Minimalist set theory

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Preface

The previous edition of this book was for use in Math 320 (Set Theory) at METU in the spring semester of 2010/11. I edited and added to the book throughout the semester; the last version for the course was dated May 17, 2011. The book was based on notes I had used in teaching the course in the past; but I had rewritten many sections. This is true now. In particular, I have split what was Chapter 2 into the two chapters 2 and 3 of this edition, and I have made many adjustments to these chapters. The material now in Chapter 7 was either missing or relegated to appendices before.

The catalogue description of Math 320 is:

> Language and axioms of set theory. Ordered pairs, relations and functions. Order relation and well ordered sets. Ordinal numbers, transfinite induction, arithmetic of ordinal numbers. Cardinality and arithmetic of cardinal numbers. Axiom of choice, generalized continuum hypothesis.

The set theory presented in this book is a version of what is called ZFC: **Z**ermelo–**F**raenkel set theory with the Axiom of **C**hoice. I call the presentation minimalist, as in the title, for several reasons:

1. The only basic relation between sets is membership; equality of sets is a *defined* notion.
2. Classes as such have no *formal* existence: they are not individuals in the theory, though we can treat them in some respects as if they were.
3. `only` Axioms are introduced only when further progress is otherwise hindered.
4. The form of many axioms, namely that such-and-such a class is a set, is used even for the Axiom of Infinity: the *class* ω of natural numbers is obtained without first assuming that it exists as a set.

See Appendix D for further discussion.

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1. Introduction

In this book, we—the writer and the reader—shall develop an axiomatic theory of sets:

1. We shall study *sets,* which are certain kinds of *collections.*
2. We shall do so by the *axiomatic method.*

There are various reasons why one might want to do this. I see them as follows.

sets

1. All concepts of mathematics can be defined by means of sets. Among such concepts are the numbers *one, two, three,* and so on—numbers that we learn to count with at an early age.

ance

2. Theorems about sets can be as beautiful, as elegant, as any other theorems in mathematics.

3. The axiomatic method is of general use in mathematics, and set theory is an example of its application.

ogic

4. Set theory provides a *fundamental* or *foundational* example of the axiomatic method, in the following sense. The field of mathematics called *model theory* can be considered as a formal investigation of the axiomatic method, as it is used in ordinary mathematics. In model theory, one defines *structures,* which are to be considered as *models* of certain *theories.* All of these notions—structure, model, theory—are defined in terms of sets.[1]

By the *axiomatic method,* I mean:

1) the identification of certain fundamental properties of some mathematical structure (or kind of structure);
2) from these alone, the derivation of other properties of the structure.

[1] In a strict sense, a *theory* is a kind of collection of formal sentences. Sentences are not normally considered as sets, though they can be. A *structure* is a set considered with certain basic *relations,* that is, subsets of the Cartesian powers of the set. The theory of a structure is the collection of *true* sentences about the structure. The determination of these sentences is made by considering the interactions of the basic relations. All of this requires some notion of sets. It is true that not everybody likes the set-theoretic conception of model theory: see Angus Macintyre's speculative paper, 'Model theory: geometrical and set-theoretic aspects and prospects' [29].

The fundamental properties are called *axioms* (or *postulates*).

For example, *groups* are studied by the axiomatic method. The structure of interest in group theory is based on the set of *permutations* or *symmetries* of a given set. If the given set is A, then the symmetries of A are just the invertible functions from A to itself. These symmetries compose a set, which we may call $\mathrm{Sym}(A)$. This set always has at least one element, the identity on A, which can be denoted by id_A or simply id. Every element σ of $\mathrm{Sym}(A)$ has an inverse, σ^{-1}. Any two elements σ and τ of $\mathrm{Sym}(A)$ have the composites $\sigma \circ \tau$ and $\tau \circ \sigma$. In short, we have a *structure* on $\mathrm{Sym}(A)$, which we may denote by

$$(\mathrm{Sym}(A), \mathrm{id}, {}^{-1}, \circ). \tag{1.1}$$

eqn:

Some fundamental properties of this structure are that, for all x, y, and z in $\mathrm{Sym}(A)$,

$$x \circ \mathrm{id} = x, \qquad x \circ x^{-1} = \mathrm{id}, \qquad x \circ (y \circ z) = (x \circ y) \circ z. \tag{1.2}$$

eqn:

We have been referring to sets, such as A and $\mathrm{Sym}(A)$; this illustrates reasons 1 and 4 above to study sets. Everything in this example can be reduced to sets as follows.

