; it considers one point, in two possible positions or cases.

In the first case, we should have Figure 11.2. Despite the lettering, first EK is dropped perpendicular to CD; *then* EF is drawn in the angle BEK. The straight line AF, or rather its "production" as Lobachevski says, "must cut CD somewhere in G. Then EF, entering the triangle ACG, must exit, and the exit point must be between K and G. Verity seems reasonably

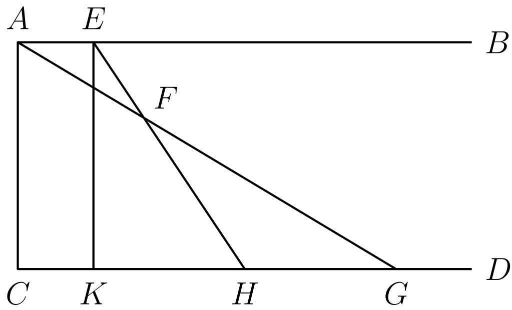


Figure 11.2. Theorem 17, first case

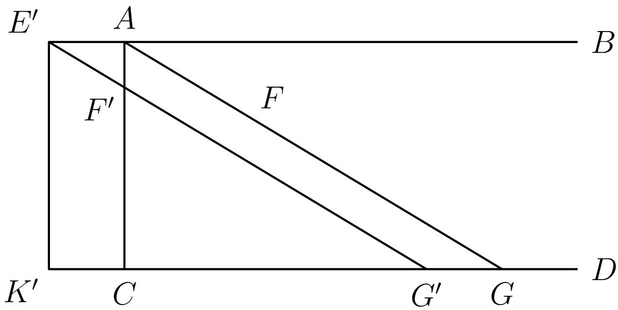


Figure 11.3. Theorem 17, second case

clear with this.

For the second case, the diagram must be considered anew, as in Figure 11.3. Here $E'K'$ is dropped perpendicular to "the production of the line CD," and then $E'F'$ is drawn,

> making so small an angle $AE'F'$ that it cuts AC somewhere in F'.

Really, $E'F'$ should be drawn at random in the angle $AE'K'$. Then it must cut either $K'C$ or AC. If it cuts $K'C$, we are done. So we suppose it cuts AC at some point, which might as well be F'. Lobachevski treats matters summarily, and I doubt that the students quite see this.

It may be that working out the literal meaning of the English is hard enough. But another approach would be to work out the *mathematics*: to understand the enunciation of a proposition, then find one's own proof, using the text for hints perhaps, but without worrying about a precise translation. I have tried to say that we care about the mathematics, not the English; but I have not suggested that students may look for their own proofs.

Meanwhile, in the second case of Theorem 17, angle FAB is made equal to $F'E'A$. the point F is *not* the same one used in the first case; but again Lobachevski does not make this clear, and therefore Verity may not fully understand it. Now $E'F'$ cannot meet AF, so it exits triangle AGC between C and G.

Charity presents **Theorem 20,** which Lobachevski enunciates as,

> *If in any rectilineal triangle the sum of the three angles is equal to two right angles, so is this also the case for every other triangle.*

The adjective "rectilineal" is confusing. I may propose *doğru kenarlı.* The qualification could be dropped: the second triangle in the enunciation is not qualified, but is presumably rectilineal as well. (Likewise lines are now always straight, but are sometimes redundantly called straight lines.)

Lobachevski's diagrams, in Figure 11.4, are misleading as well: in 11.4a, AB and BC need not be equal; in 11.4b, AB need not be the same multiple of p that AD is of q. It would be *sufficient* to require the multiples to be the same; but the text does not require this.

Charity herself is confused by the sentence,

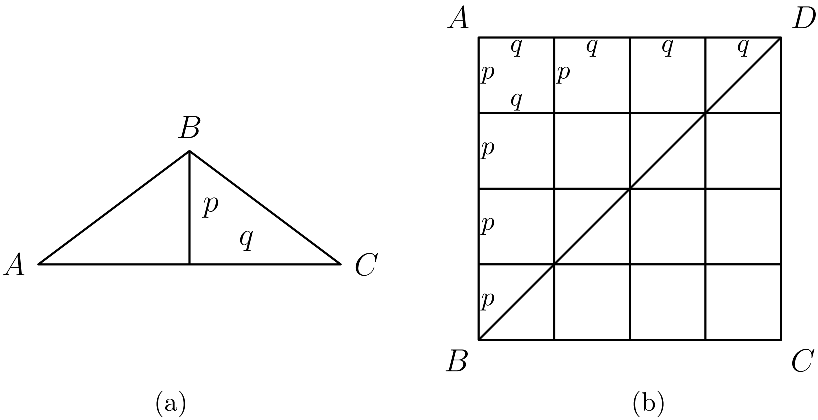


Figure 11.4. Theorem 20

> The numbers n and m can be taken sufficiently great for the right-angled triangle ABC (Fig. [11.5]) whose perpendicular sides $AB = np$, $BC = mq$, to enclose within itself another given (right-angled) triangle BDE as soon as the right-angles fit each other.

The last qualification in particular is confusing. The point is that if the angles in the original triangle ABC of Figure 11.4a add up to π, then when we drop the perpendicular from the largest angle (so that it cuts the opposite side) we get two right triangles whose angles separately add up to π. One of these, rotated and added to itself, yields a rectangle. Given the right triangle BDE of Figure 11.5, we can cover it with the rectangles, one of them sharing one of its angles with the right angle DBE.

Let me note finally that, having three different figures, Theorem 20 tends to contradict my suggestion that, in Theorem 17, Lobachevski combines two diagrams into one to save on

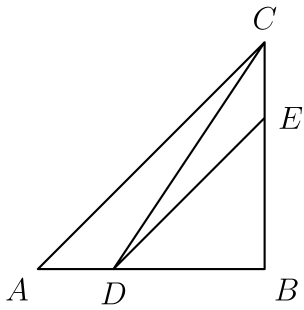


Figure 11.5. Theorem 20 still

printing costs.

12. December 15

On Saturday, Sparkle sent me an email, asking to meet on Monday. She was working on Theorem 22. She did not understand the expression, "by revolving the line AF away from the perpendicular AC." On Sunday, I saw the email and answered, saying I was teaching in the morning, but could meet at 13:00. (Ayşe being in Ankara, I was teaching her course as well as my other course.) On Monday, Sparkle saw me at noon as I was going to lunch; then she found me again when I came back. We went over Theorem 22. She told me that she had missed the last class from having been tired out from the housecleaning that she had had to do with her mother. She lived alone with her mother; her parents were divorced.[1]

At home I started writing up an account of the Poincaré half-plane model of Lobachevskian geometry. At the beginning of class, nobody else being present, I start writing some of it on the board. Eventually Verity comes,[2] then Sparkle, then Charity (unusually late), and finally Lucky at around 9:40. No Eve. Meanwhile, I recall Euclid's postulates:

1. From a given point to a given point, one (and only one) straight line can be drawn.
2. Any straight line can be extended.
3. With a given point as center, passing through any other

[1]She may have mentioned a brother living elsewhere, but I am not sure.

[2]She must have been first, because I felt free to ask her whether her full given name was written as two words; it was not.

given point, a circle can be drawn.
4. All right angles are equal to one another.
5. If a straight line cuts two straight lines, making the interior angles on the same side less then two right angles, then the two straight lines intersect.

Lobachevskian geometry rejects the fifth postulate. In the Poincaré half-plane model:

- The points are the points one one side of a given infinite straight line in the Euclidean plane.
- The straight lines are
  - straight lines perpendicular to the given straight line, or
  - arcs with center on the given straight line.
- The circles are the "real" circles.
- The right angles are the "real" right angles.

I indicate with a diagram that two arcs are at right angles if the tangents at the point of intersection are at right angles. It is necessary to show that the circles of the model have centers. In Euclidean geometry, a chord of a circle passes through the center if and only if the chord is at right angles to the circle. Thus we have to show that, in the Poincaré model, all chords of a circle have a common point of intersection. In Figure 12.1, the center of the circle is C, where

$$AB : AC :: AC : AD.$$

I say that I have a proof (which I do, using trigonometry), but I want to find a simpler one. (In fact the trigonometric proof will be the simplest that I can find; see page 136.)

In the Poincaré model, a horizontal straight line is not straight, but is a "boundary line": the perpendicular bisectors of all of its chords are parallel to one another, as in Figure

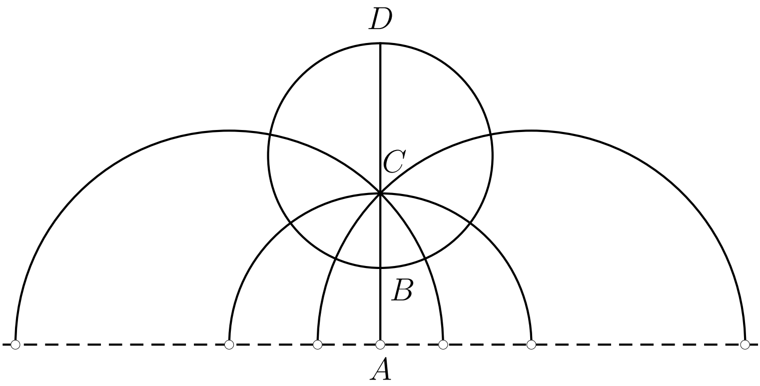


Figure 12.1. Center of circle in Poincaré model

12.2. I ask whether there are any other boundary lines in the Poincaré model. (There are: the "real" circles that are tangent to the boundary line of the model.)

Sparkle's presentation of **Theorem 22** is reasonably clear in detail. See Figure 12.3. Sparkle draws triangle ACE separately, indicating that its defect is α, as in Figure 12.4. Then she draws ACF separately as having defect β, before she corrects herself and makes the triangle AEF.

In Figure 12.3, we assume AB and CD are perpendicular to AC, but are also parallel. If ACE is defective by α, then AEF is defective by some β, so the defect of ACF is $\alpha + \beta$. But the defect is also $a - b$: Lobachevski does not spell out why, but the reasoning may simply be that the defect is what angle CAF lacks from being right, less angle AFC. In any case, we can make a as small as we like, and so α can only be 0.

Thus there is a defectless triangle. Sparkle is not quite clear on what this means. From 20, we know that if any triangle is defective, all are defective. In 22, we show that if $\Pi(p) = \pi/2$

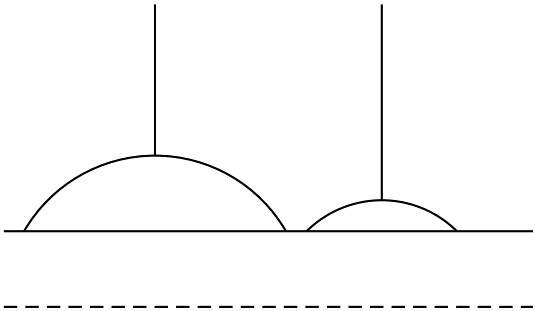


Figure 12.2. A boundary line in the Poincaré model

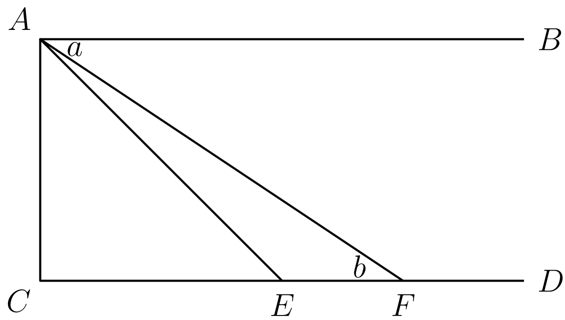


Figure 12.3. Theorem 22

for some p, then there is a defectless triangle; but then all triangles are defectless.