1. Every element of $\mathrm{Sym}(A)$ is a certain kind of subset of the *Cartesian product* $A \times A$.
2. This product is the set $\{(x, y) \colon x \in A \mathbin{\&} y \in A\}$ of *ordered pairs* of elements of A.
3. An ordered pair (a, b) is the set $\{\{a\}, \{a, b\}\}$.
4. This set has just two elements, namely $\{\{a\}$ and $\{a, b\}\}$.
5. These elements are themselves sets, $\{a\}$ having the unique element a, and $\{a, b\}$ having just two elements, a and b (which may however be the same).
6. id or id_A is the set $\{(x, x) \colon x \in A\}$.
7. ${}^{-1}$ is the set $\{(x, x^{-1}) \colon x \in \mathrm{Sym}(A)\}$.
8. σ^{-1} is the set $\{(y, x) \colon (x, y) \in \sigma\}$.
9. $\circ$ is the set $\{((x, y), x \circ y) \colon (x, y) \in \mathrm{Sym}(A) \times \mathrm{Sym}(A)\}$.
10. $\sigma \circ \tau$ is the set $\{(x, z) \colon \exists y\, ((x, y) \in \tau \mathbin{\&} (y, z) \in \sigma)\}$.

The properties in (1.2) are the *group axioms.*[2] Suppose $(G, \mathrm{e}, {}^*, \cdot)$ is a structure satisfying these axioms: that is, for all x, y, and z in G,

$$x \cdot \mathrm{e} = x, \qquad x \cdot x^* = \mathrm{e}, \qquad x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

Then $(G, \mathrm{e}, {}^*, \cdot)$ is called a *group.* There is a set A such that $(G, \mathrm{e}, {}^*, \circ)$ *embeds* in $(\mathrm{Sym}(A), \mathrm{id}, {}^{-1}, \circ)$; that is, the former structure can be considered as a *substructure* of the latter. Indeed, A can be chosen as G itself, and the embedding of G in $\mathrm{Sym}(G)$ is the function that takes an element g of G to the symmetry $\{(x, g \cdot x) : x \in G\}$ of G. This result is known as *Cayley's Theorem.* It shows that all groups, as defined by the axioms, have the structural properties of groups of permutations. Thus the theorem provides an example of complete success with the axiomatic method.

Axiomatic set theory does not have the same success. Here there is no result corresponding to Cayley's Theorem. We can *prove* that there is no such result, by proving the *Incompleteness Theorem* (which we shall do in §7.2). This theorem can be taken to illustrate reason 2 to study set theory. Sets are logically *prior* to the rest of mathematics; we cannot expect to identify all of their properties, even implicitly. We shall identify enough properties for some wonderful consequences though, such as the existence of *transfinite* ordinal and cardinal numbers (covered in Chapters 4, 5, and 6).

Euclid's *Elements* [13] is the world's most popular textbook, having been in use for over two thousand years. It has been taken as the prototypical example of the axiomatic method. Before Euclid, many theorems of geometry were known, such as:

1. the so-called Pythagorean Theorem (Fig. 1.1): the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides;
2. the existence of just five so-called Platonic solids: the cube, tetrahedron, octahedron, icosahedron, and dodecahedron (Fig. 1.2).

Euclid's innovation is to arrange all of these theorems in a *system.* Euclid starts with some basic facts, which we call axioms or postulates:[3] for example,

[2] Usually two more axioms are given, namely $\mathrm{id} \circ x = x$ and $x^{-1} \circ x = \mathrm{id}$; but these can be derived from the others.

[3] Euclid calls them αἰτήματα, which is the plural of αἴτημα. (The Greek text of Euclid as established by Heiberg [12] can be found in various places on the Web. See

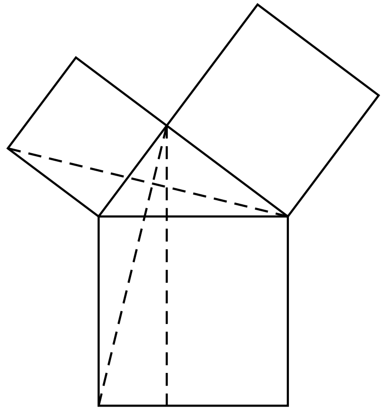

fig:P

Figure 1.1. The figure of the Pythagorean Theorem

- It is possible to draw a straight line from one point to another.
- All right angles are equal.

Euclid uses these axioms to prove some theorems. (His first theorem is that it is possible to construct an equilateral triangle with a given side.) He uses these theorems to prove other theorems, and so on.

Today it is often believed that Euclid's axioms are insufficient to the task of establishing all of his theorems of geometry. I would say rather that Euclid's methods of proof are not the same as the methods we use today.[4] In any case, the development of axiomatic set theory has been inspired in part by an analysis of Euclid.

Appendix A for the Greek alphabet.) The ordinary meaning of αἰτήμα is *request, demand.* A Latin translation of the word is *postulatum,* and the English translation *postulate* is derived from this. The Greek noun comes from the verb αἰτέω *ask.* The Greek ἀξίωμα means *that which is thought fit,* from the adjective ἄξιος *worthy,* from the verb ἄγω meaning *lead* etc. Several Greek compounds using this verb have found their way into English; one example is *pedagogy.*

[4]See Avigad *et al.,* 'A formal system for Euclid's *Elements*' [1], for a modern analysis of Euclid's methods. The abstract of this paper of 2009 reads: 'We present a formal system, E, which provides a faithful model of the proofs in Euclid's *Elements,* including the use of diagrammatic reasoning.' There is a sense in which the model is not faithful: its reliance on special symbolism—its very *formalism*—is foreign to the spirit of Euclid.

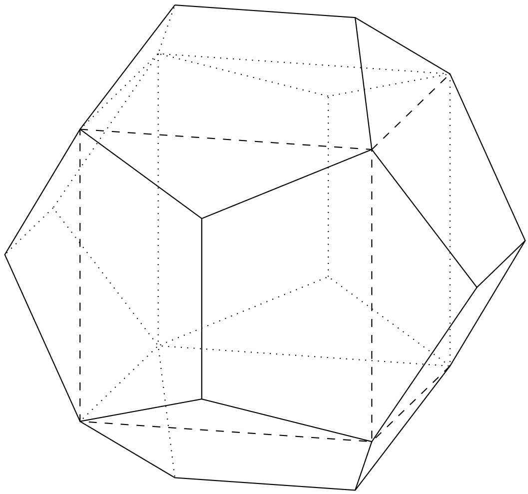

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Figure 1.2. The dodecahedron, about a cube

The text of Euclid's *Elements* that we have today begins, not with axioms, but with *definitions* of objects like points and lines. These definitions are never explicitly used to prove anything, and it is possible that they have been added to Euclid's original text by later editors. Let us call them *informal* definitions. We shall start our own work with some informal definitions.

One of my typographical conventions in this work is to put important technical terms in **boldface** when they are being defined, or when an an important example of their use is being given. A technical term may be in *italics* if it is of less importance, or if it is not yet being defined.

Some writers use the expression *if and only if* when making definitions. For example, they may write,

> An animal that walks on two legs is a human if and only if it has no feathers.[5]

[5] In the *Lives of the Eminent Philosophers* [11, 6.2.40], Diogenes Laërtius wrote the following about Diogenes of Sinope (today's Sinop): 'Plato had defined Man as an animal, biped and featherless, and was applauded. Diogenes plucked a fowl and brought it into the lecture-room with the words, "Here is Plato's man." In consequence of which there was added to the definition, "having broad nails."' I

However, if one knows that this is a definition, then the *and only if* is not needed; it is enough to say,

> An animal that walks on two legs is a **human** if it has no feathers.