It is also the case that if $\Pi(p) < \pi/2$ for some p, then there is a defective triangle. This is by Theorem 21, though I fail to observe this in class. We henceforth assume $\Pi(p) < \pi/2$ for some p, and therefore for all p.

In the absence of Eve (who will write to say she is still sick), I prove **Theorem 23,** that every acute angle is $\Pi(p)$ for some p. Thus if we are given angle BAC as in Figure 12.5, and some

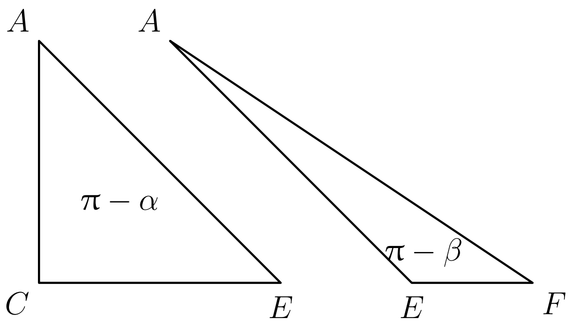


Figure 12.4. Theorem 22 detail

perpendicular DE to AC cuts AB, then triangle ADE has a positive defect α. If $DF = AD$, then triangle AFE has defect 2α, so AFG has greater defect, and so on. Eventually we find a perpendicular, as CK, that does not cut AB. Then we may assume CK is a boundary between perpendiculars that cut AB and those that do not. Then CK must be parallel to AB, for the perpendicular dropped from any inclined straight line such as CL must cut AB, and then CL also cuts.

Charity has not prepared 24. She seems to think she needed only be ready for next week; but Sparkle says this was not the case. Lucky says he has not prepared 25, but can try to do it anyway. I skip him for now and go to Verity for **Theorem 29:** if two of the perpendicular bisectors of a triangle meet, then they all meet at the same point. First she thinks the proposition is about *medians,* or in Turkish *kenarortaylar* (and it is possible that, in talking to her about the proposition earlier, I was confused about the meaning of this term). In the diagram in the text, it is not clear that the lines drawn are not medians. I draw a triangle with an obtuse angle, asking Verity to use

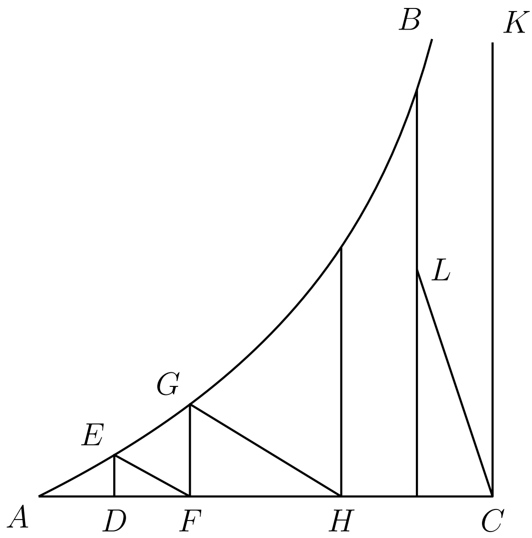


Figure 12.5. Theorem 23

it; this brings out the confusion. Corrected, she is able to proceed. She seems unclear, though, about when the intersection point lies outside the triangle. For example, she has a diagram looking as in Figure 12.6, where DG, EG, and FG are indeed to be understood as perpendicular bisectors. Also confused is Charity, who wonders when the point of intersection will lie on a side of the triangle.

I observe that the proposition is really Euclidean.

Lucky has had ten or 15 minutes to study 25 in class, but this turns out not to be enough. He draws the whole diagram first. He does not seem to state clearly which lines are parallel to which. I ask him to write it out. When he starts using words, I suggest using symbols; then he does write correctly

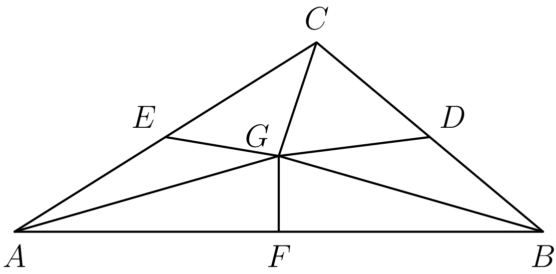


Figure 12.6. Theorem 29

that we are first going to prove

$$AB \parallel EF \ \& \ CD \parallel EF \implies AB \parallel CD.$$

He does not see the point of Lobachevski's auxiliary lines, so I go the board to explain, as time is running out. He says he can finish next week.

It is understood that we shall finish next week. It seems to be generally understood that students will seek me out on Monday afternoon with questions about their propositions. Charity may want to meet on Monday morning. I say it will have to be at 9. She says she will email me if she wants to meet.

13. December 22

Nobody wrote me. I was in the office at 9 yesterday morning, just in case. On Monday afternoon, after my other course, I saw Sparkle and Lucky sitting with somebody else in the common area, so I went to see them briefly. A bit later, in my office, Sparkle, Charity, and Verity came, though not simultaneously. They all seemed to go away, reasonably content, though I was not necessarily content, particularly about what Sparkle was working on: **Theorem 33.**

The point of 33 is to establish that, in a figure bounded by boundary lines AB and $A'B'$ and their axes AA' and BB', as in Figure 13.1a, if CC' is another axis, and lengths are as indicated, then

$$\frac{t'}{t} = \frac{s'}{s}. \tag{13.1}$$

First note that the figure exists because of Theorem 32, that a boundary line is a circle of infinite radius, the axes being radii of this circle. The claim is clear in case t/s is a rational number p/q, since then we can divide the figure into q congruent strips by means of new axes, and one of these will be CC'. I think Lobachevski confuses the matter by assuming also s/s' is a rational number n/m. He then divides AB into nq parts. It follows that $A'B'$ contains mq of the same equal parts, and AC contains np of them. Lobachevski observes this, but does not seem to use it to conclude (13.1). he uses only the construction of the equally spaced axes, except that they divide AB into nq parts, not just q of them.

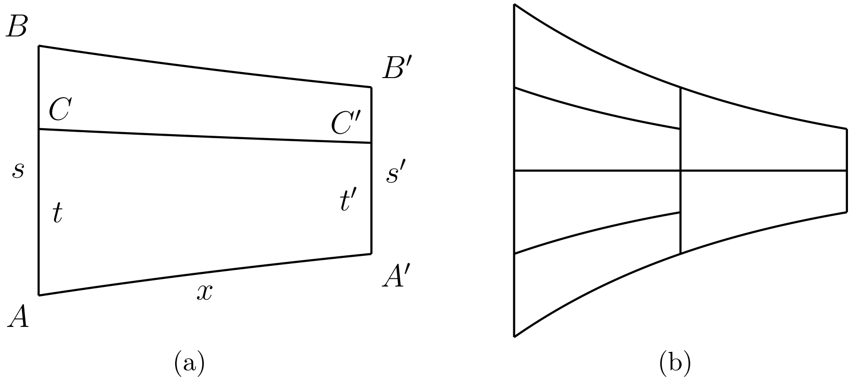


Figure 13.1. Theorem 33

Lobachevski does not say that (13.1) holds generally by continuity. The proportion is clear from the Eudoxan definition in Euclid, though Lobachevski does not allude to this, nor did I in talking with Sparkle.

No argument is given for why s'/s depends exponentially on x. I tried to suggest this dependence to Sparkle by pictures as in Figure 13.1b, where all six of the small quadrilaterals are congruent.

This morning I finished working out in detail the proofs of the remaining propositions that I hoped to cover: 30–3 and 36. I also confirmed the formulas of 33 and 36 for the Poincaré half-plane.

When I go to class, Charity is already there, continuing to think about **Theorem 36.** Yesterday she left my office at the point where Lobachevski was just about to obtain the formula

$$\left(\tan\frac{\Pi(c)}{2}\right)^2 = \tan\frac{\Pi(c-\beta)}{2}\cdot\tan\frac{\Pi(c+\beta)}{2}. \qquad (13.2)$$

from

$$\cos \Pi(c) = \frac{\cos\left(\frac{1}{2}\Pi(c-\beta) + \frac{1}{2}\Pi(c+\beta)\right)}{\cos\left(\frac{1}{2}\Pi(c-\beta) - \frac{1}{2}\Pi(c+\beta)\right)}. \tag{13.3}$$

Lobachevski does not give a derivation though. All I can see to do is to write the latter formula as

$$\cos\vartheta = \frac{\cos(\varphi+\psi)}{\cos(\varphi-\psi)}. \tag{13.4}$$

Since

$$\tan\frac{\vartheta}{2} = \frac{\sin\vartheta}{1+\cos\vartheta} = \frac{1-\cos\vartheta}{\sin\vartheta},$$

so that

$$\left(\tan\frac{\vartheta}{2}\right)^2 = \frac{1+\cos\vartheta}{1-\cos\vartheta},$$

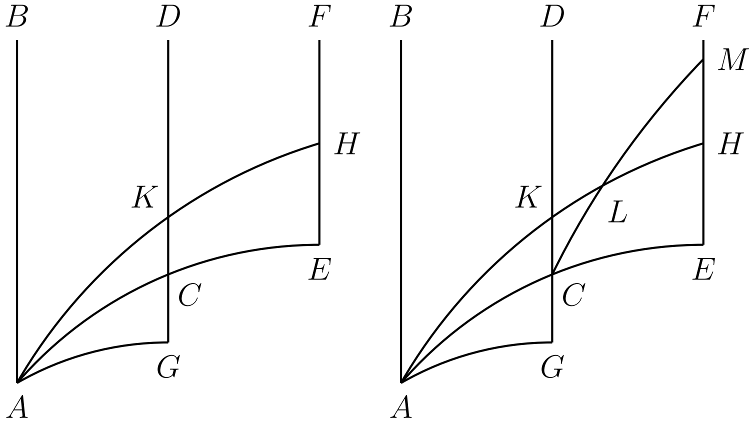
we obtain at present

$$\left(\tan\frac{\vartheta}{2}\right)^2 = \frac{\cos(\varphi-\psi)-\cos(\varphi+\psi)}{\cos(\varphi-\psi)+\cos(\varphi+\psi)} = \frac{\sin\varphi\cdot\sin\psi}{\cos\varphi\cdot\cos\psi}$$
$$= \tan\varphi\cdot\tan\psi.$$

In my own notes I have worked this out by first expanding (13.4) using the angle addition formula. In any case, Charity has not seen where (13.2) comes from, so I show her my notes.