Some (but not all) important definitions in this book are explicitly labelled as such:

Definition 0. An animal that walks on two legs is a **human** if it has no feathers.

Even without the labelling, the boldface typography of the key word should be enough to distinguish a definition.

If a theorem is given without a proof or with a sketchy proof, it is usually assumed that the reader can supply a proof or the missing details of the proof.

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do not take this anecdote to have much value beyond amusement value; Diogenes Laërtius lived some centuries after Diogenes of Sinope, and he does not document his claims. (The quotation was taken from the Perseus Digital Library `http://www.perseus.tufts.edu/`, February 17, 2011.)

2. Logic

ogic

2.1. Collections

sets

One of our earliest mathematical activities is *counting.* Counting is an activity involving a *thing* that is also many *things.* We may count the days in a week: the days are many things, but the week that they make up is one thing. We cannot count things unless we can also consider them together as one thing. I propose to refer to such a thing as a **collection.** A collection is made up of **individuals.** In counting a collection, we take its individuals one by one, while uttering words like *one, two, three,* and so on.

tion

The word *collection* is a **collective noun,** and I propose to use it as the most general collective noun. Other collective nouns are words like *pair, flock, deck* (of cards), *number* (of things), *group, family,* and so on.[1] In English, such nouns can be used as subjects of singular or plural verbs:

It's where my family lives.
It's where my family live.[2]

The individuals that make up a collection will be called **elements** or **members** of the collection. They are **in** the collection; they **belong to** the collection; and the collection **contains** them. Collections will be

[1] Despite the earlier example of days in a week, I do not think that *week* is a collective noun. We didn't count the week; we counted the collection of days in a week. Likewise, *meter* is not a collective noun, even though a meter is made up of 100 centimeters.

[2] From Evelyn Waugh's 1945 novel *Brideshead Revisited* (text taken from `http://www.en8848.com.cn/fiction/Fiction/Classic/2008-03-20/59319_4.html`, February 18, 2011):

'Well?' said Sebastian, stopping the car. Beyond the dome lay receding steps of water and round it, guarding and hiding it, stood the soft hills. 'Well?'

'What a place to live in!' I said.

'You must see the garden front and the fountain.' He leaned forward and put the car into gear. 'It's where my family live'; and even then, rapt in the vision, I felt, momentarily, an ominous chill at the words he used—not, 'that is my house', but 'it's where my family live'.

'Don't worry,' he continued, 'they're all away. You won't have to meet them.'

allowed to have just one element or no element. A collection is said to **consist of,** or **comprise,** its members, and the members are said to **compose** the collection.[3]

The members of a collection share some **property** with one another, but with nothing else. Two collections have the same **extension,** or are the same in extension, if they have exactly the same members. Two collections A and B differ in **intension** if the property shared by elements of A as such is different from the property shared by elements of B as such. Collections that differ in extension must differ in intension.

The converse fails. There may be two apparently different properties that are shared by exactly the same individuals. An example of two such properties is

1) being in Washington,
2) being in the District of Columbia.

The city of Washington, which is the capital of the United States of America, lies within a region called the District of Columbia, and this is why the city is referred to as Washington, D.C. Originally, the District also contained two other cities (namely Alexandria and Georgetown), along with unincorporated land. Today, the city of Washington has been enlarged, and the District shrunken, so that they have the same boundaries (which include Georgetown, but not Alexandria). The city and the District are today the same in extension. However, they differ in intension. By *Washington,* we refer to the capital of the USA; by *District of Columbia,* we refer merely to the area in which that capital lies.[4] Hence the collection of people living in Washington differs in intension, but not in extension, from the collection of people living in the District of Columbia.

Since a collection is a thing, it may itself be in a collection. There is no end to the creation of collections, even in extension, as the following theorem shows; Bertrand Russell wrote a version of this theorem in a letter [33] to Gottlob Frege in 1902.