I have finally brought to class my two Escher books, for their reproductions of Escher's "Circle Limit" pictures. I have also brought my slide rule.

Verity and Sparkle come. The latter says Eve and Lucky are stuck in traffic. They are supposed to present 30 and 31 respectively. I ask Verity to have a go at 32. She starts; but



(a) $AB \parallel EF$ & $CD \parallel EF$

(b) $AB \parallel CD$ & $CD \parallel EF$

Figure 13.2. Theorem 25

despite our discussion yesterday, she says she has not understood the boundary line. Meanwhile Lucky has arrived, so I suggest that he might explain it.

He first wants to present **Theorem 25,** which he could not do properly last week. I let him try again, but he still has not understood. Again he draws the whole diagram. He writes

$$\angle DCE < \pi/2 \qquad \textit{Teorem 22'den} \qquad EF \parallel CD.$$

I let this go for a while, but then must come back to it.

Three straight lines AB, CD, and EF are given, in that order. If two are parallel the third, then they will be shown parallel to one another. In the first case, $AB \parallel EF$ and $CD \parallel EF$, as in Figure 13.2a, and we want to prove $AB \parallel CD$. Though I do not check the English until later, what Lucky has written comes from a misreading of the end of Lobachevski's sentence,

> In order to prove this, let fall from any point A of the outer line AB upon the other outer line FE, the perpendicular AE, which will cut the middle line CD in some point C (Theorem 3), at an angle $DCE < \frac{1}{2}\pi$ on the side toward EF, the parallel to CD (Theorem 22).

I shall not remember exactly how I tell Lucky just to stop. The point that he apparently does not see is that when the perpendiculars AE and AG are dropped to EF and CD respectively, then, since $CD \parallel EF$, the angle ECD must be acute, and so the point G will be on the other side of C from D. Then a straight line drawn out from A into the angle BAG will be on one side or other of AC. If it is in angle CAG, then it must cut GC. If it is in angle BAC, then (since $AB \parallel EF$) it must cut EF at a point H, and therefore it must cut CD at a point K, or rather CD must cut *it.*

Now suppose $AB \parallel CD$ and $CD \parallel EF$, as in Figure 13.2b. Then a straight line drawn from A in the angle BAE will cut CD at a point K. I do not know why Lobachevski does not appeal to Theorem 17 here in order to conclude that AK must also cut EF. Instead he repeats the proof, extending AK to L, then observing that CL must cut EF at a point M, so that AL must cut EF also.

I do not spend time in class to clarify the proof of 25. I just ask Lucky to go on to 31. He does not understand that either. I observe that he was just hanging out yesterday afternoon when I saw him. His classmates came to talk to me then, but he did not. He suggests that he can study 33 and present it in the next class. There is no next class, I say. He says he means the next hour.

I mutter something about how this is all like a joke (*şaka gibi*). I heard a few weeks ago that Lucky was having trouble with his girlfriend. Also it seems his parents are recently divorced. But if he finds no solace in mathematics when he is troubled, then he should study something else. I wonder if what really troubles him is that he is not as smart as he likes to imagine himself to be.

I don't know what to do. "What shall we do?" I ask (*Ne yapalım?*). Nobody makes a suggestion. Eventually I go to write on the board the saying that I showed Sparkle yesterday, when she whined a bit about some difficulty:

ΧΑΛΕΠΑ ΤΑ ΚΑΛΑ.

I invite her to say the translation:

Zordur güzeller.

Then I start talking about Escher, using the books that I have brought. His pictures are beauties, though *not* hard to look at; but some at least are based on the mathematics that we are doing.

Eve has come in at some point. But the time is about 9:50, so I take a break, as if most students have not already taken their personal breaks by arriving late.

Eve has written out a translation of **Theorem 30,** but freely and cheerfully admits that she does not understand it. In triangle ABC in Figure 13.3, we consider the perpendicular bisectors DE, FG, and HK of the sides. First assuming $DE \parallel FG$, we aim to prove $HK \parallel DE$. However, after the diagram (which is oriented oppositely and is drawn with straight lines only in the text), $HK \parallel DE$ is all that Eve writes on the board. She cannot explain whether it is hypothesis or conclusion. After some discussion, I am reduced to pointing out that in most theorems, there is an hypothesis and a conclusion. I

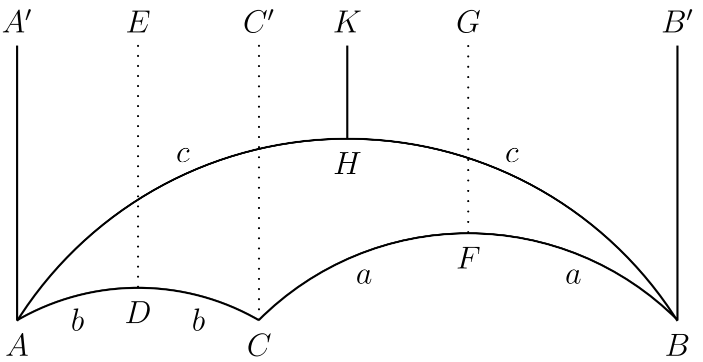


Figure 13.3. Theorem 30

wonder what the students have been learning in their other classes, if they have never picked up this basic fact.

Somehow we work out that, on the hypothesis $DE \parallel FG$, the conclusion $HK \parallel DE$ is obvious, since by Theorem 29, HK cannot *meet* DE.[1] After this conclusion, the text may be confusing; for before explicitly turning to the converse, Lobachevsky draws further conclusions. If the sides opposite the points A, B, and C are respectively $2a$, $2b$, and $2c$, and the letters of the points designate the angles at those points, then

$$A = \Pi(b) - \Pi(c), \quad B = \Pi(a) - \Pi(c), \quad C = \Pi(a) + \Pi(b).$$

Eve can work this out, with the help of the auxiliary lines AA', BB', and CC', drawn parallel to the perpendicular bisectors, or rather to HK in particular.

Now we assume $HK \parallel FG$. As before, if $DE \nparallel HK$, then as it cannot cut HK, it must cut AA'. In this case

$$B = \Pi(a) - \Pi(c), \qquad C > \Pi(a) + \Pi(b).$$

[1] We use also the assumption that HK lies between DE and FG.

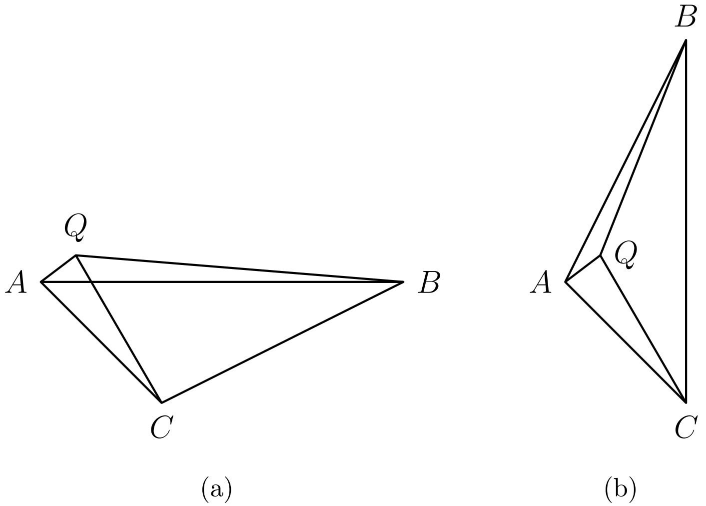


Figure 13.4. Theorem 30 continued

If we rotate CA about C into a position CQ as in Figure 13.4, so that $\angle QCB = \Pi(a)+\Pi(b)$, then $\angle QBC > \Pi(a)-\Pi(c)$. But we now have a triangle QBC to which the earlier arguments apply,[2] so that, if $QB = 2c'$, then

$$\Pi(a) - \Pi(c') > \Pi(a) - \Pi(c),$$
$$\Pi(c') < \Pi(c),$$
$$c' > c.$$

But $AC = QC$, and so $AB > QB$ (see below), that is, $c > c'$. Lobachevski stops abruptly here, leaving it to the reader to note the contradiction. Eve copies some of this onto the board,

[2]Namely that the perpendicular bisector of QB is parallel to the other two.

but not with any understanding that I can see. Since time is running out, we move along anyway.

Meanwhile, let us note that Lobachevski considers the configuration of Figure 13.4a only, not 13.4b. Since $AC = QC$, we have

$$\angle CAQ = \angle CQA,$$

and therefore $\angle BAQ < \angle BQA$, and so $BA > BQ$. According to Lobachevski, this is by Theorem 9:

> In a rectilineal triangle, a greater side lies opposite a greater angle. In a right-angled triangle the hypothenuse is greater than either of the other sides, and the two angles adjacent to it are acute.

The first part of this is worded like Heath's rendition of Euclid's I.18:

> In any triangle the greater side subtends the greater angle.

This means $BA > BQ \implies \angle BAQ < \angle BQA$, when what we want is the converse, I.19.

In class, I ask Lucky if he can explain **Theorem 31** now. He says he doesn't think so (*zannetmiyorum*); so I do it. Again Lobachevski's explanation is not as clear as I think it could be. It starts with what is formally a definition, though it is italicized as it enunciated a theorem:

> We call boundary line (oricycle) that curve lying in a plane for which all perpendiculars erected at the mid-points of chords are parallel to each other.

Where the bizarre form "oricycle" comes from, I do not know. The normal form in English would be horocycle, as if derived from ὁροκύκλος; I point out some time that ὅρος is what is

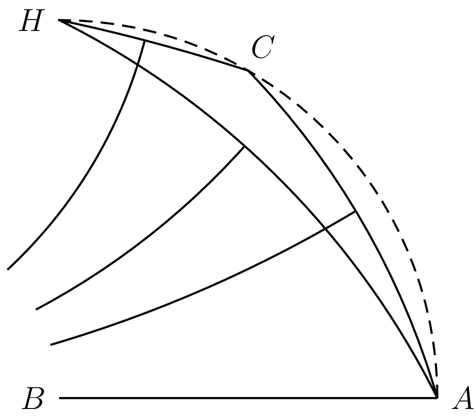


Figure 13.5. Theorem 31: the horocycle

used in Euclid's *Elements* for what we call definitions.[3] In any case, the theorem would be that the horocycle actually exists. Given a straight line AB and two acute angles α and β, we may draw AC and AH as in Figure 13.5, so that

$$\angle BAC = \alpha = \Pi(a), \qquad AC = 2a,$$
$$\angle BAH = \beta = \Pi(b), \qquad AH = 2b.$$

(Assuming $\alpha > \beta$, I ask which of AC and AH will be greater, but do not get the right answer at first.[4]) The perpendicular bisectors of AC and AH are parallel to AB; therefore, by Theorem 30, the perpendicular bisector of CH is also parallel to AB. Thus all such points as C and H fill out a curve, the **boundary line,** the perpendicular bisector of whose every

[3]See Appendix E.4 page 150.

[4]I do not recall using the letters a and b here, but I think I did use the Π. It is therefore possible that I wrote nonsense like $\Pi(\alpha)$; but in that case, nobody corrected me.

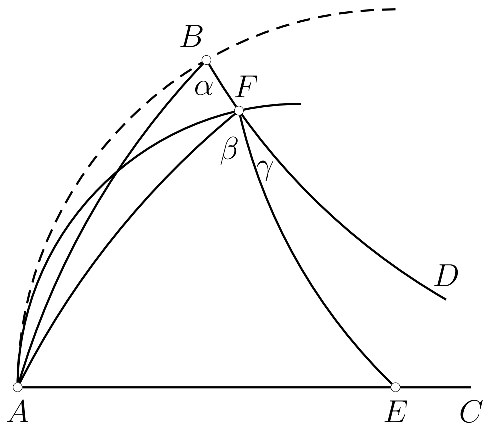


Figure 13.6. Theorem 32

chord is parallel to AB. All of these parallels are called **axes** of the boundary line.

Verity now tries again with **Theorem 32:** that the boundary line is, so to speak, a circle of infinite radius. She is able to work it out pretty much as Lobachevski does, though she is still a bit confused. At least she can think at the board and correct herself. In Figure 13.6 then, the straight line AB is a chord of a boundary line with axes AC and BD (Lobachevski calls the latter BF). The angles BAC and ABD have the same value α. Lobachevski justifies this by a reference to Theorem 31, though the fact is not made explicit there. It must be understood that $\alpha = \Pi(a)$, where $2a$ is the length of AB. If E is chosen at random on AC, then the circle with center E passing through A cuts BD at a point F. Lobachevski has not yet mentioned the point D, but has always used F. He does not note explicitly that F must be on the indicated side of the boundary line, and Verity does not worry about this (and I do not worry her about this).[5] However, wherever F falls on BD, the angle AFE or β will be less than α, and this means $\angle FAC < \angle BAC$, so the diagram is correct.

Lobachevski also does not make it quite clear that as the center E moves away from A, so the point F moves towards B. But when E moves away from A to E', then $E'F < E'A$, by Theorem 9 as interpreted earlier (page 113). This shows that the circle through A having center E' lies outside that with center E. Thus F must move towards B as E moves away from A.

Lobachevski does note that as E moves, γ becomes smaller; it becomes as small as we like, by Theorem 21. But then he makes an obscure reference to 22:

> ... the angle γ approaches the limit 0, as well in consequence of a moving of the center E in the direction AC, when F remains unchanged, (Theorem 21), as also in consequence of an approach of F to B on the axis BF, when the center E remains in its position (Theorem 22) ...

The point is that, by 21, γ approaches 0, even if F is fixed; but as F moves towards B, γ only becomes smaller still. Theorem 22 seems irrelevant. In any case, as Verity observes (but Lobachevski does not explicitly), the sum of the angles in triangle ABF is

$$\alpha - \beta + \alpha + (\pi - \beta - \gamma),$$

which is $2\alpha - 2\beta - \gamma + \pi$; since this is less than or equal to π, we obtain

$$\alpha - \beta \leqslant \frac{1}{2}\gamma.$$

[5] The intransitive and transitive forms of "to worry" would be different in Turkish: *merak etmek* and *merak ettirmek*.

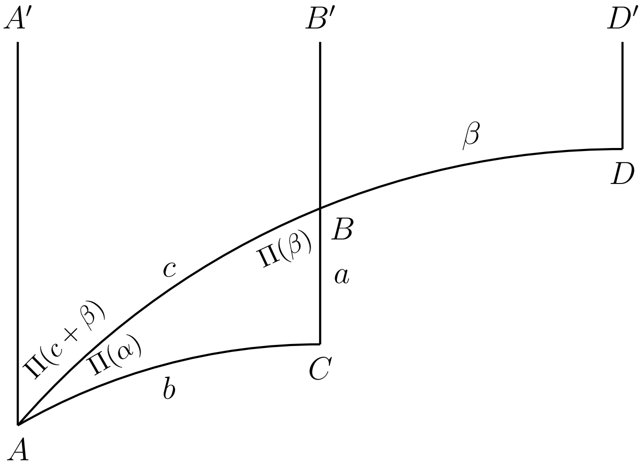


Figure 13.7. Theorem 36, first case

Lobachevski himself says first $\alpha - \beta < \beta + \gamma - \alpha$, the inequality being strict by Theorem 22, which he cites, though the strictness is not needed.

Time is running out. Sparkle goes through **Theorem 33,** reproducing Figure 13.1b (page 106) as I suggested.

I have told the students that the final examination will concern the exercises that I gave out on November 3 (page 58). I told Verity last week that I would type them up and put them on the web; this I have done, as I now make clear.

It is 10:50 or even later, and everybody else leaves; but Charity is pumped up about **Theorem 36,** and she has no class now, and there is no class in our room, so we stay on so that she can present 36 to me. We are given triangle ABC as in Figure 13.7; the angle at C is right, and for some *distances* α and β, the angles at A and B are $\Pi(\alpha)$ and $\Pi(\beta)$ respectively. The sides opposite A, B, and C are a, b, and c respectively. One must be clear that a is not α, though both are distances.

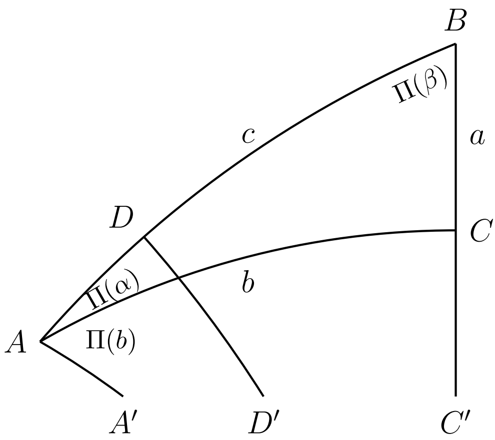


Figure 13.8. Theorem 36 when $\beta < c$

AB is extended a distance β to D. Then the perpendicular DD' to AD is parallel to CB (which is in turn extended to B'). If the parallel AA' to BB' is also drawn, then, considering that AA' is parallel to two different straight lines to which perpendiculars are dropped from A, we have

$$\Pi(b) = \Pi(\alpha) + \Pi(c + \beta). \tag{13.5}$$

We derive a related equation by measuring β along BA in the other direction. There are three cases. If $\beta < c$, we have Figure 13.8, from which we can infer

$$\Pi(c - \beta) = \Pi(\alpha) + \Pi(b). \tag{13.6}$$

In case $\beta = c$, the diagram is as in Figure 13.9a, and then

$$\Pi(\alpha) + \Pi(b) = \frac{1}{2}\pi;$$

but now $\Pi(c - \beta) = \Pi(0) = \frac{1}{2}\pi$ by definition; so again (13.6) holds. Finally, if $\beta > c$, then as in Figure 13.9b,

$$\Pi(\beta - c) + \Pi(b) + \Pi(\alpha) = \pi,$$

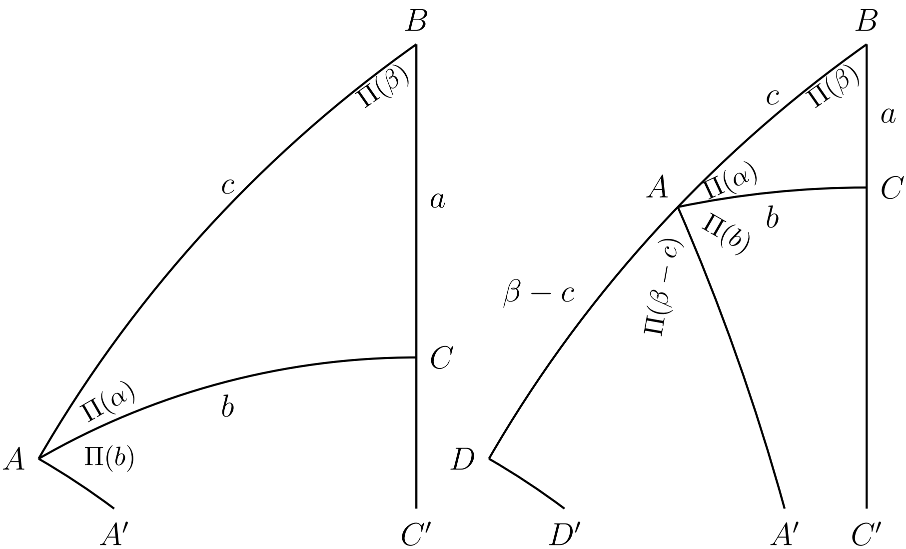


Figure 13.9. Theorem 36 when $\beta \geqslant c$

so (13.6) still holds since $\pi - \Pi(\beta - c) = \Pi(c - \beta)$ by definition.

I need to point out to Charity that it is by solving the system of (13.5) and (13.6) that Lobachevski obtains

$$2\Pi(b) = \Pi(c-\beta) + \Pi(c+\beta),$$
$$2\Pi(\alpha) = \Pi(c-\beta) - \Pi(c+\beta).$$

This yields immediately

$$\frac{\cos \Pi(b)}{\cos \Pi(\alpha)} = \frac{\cos\left(\frac{1}{2}\Pi(c-\beta) + \frac{1}{2}\Pi(c+\beta)\right)}{\cos\left(\frac{1}{2}\Pi(c-\beta) - \frac{1}{2}\Pi(c+\beta)\right)}.$$

Now we use a result of Theorem 35 without proof, namely

$$\frac{\cos \Pi(b)}{\cos \Pi(\alpha)} = \cos \Pi(c).$$

This is a standard result from spherical geometry, and 35 shows that there is no change in this geometry when one eliminates Euclid's fifth postulate. Now we have (13.3) on page 107, from which we obtain (13.2), namely

$$\left(\tan\frac{\Pi(c)}{2}\right)^2 = \tan\frac{\Pi(c-\beta)}{2}\cdot\tan\frac{\Pi(c+\beta)}{2},$$

as shown. Now Lobachevski proposes replacing β with c, $2c$, $3c$, and so forth. One can do this; that is, one can use induction to obtain

$$\left(\tan\frac{\Pi(c)}{2}\right)^n = \tan\frac{\Pi(nc)}{2}.$$

But it seems neater to me to rewrite (13.2) as

$$\frac{\tan(\Pi(c)/2)}{\tan(\Pi(c-\beta)/2)} = \frac{\tan(\Pi(c+\beta)/2)}{\tan(\Pi(c)/2)};$$

for since $\tan\bigl(\Pi(0)/2\bigr) = 1$ (as Charity tells me), we have

$$\left(\tan\frac{\Pi(c)}{2}\right)^n = \prod_{k=1}^{n}\frac{\tan\bigl(\Pi(kc)/2\bigr)}{\tan\Bigl(\Pi\bigl((k-1)c\bigr)/2\Bigr)}.$$

We can define the unit so that $\tan\bigl(\Pi(1)/2\bigr) = \exp(-1)$; and then

$$\exp(-x) = \tan\frac{\Pi(x)}{2},$$

so $\Pi(x) = 2\arctan\exp(-x)$.