Theorem 1 (Russell Paradox)**.** *For every collection A of collections,*

[3] Unfortunately the words *comprise* and *compose* are confused, even by native English speakers. The former, cognate with *comprehend,* has the root meaning of *take together;* the latter, *put together.*

[4] I do not know of a current *legal* distinction between the terms *Washington* and *District of Columbia.* In my experience, *Washington* may refer to the metropolitan region of which the city is the center; then *District* may be used to refer to the city itself.

there is some collection that differs in extension from every collection in A.

Proof. Let B be the collection of all collections in A that do not contain themselves. Suppose C is in A. We show that B differs in extension from C. If C contains itself, then B does not contain C. If C does not contain itself, then B contains C. In either case, C and B differ in extension. Thus A does not contain B. □

2.2. The class of all sets

Because of the Russell Paradox, it does not appear that we can develop a theory of all collections. We shall instead develop a theory of a certain kind of collection—a kind of collection that will be called a **set.** Sets will be certain collections considered in extension, not intension: two sets with the same members will be considered as the same set. Sets will have the peculiarity that all of their members are themselves sets.[5]

We shall start with a set with no members. There will be only one such set, called the *empty set,* denoted by $\varnothing$ or 0. From this we shall obtain the set, denoted by $\{0\}$, whose sole element is 0. This new set will also be called 1. Then we shall be able to obtain the set $\{0, 1\}$, also called 2, whose elements are 0 and 1. Continuing this way, we shall obtain *formal* definitions of each of the so-called *natural numbers.* A natural number n will be the set that we can denote by $\{0, \dots, n-1\}$. There will be other sets, such as $\{2\}$ and $\{1, 3\}$, which are not natural numbers.

The question will arise: How can we legitimately refer to the *collection* of natural numbers? What property do the the natural numbers share with one another?

Meanwhile, let us note another peculiarity of sets. Again, the number 2 will be the set $\{0, 1\}$, that is, $\{0, \{0\}\}$. In the latter expression, 0 occurs twice. But 0 is *one* set. The set 0 is both a member of, and a member of a member of, the set 2. However, 0 is *not* a member of the set $\{\{0\}\}$; the only member of *this* set is $\{0\}$, that is, 1. Such a situation does not often arise in ordinary life. If I put a spoon in a teacup, and the teacup

[5]Some writers allow sets to have members that are not sets. In that case, the sets that *we* shall use are called something like *hereditary sets* [26, p. 9] or *pure sets* [34, §9.1, p. 238].

in a cupboard, then the spoon is automatically counted as being in the cupboard. But sets do not work this way.

If b is a set, and a is a member of b, we write

$$a \in b;$$

if a is not a member of b, we write

$$a \notin b.$$

if $a \in b$, then a must be a set itself (since all members of sets are sets); even if $a \notin b$, we shall understand a to be a set in our system.

The symbol $\in$ is derived from the Greek minuscule letter epsilon (ε): this is the first letter of the Greek verb ἐστί, which just means *is.* The original idea[6] was that $a \in b$ means a is b, in the sense that a cat *is* a mammal: a is one of the b. This way of thinking is potentially ambiguous; for us, $a \in b$ means simply a is *in* b.

The symbol $\in$ by itself can be understood as denoting the *binary relation* of **membership** of one set to another. That is, a longer way to say that a is a member of b, or $a \in b$, is to say that a has the relation of membership to b.

We cannot expect the collection of all sets to be a set itself. It will be called a *class.* More precisely, the collection of all sets is the **universal class,** and we shall denote it by

$$\mathbf{V}.$$

This class will be our object of study. More precisely, the *ordered pair*

$$(\mathbf{V}, \in)$$

—the universal class, equipped with the relation of membership—will be our object of study. (We shall not try to say exactly how the class $\mathbf{V}$ and the relation $\in$ are put together to create the one thing denoted by $(\mathbf{V}, \in)$.)