Part III.

Appendices

A. Attendence

		Verity	Eve	Sparkle	Charity	Lucky
1	Sep 29	X				X
2	Oct 6	X	X	X	X	X
3	Oct 13					
4	Oct 20	X	X	X	X	X
5	Oct 27	X		X	X	
6	Nov 3	X			X	
7	Nov 10	X		X	X	X
8	Nov 17	X		X	X	X
9	Nov 24			X	X	
10	Dec 1	X	X	X	X	X
11	Dec 8	X			X	
12	Dec 15	X		X	X	X
13	Dec 22	X	X	X	X	X
		11	4	9	11	8

Here a student is counted if she or he showed up at all, even if it was 40 minutes late. In the accounts of the days themselves, I tried to give some indication of who was late; but I was not systematic in keeping such records. In another year, I would try to keep the class from being scheduled at 9 a.m., since this is perhaps the worst time to need to be anywhere in Istanbul, as far as traffic is concerned. However, presumably people do manage to be where they should be then, if their livelihoods depend on it. In any case, I ought to have a clear policy on lateness.

B. Final examination

The final examination was scheduled for 13:30–15:00, Wednesday, January 6, 2015. On the last day of class (page 117), I said that the exam would be based on the exercises that I had announced.

B.1. Preliminary meeting

Sparkle sent me an email on the day before the exam, asking if she could meet me. Later in the day, she and Eve did come to my office. Because of their questions, I inferred that, from the exercise sheet, it was not clear that problems 4 and 5 were based on Lemma IV of Pappus. I made this clear in person. The students did not have Pappus with them, and I did not have my paper copy; so we looked at the image on my computer screen.

Problem 5 was the *converse* of Lemma IV, but much of the proof could be the same. However, I was not prepared to go through this with the students. I just worked out an argument as follows.

In the given diagram (as in Figure B.1, but without HMN), we are to show

$$\frac{AF \cdot BC}{AB \cdot CF} = \frac{AF \cdot DE}{AD \cdot EF}. \tag{B.1}$$

We note that AF is common to both sides. From Problem 4, we know

$$\frac{CF}{FE} = \frac{KC}{HK} \cdot \frac{HL}{LE} \tag{B.2}$$

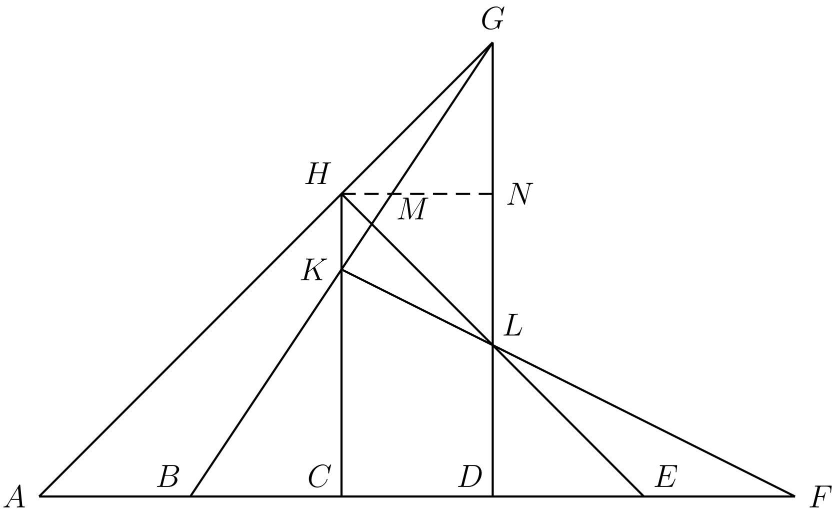


Figure B.1. Homework Problem 5

The FE appears in the right member of (B.1), while CF is on the left. Thus, keeping AF, we shall manipulate the left member of (B.1) so as to obtain, as a factor, the right member of (B.2). We do this by first adding to the diagram HMN, which is parallel to AE. Then

$$\begin{aligned}
\frac{AF\cdot BC}{AB\cdot CF} &= \frac{AF}{CF}\cdot\frac{BC}{AB}\\
&= \frac{AF}{CF}\cdot\frac{BC}{HM}\cdot\frac{HM}{AB}\\
&= \frac{AF}{CF}\cdot\frac{CK}{KH}\cdot\frac{HN}{AD}\\
&= \frac{AF}{CF}\cdot\frac{CK}{KH}\cdot\frac{HN}{DE}\cdot\frac{DE}{AD}\\
&= \frac{AF}{CF}\cdot\frac{CK}{KH}\cdot\frac{HL}{LE}\cdot\frac{DE}{AD}
\end{aligned}$$

$$= \frac{AF}{CF} \cdot \frac{CF}{FE} \cdot \frac{DE}{AD}$$
$$= \frac{AF \cdot DE}{AD \cdot FE},$$

as desired.

B.2. The examination itself

The exam itself is indeed based on the exercises. It uses "rational" notation, rather than fractional notation; but I note the equivalence at the beginning ("instead of $X : Y$, one can write X/Y or $\frac{X}{Y}$"). Translated, the exam problems are as follows. Only one diagram is supplied, and that for Problem 4.

Problem 1. Let $ABCD$ and $AEFG$ be straight, and let

$$BE \parallel DG, \qquad BF \parallel CG.$$

Show the parallelism $CE \parallel DF$.

Problem 2. Of the quadrilateral $ABCD$, let the sides AB and DC meet at E; sides DA and CB, at F. Prove the proportion

$$AE : EB :: AD : DF \ \ \& \ \ FC : CB.$$

Problem 3. If $ABCD$ is straight, the **cross ratio** of these four points can be defined as the compound ratio

$$AC : CB \ \ \& \ \ BD : DA \quad \text{or} \quad AB : BC \ \ \& \ \ CD : DA.$$

Show the equivalence of these two definitions. That is, supposing $EFGH$ is straight, prove that the proportion

$$AC : CB \ \ \& \ \ BD : DA :: EG : GF \ \ \& \ \ FH : HE$$

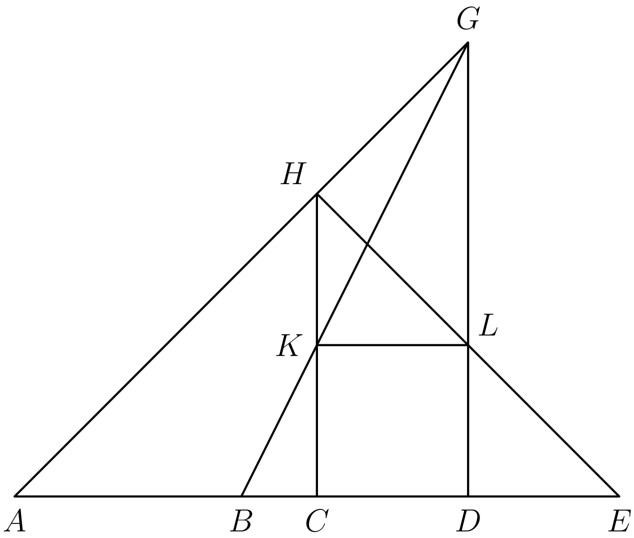


Figure B.2. Exam Problem 4

implies the proportion

$$AB : BC \;\&\; CD : DA :: EF : FG \;\&\; GH : HE,$$

and conversely.

Problem 4. In the diagram [in Figure B.2], the lines that appear straight are straight, and

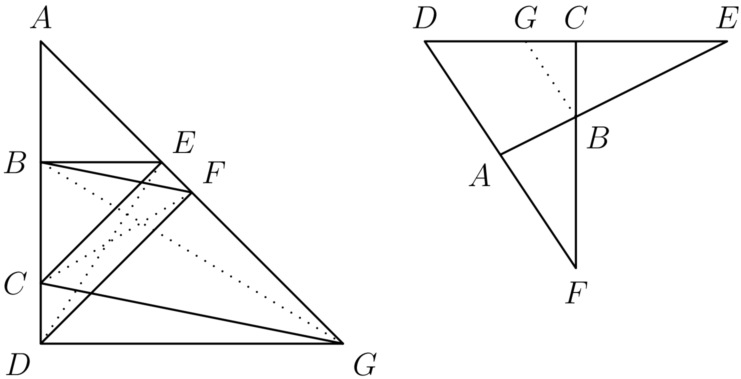
$$KL \parallel AE.$$

Show the proportion

$$AB : BC :: AD : DE.$$

B.3. My solutions

My own solutions (prepared in Turkish before the exam) are as follows.



(a) Problem 1 (b) Problem 2

Figure B.3. Diagrams for exam solutions

1. In Figure B.3a,

$$\begin{aligned}
DBE &= GBE, && [\text{because } BE \parallel DG]\\
DAE &= GBA, && [\text{adding } ABE]\\
GBF &= CBF, && [\text{because } BF \parallel CG]\\
GBA &= CAF, && [\text{adding } ABF]\\
DAE &= CAF, && [\text{because } DAE = GBA]\\
DCE &= CEF, && [\text{removing } CAE]
\end{aligned}$$

wherefore $CE \parallel DF$.

2. In Figure B.3b, let $BG \parallel AD$. Then

$$\frac{AE}{EB} = \frac{AD}{GB} = \frac{AD}{DF} \cdot \frac{DF}{GB} = \frac{AD}{DF} \cdot \frac{FC}{CB}.$$

3. Translating into fractional notation, we have

$$\frac{AC \cdot BD}{CB \cdot DA} = \frac{EG \cdot FH}{GF \cdot HE}$$

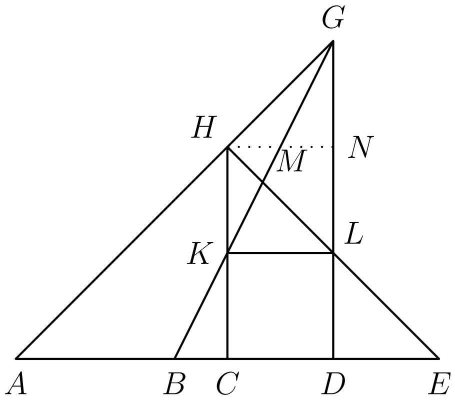


Figure B.4. Solution for Problem 4

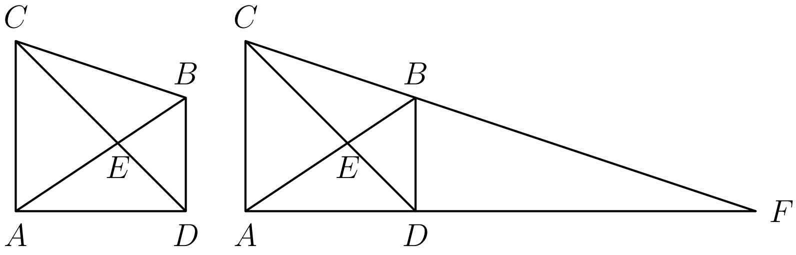
$$\begin{aligned}
&\iff \frac{AC \cdot BD}{BC \cdot AD} = \frac{EG \cdot FH}{FG \cdot EH} \\
&\iff \frac{AC \cdot BC + AC \cdot CD}{BC \cdot AC + BC \cdot CD} = \frac{EG \cdot FG + EG \cdot GH}{FG \cdot EG + FG \cdot GH} \\
&\iff \frac{AC \cdot CD - BC \cdot CD}{BC \cdot AC + BC \cdot CD} = \frac{EG \cdot GH - FG \cdot GH}{FG \cdot EG + FG \cdot GH} \\
&\iff \frac{AB \cdot CD}{BC \cdot AD} = \frac{EF \cdot GH}{FG \cdot EH}.
\end{aligned}$$

4. In Figure B.4, let $HMN \parallel AE$. Then

$$\frac{AB}{BC} = \frac{AB}{HM} \cdot \frac{HM}{BC} = \frac{AG}{HG} \cdot \frac{HK}{KC} = \frac{AD}{HN} \cdot \frac{HL}{LE} = \frac{AD}{HN} \cdot \frac{HN}{DE}.$$

B.4. Examination day

Before the exam, I see Charity in a classroom, working at the whiteboard, apparently with Hapless. Hapless tries to enter the exam, but I do not allow him. I point out that 70% attendance was required, and anyway, he never come back to talk to me as I had told him (page 70).