[6] It was Peano's idea [31]. Writing in Latin, he said he would use ε to mean *est*—the Latin for *is.*

2.3. Logic

ogic

A general term for an object like $(\mathbf{V}, \in)$ is **structure.** High-school algebra and calculus can be understood as studies of the structure $(\mathbb{R}, +, \cdot, <)$, which is the *ordered field* of real numbers. (The reader is not required to know the definition of an arbitrary ordered field.) There is a certain priority of the former structure over the latter: In §6.7 we shall show how to define $\mathbb{R}$ as a particular set in $\mathbf{V}$. Indeed, every mathematical structure commonly encountered can be understood in terms of $(\mathbf{V}, \in)$, but not the other way around.

Every structure can be studied by means of a **(formal) logic.** The logic provides a precise way to state and prove what is *true* about a structure. We are interested mainly in a logic for $(\mathbf{V}, \in)$; but we shall make some comparisons with a logic for $(\mathbb{R}, +, \cdot, <)$.

A logic is a combination of **syntax**—formal rules for manipulating symbols—and **semantics**—ways that those symbols can have *meaning.*

Syntax

On the syntactic side, the logic has mechanical rules for constructing certain strings of symbols. Such strings will be called *formulas.* By additional mechanical rules, some formulas will be called *sentences.* By yet more mechanical rules, called **rules of inference,** certain strings of sentences will be called **formal proofs.** Each occurrence of a sentence in a formal proof can be called a **line** of the proof. (We thus imagine the formal proof as being written vertically, from top to bottom; each sentence occurring in the formal proof is written horizontally, left to right.) Certain lines of a formal proof will be its **hypotheses;** the last line will be the **conclusion** of the formal proof. The conclusion is then said to be **derivable** from the hypotheses. A line that is not an hypothesis is obtained from earlier lines by application of a rule of inference. In particular, an *initial segment* of a formal proof must be a formal proof. A conclusion that is derivable without any hypotheses can be called a **logical theorem.** If there are to be any logical theorems at all, there must be logical theorems that are one-line proofs of themselves; these are the **logical axioms.**

If a sentence σ is derivable from the collection Γ of sentences, we may write

$$\Gamma \vdash \sigma.$$

Here the symbol $\vdash$ can be called the **syntactic turnstile.**[7] In particular, if σ is a logical theorem, we may write $\vdash \sigma$.

Semantics

On the semantic side, the formulas are given some relation to the structure under study; in particular, each sentence is either *true* or *false* in the structure. But there is more. The structure to be studied by means of a given logic is an *interpretation* of the logic. In particular, it is the **intended interpretation** of the logic; but there will be other interpretations. For example, the logic of $(\mathbb{R}, +, \cdot, <)$ also has, as an interpretation, the *ordered semi-ring* $(\mathbb{N}, +, \cdot, <)$ of natural numbers. (The reader is not required to know the definition of an ordered semi-ring.) A sentence is **logically true** if it is true in every possible interpretation. An interpretation is a **model** of a collection of sentences if each of those sentences is true in the interpretation. A sentence is a **logical consequence** of a collection of sentences if the sentence is true in every model of the collection. In particular, a logically true sentences is just a logical consequence of the empty collection of sentences.

If a sentence σ is a logical consequence of the collection Γ of sentences, we may write

$$\Gamma \models \sigma.$$

Here the symbol $\vdash$ can be called the **semantic turnstile.** In particular, if σ is logically true, we may write $\models \sigma$.

A logic is **sound** if every sentence derivable from some collection of hypotheses is a logical consequence of those hypotheses. So a logic is sound if and only if, in all cases,

$$\text{if } \Gamma \vdash \sigma, \text{ then } \Gamma \models \sigma.$$

We shall want our logic for sets to be sound; but this will be easy to show. A sound logic is **complete** if, conversely, every logical consequence of some hypotheses is derivable from those hypotheses. So a sound logic is complete if and only if, in all cases,

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[7]It is called a turnstile simply for its appearance. The symbol appears to be derived from the notation of Frege [17].

$$\text{if } \Gamma \models \sigma, \text{ then } \Gamma \vdash \sigma.$$

Thus, in a complete sound logic, the symbols $\vdash$ and $\models$ have the same *extension.* We shall want our logic for sets to be complete as well. It will be; but we shall not prove this until §7.4.