(a) Initial (b) Final

Figure B.5. Sparkle's diagram for Problem 2

Verity tells me something I do not understand; it may be about Lucky. I ask her if he is coming. She calls him and reports that he is *not* coming.

So four students take the exam. There is initial confusion about Problem 1: students do not recognize that the point A is shared by the two straight lines. On Problem 2, Sparkle draws a diagram as in Figure B.5a, but does not see how DA and CB can also cross. I indicate that these sides could be extended. Her completed paper will have a diagram as in Figure B.5b.

B.5. Students' solutions

Results are as follows.

Problem 1. Charity is quite correct. She makes one slip, which need not be counted. Having shown $ABG = FAC$, she shows $AED = ABG$ and underlines it. Recalling that $ABG = FAC$, she repeats it and underlines it, when evidently she means $AED = FAC$.

• Verity correctly establishes first $BED = BEG$, and then $ADE = BAG$. But then she wrongly claims that

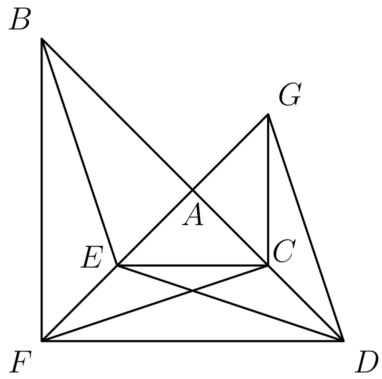


Figure B.6. Sparkle's diagram for Problem 1

$ECG = BFD$ because $BF \parallel CG$.

• Sparkle's diagram is as in Figure B.6 (without BG drawn). Her first claim is that since BED and GED have a common base, and $BE \parallel GD$, it follows that the two triangles are equal. This is the same mistake that Verity made originally (page 28).

• Eve wrongly has the straight lines BG, CF, and DE meeting at a common point M. But there are three points to consider, as in Figure B.7. Eve's argument would be correct, if M, N, and P were one. The argument can be corrected:

$$BDG = EDG,$$
$$BMD = EMG,$$
$$BFG = BFC,$$
$$FNG = BNC,$$
$$BMD - BNC = EMG - FNG,$$
$$CPD - MNP = EPF - MNP,$$

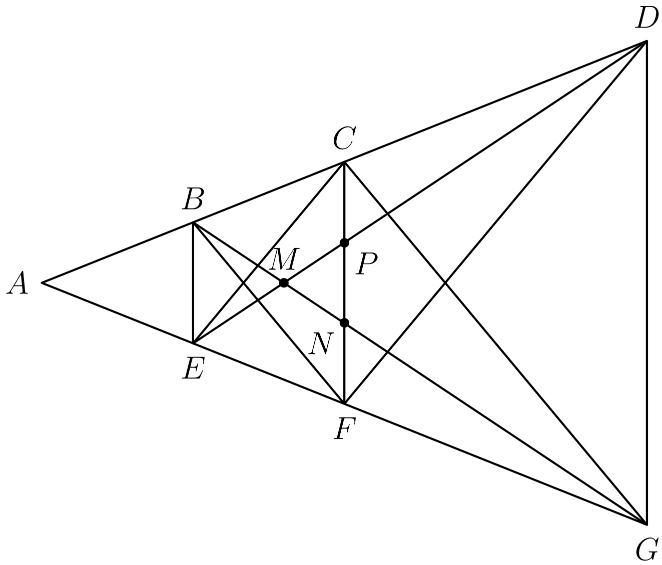


Figure B.7. Correction of Eve's diagram for Problem 1

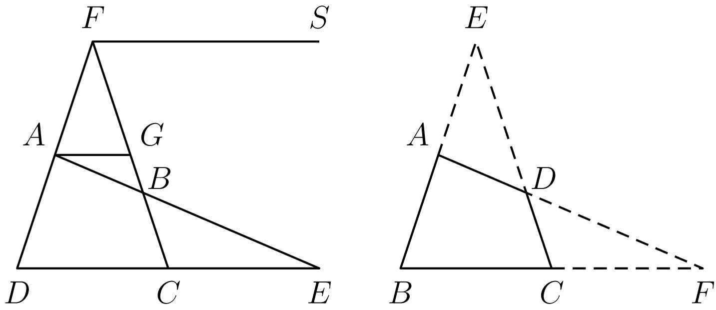
$$CPD = EPF,$$
$$CFD = FED.$$

Problem 2. Eve has this correct (unless it matters that AD and BC meet in the direction of D and C). The others have not understood that the ampersand stands for multiplication.

- Sparkle writes

$$\frac{AE}{EB} = \frac{AD}{DF} \qquad \text{ve} \qquad \frac{FC}{CB},$$

draws the diagram in Figure B.5b as described, repeats the verbal description of the intersections (changing the



(a) Charity (b) Verity

Figure B.8. Student diagrams for Problem 2

imperative *kesişsin* to the declarative *kesişir*), but says no more.

• As what is to be proved (because it is followed by *Kanıt* “Proof”), Charity writes

$$\frac{AE}{EB} = \frac{AD}{DF} \ \& \ \frac{AE}{EB} = \frac{FC}{CB},$$

and has a diagram as in Figure B.8a. I translate what she gives as a proof. The style is correct. It appears Charity can write mathematics, which is something that the best students at METU might not do well, even if they ended up being research mathematicians. But the mathematical content of her proof is almost completely wrong; I see only one correct proportion:

> From the point A, parallel to DE, let us produce the straight line AG. Since $AG \parallel CE$ (because AG is parallel to the whole of DE), we obtain the equality $CE : AG :: EA : EB$. Let us look at the triangle

FDC. Since $AG \parallel DC$,

$$\frac{AG}{DC} = \frac{FD}{DA} = \frac{FC}{GC},$$
$$\frac{AG}{DC} = \frac{GB}{BC},$$

therefore

$$\frac{FD}{DA} = \frac{GB}{BC}.$$

Since $AG \parallel CE$,

$$\begin{array}{ccc} \dfrac{BG}{BC} & = & \dfrac{AE}{BE} \\ \Vert & & \\ \dfrac{FD}{DA} & = & \dfrac{AE}{BE} \end{array}$$

I wonder if the observation about the "whole of DE" shows the influence of Lobachevski.

• Verity writes few words, and her diagram, as in Figure B.8b, has no auxiliary lines. She starts with the enunciation,

$$\frac{AE}{EB} = \frac{AD}{DF} \qquad \& \qquad \frac{FC}{CB}$$

and proceeds immediately to write nonsense:

$$\frac{AE}{EB} = \frac{ED}{EC} = k, \qquad \frac{AD}{DF} = \frac{BC}{FC} = \ell \qquad \textit{diyelim}$$

"let us say." She never uses k or ℓ again. The next thing she writes is

$$\frac{AE \cdot DF}{EB \cdot DF} = \frac{ED \cdot DF}{EC \cdot DF} \implies \frac{AD \cdot EF}{DF \cdot EB} = \frac{ED \cdot DF}{EC \cdot DF},$$

which might be said to have one correct feature: the protasis would be correct, if the fractions set equal to k were indeed equal to one another.

Problem 3. Eve is mostly correct. At least she has the computations correct, roughly as in my own solution. Indeed, she quotes what I told her:

$$\frac{A}{B} = \frac{C}{D} \implies \frac{A-B}{B} = \frac{C-D}{D}.$$

I should think my own arrow would have been two-headed. During the exam, she asks if she has to write everything again for the converse. I draw the double-headed arrow on the board and suggest that her steps ought to be reversible. I think she does not really understand this, though she does write out her conclusion with the double-headed arrow.

- As before, Verity and Sparkle misunderstand the ampersand. Sparkle writes nothing new; Verity, nonsense. Charity has nothing.

Problem 4. Eve draws the correct auxiliary straight line (hers is called HST, where mine is HMN), and she writes some correct proportions (as equations of fractions); but then she says

$$\begin{aligned}\frac{AB}{BC} &= \frac{AB}{HS} \cdot \frac{HS}{BC} \\ &= \frac{HT}{DE} \cdot \frac{HS}{BC} \\ &= \frac{HT}{DE} \cdot \frac{AD}{HT} = \frac{AD}{DE}.\end{aligned}$$

with two incorrect substitutions, although has already written correctly

$$\frac{AB}{HS} = \frac{AD}{HT}.$$

• Sparkle extends LK to meet AG at a point T, and she draws BZ parallel to DG, but she cannot make use of these. She writes three correct proportions and two incorrect proportions, but uses no compound ratios.
• Verity draws no auxiliary lines. Her words suggest that she does not understand how KL is constructed. She observes correctly $HK/HC = KL/CE$, but cannot continue.
• Again Charity has nothing.

These results can be scored as follows (accompanied by attendence figures as on page 122):

	1	2	3	4	total	4/13 × attendence
Verity	0.5	0.25	0	0.5	1.25	3.38
Eve	0.5	1	0.75	0.75	3	1.23
Sparkle	0.25	0.25	0	0.25	0.75	2.77
Charity	1	0.5	0	0	1.5	3.38
Lucky	—	—	—	—	—	2.46

C. Centers of hyperbolic circles

Here is the proof, referred to on page 99, for why the putative diameters of circles in the Poincaré half-plane model do actually meet at one point.

In the Euclidean plane, on one side of a straight line AB, let a circle with center C be drawn, as in figure C.1 or C.2. Drop the perpendicular CA to AB. Let the straight line CD, perpendicular to CA, cut the circle at D, and let the rectangle $ACDE$ be completed. Then DE is tangent to the circle. Let the circle with center E passing through D cut AC at F. Then

$$EF = ED,$$

and the arc DF is at right angles to the circle.

Now let a random point G be taken on the circle. Let the radius CG be drawn, and let GB, at right angles to this, cut AB at B. Then GB is tangent to the circle, so the circle with center B passing through G is at right angles to the original circle. We shall prove that the new circle passes through F, that is,

$$BF = BG.$$

To this end, let the rectangle $ABHK$ be drawn so that HK passes through G. Make the following definitions:

$$CD = 1, \qquad AC = a, \qquad \angle DCG = \vartheta.$$

Then

$$KG = \cos\vartheta, \quad CK = \sin\vartheta, \quad BH = AK = AC + CK = a + \sin\vartheta.$$

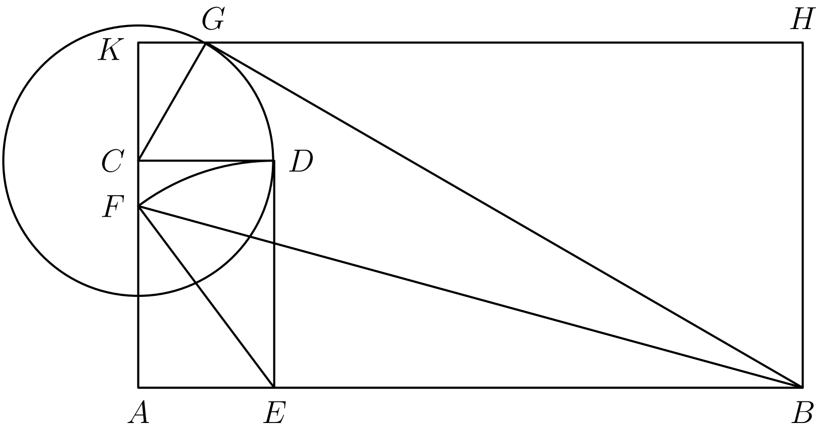


Figure C.1. Hyperbolic circle

Here ϑ and CK can be negative, but $\pi/2 < \vartheta < \pi/2$, and KG and BH are positive. Moreover,

$$\frac{|BG|}{BH} = \frac{|CG|}{KG} = \frac{1}{\cos\vartheta},$$

wherefore

$$BG^2 = \frac{(a + \sin\vartheta)^2}{\cos^2\vartheta}. \tag{C.1}$$

Moreover, $BF^2 = AF^2 + AB^2$, where

$$AF^2 = EF^2 - AE^2 = ED^2 - CD^2 = AC^2 - CD^2 = a^2 - 1,$$
$$AB^2 = KH^2 = (KG + GH)^2.$$

To calculate the latter we use

$$\frac{GH}{BH} = \frac{CK}{KG} = \tan\vartheta,$$

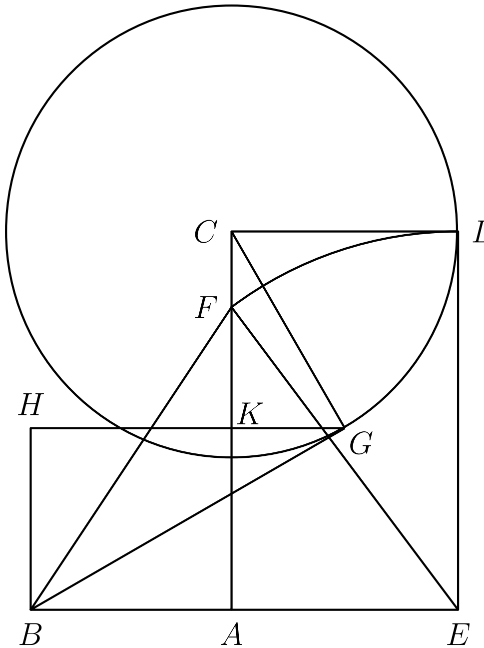


Figure C.2. Hyperbolic circle

so that

$$AB^2 = \bigl(\cos\vartheta + (a+\sin\vartheta)\cdot\tan\vartheta\bigr)^2$$
$$= \frac{\bigl(\cos^2\vartheta + (a+\sin\vartheta)\cdot\sin\vartheta\bigr)^2}{\cos^2\vartheta} = \frac{(1+a\cdot\sin\vartheta)^2}{\cos^2\vartheta},$$

wherefore

$$BF^2 = AF^2 + AB^2 = a^2 - 1 + \frac{(1+a\cdot\sin\vartheta)^2}{\cos^2\vartheta}$$
$$= \frac{a^2\cdot\cos^2\vartheta - \cos^2\vartheta + 1 + 2a\cdot\sin\vartheta + a^2\cdot\sin^2\vartheta}{\cos^2\vartheta}$$

$$= \frac{a^2 + \sin^2 \vartheta + 2a \cdot \sin \vartheta}{\cos^2 \vartheta},$$

which by (C.1) is BG^2. This completes the proof.

D. Hyperbolic diagrams

In the diagrams in our text of Lobachevski, straight lines are drawn as straight. This is reasonable; but it means that straight lines supposed to be parallel will "obviously" meet if extended. In the present notes, I have redrawn many of Lobachevski's figures in the Poincaré half-plane.

For me this activity has been something of a reversion to childhood. At around the age of twelve, I plotted various mathematical curves, first using the tables and equations in the analytic geometry textbook [12] that my mother had used in college, and then using one of the scientific pocket calculators that were coming on the market. I noticed that, when one measured angles in radians, the graph of the sine function was like $y = x$ near the origin. On my own though, I could not quite recognize then that the cosine function was the derivative of the sine function.

Now I am returning to the plotting of mathematical curves, but with somewhat more knowledge. However, my knowledge of plotting with the aid of a computer extends only as far as the use of `pstricks`. With this TeX package, figures such as 13.5 on page 114 can be constructed by means of the computations illustrated in Figure D.1. In the usual coordinate grid, the horocycle is chosen as unit circle centered at $(1, 0)$, and the boundary of the half-plane is the y-axis. If two points A and B are chosen on the horocycle, with coordinates $(1+\cos\alpha, \sin\alpha)$ and $(1+\cos\beta, \sin\beta)$ respectively, then the arc centered on the y-axis passing through A and B has center $(0, -\tan\frac{1}{2}(\alpha+\beta))$.

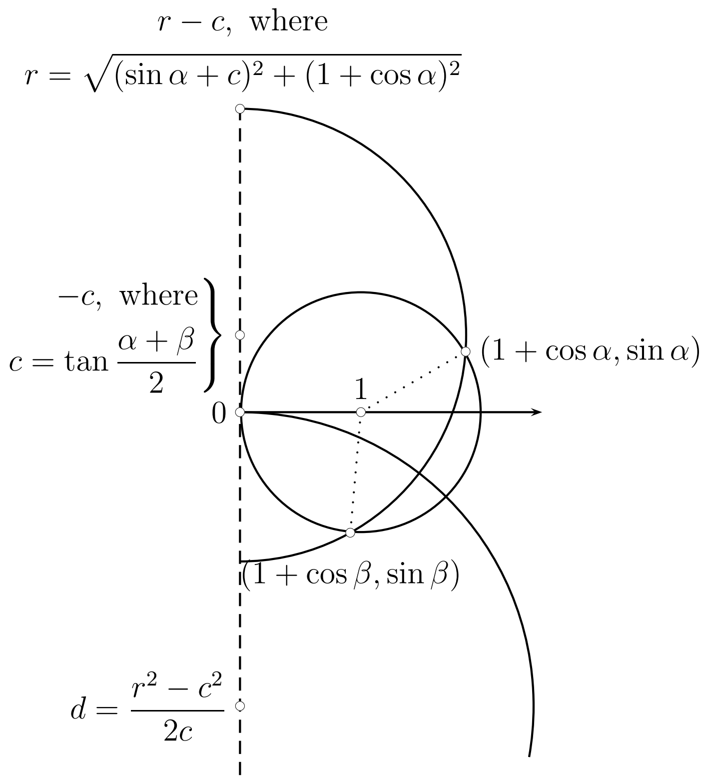


Figure D.1. Computations in the Poincaré half-plane

Call this $(0, -c)$. Then the radius of the arc is

$$\sqrt{(c + \sin\alpha)^2 + (1 + \cos\alpha)^2}.$$

Call this r. The arc centered on the y-axis, passing through the origin, and orthogonal to the first arc, will have a center $(0, d)$. Then

$$(c + d)^2 = r^2 + d^2,$$
$$c^2 + 2cd = r^2,$$

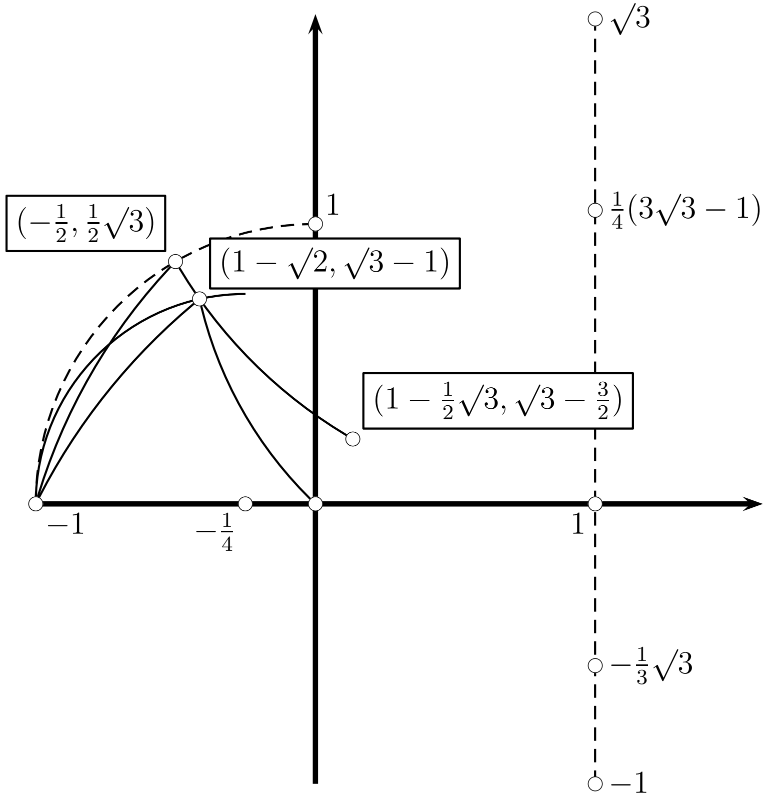


Figure D.2. Theorem 32 figure coordinates

$$d = \frac{r^2 - c^2}{2c}.$$

The construction of Figure 13.6 on page 115 led to curious discovery. The points of the figure have the coordinates shown in Figure D.2. I found these as follows. I chose the boundary of the half-plane to be the straight line given by $x = 1$. The boundary line AB was the unit circle centered at the origin, and its axis AC was the x-axis. The foot B of the other axis was $(1; \pi/3)$ in polar coordinates. This meant that the

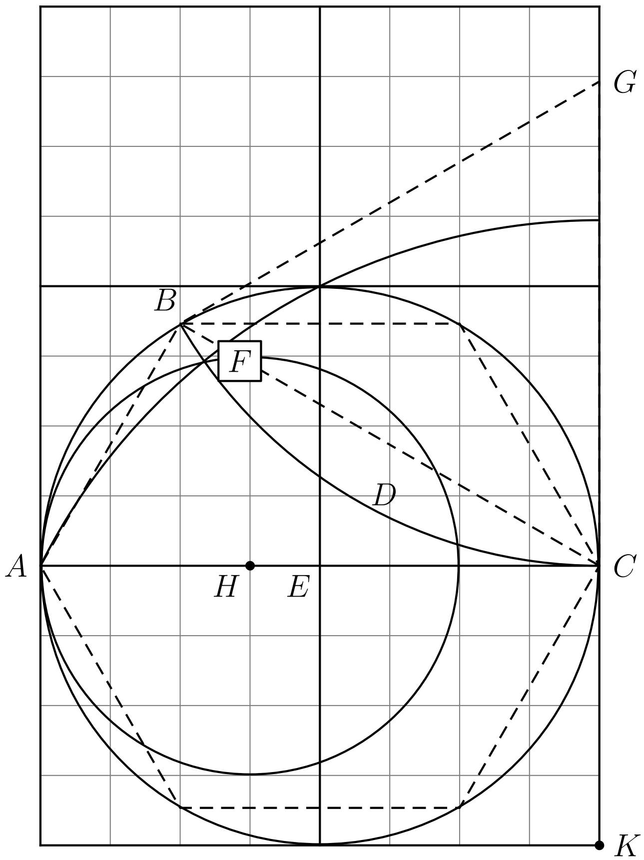


Figure D.3. Theorem 32 figure construction

axis BD itself was drawn as a circle with center G having coordinates $(1, \sqrt{3})$. This can all be seen in Figure D.3. The circle through A with center on the axis AC was drawn as the real circle with center H having coordinates $(-\frac{1}{4}, 0)$. This circle cut the axis BD at F.

But where was F? I resorted to computing it by hand, approximately, starting the lengths shown in Figure D.4. I used the rule that, in a triangle with sides measuring a, b, and

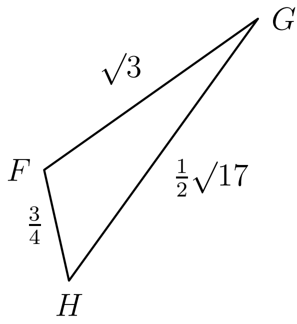


Figure D.4. Theorem 32 figure triangle

c respectively, the angle α subtended by a is given by

$$\tan \frac{1}{2}\alpha = \sqrt{\frac{(a-b+c)(a+b-c)}{(a+b+c)(-a+b+c)}}.$$

The rule is given in the form

$$\tan \frac{1}{2}\alpha = \frac{r}{s-a},$$

where

$$s = \frac{1}{2}(a+b+c), \qquad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

in the handbook [4], a book that in my youth I obtained from a West Virginia junk shop, but now have obtained from my father-in-law in Ankara. In any case, I ultimately found the coordinates of F to be about $(-0.41, 0.73)$.

This looked like $(1-\sqrt{2}, \sqrt{3}-1)$, and so I completed the diagram on the assumption that F had these coordinates. In particular, the chord AF was drawn as a circle with center K

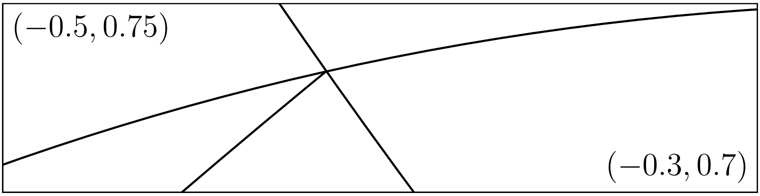


Figure D.5. Theorem 32 figure around F

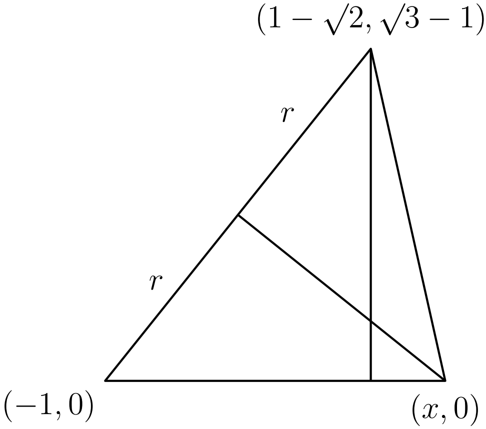


Figure D.6. Theorem 32 figure computations

having coordinates $(1,-1)$. On this basis, the three lines that come together at F are shown in Figure D.5. However, let $2r$ be the distance between $(-1,0)$ and $(1-\sqrt{2}, \sqrt{3}-1)$, as in Figure D.6. Then

$$\begin{aligned}(2-\sqrt{2})(x+1) = 2r^2 &= \frac{1}{2}\big((2-\sqrt{2})^2 + (\sqrt{3}-1)^2\big) \\ &= \frac{1}{2}(6 - 4\sqrt{2} + 4 - 2\sqrt{3}) \\ &= 5 - 2\sqrt{2} - \sqrt{3},\end{aligned}$$

so

$$
\begin{aligned}
2(x+1) &= (2+\sqrt{2})(5-2\sqrt{2}-\sqrt{3}) \\
&= (10+\sqrt{2}-4-2\sqrt{3}-\sqrt{6}) \\
&= 6+\sqrt{2}-2\sqrt{3}-\sqrt{6},
\end{aligned}
$$

which is about 1.5006, but is not exactly the 1.5 that I took it to be.

E. Some words

E.1. Etymology of "man"

I record here some etymological observations about what to call the first-year students mentioned in the Introduction. The observations have little relevance to the rest of this document; but I did do the research while I was writing these notes, and I do not want to lose it.

The traditional English term for a first-year student was "freshman." It was also a tradition that these students would be male. Today, at Mimar Sinan, not only are some of our mathematics students female: most of them are. Any one of our first-year students may still be called a freshman, provided the second component of this term is understood to mean simply a human being. This is the "prominent sense" of the word *man* in Old English, the language spoken in England before the Norman Invasion of 1066 [7, man, p. 279].

In Old English, male and female specimens of the human species were *wer* and *wīf* respectively. Strictly, the latter word was *wif*; it is modern scholars who now mark the vowel with a macron, indicating length [23, §6, p. 4]. The marking may be useful to show distinctions between words originally spelled the same, such as *gōd* "good" and *god* "god." The word *wīf* became "wife" in Modern English. Meanwhile, *wīf* also became part of the compound *wīfman*, which was first masculine in gender, then feminine [7, woman, p. 544]. The compound became *wimman* in the tenth century, with the plural *wim-*

men. We have retained the pronunciation of the plural for today's "women"; in the twelfth century, the singular *wimman* became *wumman,* giving us today's pronunciation of "woman" [22, woman, p. 614].

The Old English *wer* is cognate with "virile" and is seen in "werewolf." The word "world" can be understood as compounded from *wer* and "eld"; the latter is an archaic noun meaning "age" in various senses, derived from the original form of the adjective "old." The James Brown song "It's a Man's Man's Man's World" (credited also to Brown's girlfriend Betty Jean Newsome) is wilfully redundant; but even to say "man's world" is redundant, etymologically speaking,

I pick up the information about the James Brown song from the Web, especially *Wikipedia.* Information about etymologies might be considered as common knowledge, obtainable from many dictionaries; I have indicated the dictionaries that I used by citing the sources of specific points not found (or perhaps not found as prominently) in other sources. I have also looked at the *OED* [11], used also in the next section.

E.2. Pronunciation of "parallelepiped"

Used on page 24, the word "parallelepiped" was once "parallelepipedon," a direct transliteration of the Greek παραλληλεπίπεδον. This is compounded of παράλληλ- "parallel" and ἐπίπεδον "plane surface." The latter Greek word is in turn compounded of ἐπί "on" and πέδον "ground." Thus, unlike the vowel O in "parallelogram," the second E in "parallelepiped(on)" is not just a linking vowel. Ignorance of this fact has led to a pronunciation of "parallelepiped" with the stress on the penultimate syllable. According to the *Oxford English Dictionary*

[11], the correct pronunciation puts the stress on the antepenult. In the sixth edition (1976) of the *Concise Oxford Dictionary of Current English* [24], the etymologically correct pronunciation is dominant. It is given, as for most entries in that dictionary, by marking up the headword; the ignorant pronunciation comes afterwards. Thus the entry begins:

părallĕlĕ′pĭpĕd (*or* -*e*pī′pĭd).

In the ninth edition (1995) of the *COD* [25], where all pronunciations are given by respelling the headword in the International Phonetic Alphabet, the correct pronunciation takes second place to the ignorant one:

parallelepiped /ˌparəlɛləˈpʌɪpɛd, parəlɛˈlɛpɪpɛd/.

(For typesetting the IPA letters and diacriticals here, I have used the `tipa` package.) I still prefer the pronunciation of "parallelepiped" that is based on knowledge. There is no geometrical figure called a "piped," and this ought to be considered by any mathematician faced with the task of pronouncing the word "parallelepiped." He or she might remember that the letter sequence "epi" is seen also in words like "epimorphism," and there might be a reason for this. One ought to have a reason (preferably a better reason than "convention") for all of the technical terms that one uses. My own distaste for the use of unexplained technical terms led to the writing of an article about some of them: "Abscissas and Ordinates" [19].

E.3. Etymology of *suistimal*

When the students taught me this word (page 79), I think the students spelled it *suistimal,* and this is the spelling of

Nişanyan [14]; but Püsküllüoğlu [21] and Redhouse [2] give the more etymologically correct *suiistimal.* In particular, Redhouse lists the word under the Arabic prefix *sui-,* which means "evil, mis-"; and *istimal* by itself is an Arabic verbal noun for "using." Of about twenty words that Redhouse lists under *sui-,* I recognize only *suikast,* which I have understood to mean "assassination"; but the meaning can apparently be broader, as *kasıt* by itself means "intention."

E.4. "Oricycle"

There is no such word ὁροκύκλος in the big Greek dictionary [10], though there are compounds like ὁροθεσία "fixing of boundaries" and ὁροφύλαξ "frontier guard" (as opposed to ὀροφύλαξ "mountain guard"). Possibly the form "oricycle" mentioned on page 113 represents the translator's faithfulness to Lobachevski's Russian. Nietzsche's *Birth of Tragedy* concerns what Walter Kaufmann translates as "the Apollinian and Dionysian," and "Apollinian" is a strange way to form an adjective from the name of Apollo; but Kaufmann explains in a footnote: "*Apollinisch* has often been rendered by 'Apollonian'; but I follow Brinton, Morgan, and the translator of Spengler's *Decline of the West* in preferring 'Apollinian'; after all, Nietzsche did not say *Apollonisch*" [13, p. 9].

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Index

This index has not been prepared systematically, except to the extent of showing in **boldface** the pages where particular Lemmas of Pappus and Theorems of Lobachevski have been *proved.*