For the intended interpretation of a logic, **(nonlogical) axioms** may be chosen: these are sentences that are known to be true in the intended interpretation. Knowledge of their truth is *not* obtained through the logic; it is just assumed. (Therefore the knowledge of the truth of the nonlogical axioms might not be called knowledge.) One may derive other sentences from the axioms; if the logic is sound, then these new sentences will also be true in the intended interpretation. This is the approach that we shall take to our understanding of $(\mathbf{V}, \in)$.

2.4. Formulas

ulas

Atomic formulas

In school algebra, one encounters *equations* like

$$ax^2 + bx + c = 0; \tag{2.1}$$

eqn:quad

in analytic geometry,

$$\ell x = y^2, \tag{2.2}$$

eqn:parab

$$a^2x^2 - b^2y^2 + c^2z^2 = 1. \tag{2.3}$$

eqn:hyp

Each of these equations is a formula in a logic for $(\mathbb{R}, +, \cdot\,)$. In such formulas, letters like a, b, c, and ℓ are used as *constants,* while letters like x, y, and z are used as *variables.*[8] Both the constants and the variables here denote real numbers. Somewhat imprecisely, we can describe the distinction between constants and variables by saying that a constant denotes a *particular* real number, while a variable denotes all real numbers—not considered as one thing, which is $\mathbb{R}$ itself, but considered as individuals. However, in (2.1) for example, we cannot say *which* particular real numbers are denoted by a, b, and c; what is of interest is that, whatever real numbers are denoted by a, b, and c, as long as they are not all 0, there are

[8]This distinction of letters at the *beginning* of the alphabet from letters at the *end* is made by René Descartes in his *Geometry* [10] of 1637.

at most two real numbers that can be denoted by x so that the equation is true.

In the logic of sets, we shall follow the same convention of using letters like a, b, and c as **constants,** and letters like x, y, and z as **variables.**[9] These letters, in their different ways, will denote *sets.* A **term** is a letter that is either a constant or a variable.[10] If t and u are terms, then the string

$$t \in u$$

is called an **atomic formula.** Now, the letters t and u here are not actually symbols of our logic; they just *denote* symbols of our logic, namely symbols such as a or x. These letters t and u then, as well as letters like φ and ψ as used below, can be called **syntactic variables.**[11] (Again, see Appendix A for all of the Greek letters.)

Examples of atomic formulas include $a \in b$ as above, but also

$$a \in a, \qquad x \in a, \qquad b \in y, \qquad x \in y, \qquad z \in z.$$

We may sometimes understand a constant like a as a syntactic variable denoting an arbitrary constant, and a variable like x as a syntactic variable denoting an arbitrary variable. For example, if we should refer to an

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[9]Thus our constants and variables are generally minuscule Latin letters. In §5.2 (p. 79), we start using minuscule Greek letters as constants for the sets called *ordinals.* In §7.1, we declare that our official constants are c_0, c_1, c_2, and so on; and variables, v_0, v_1, v_2, and so on. In §7.3 (p. 136), we start using (plainface) capital Latin letters as constants.

[10]I shall occasionally use the word *term* also as it is used in ordinary speech, as a word for a word or phrase that has a precise definition. I used *term* in this way in the Introduction.

[11]Or *syntactical variables.* Older logic books like Shoenfield [34, p. 7] and Church [7, p. 60] use this terminology; a newer book like Chiswell and Hodges [6] uses *metavariables.* Syntactical variables are part of the *syntax language;* Church [7, p. 60] traces the latter term to Carnap's *Logische Syntax der Sprache* (1934). In the 1937 translation of Carnap, one finds [5, §1, p. 4]: '. . . we are concerned with two languages: in the first place with the language which is the object of our investigation—we shall call this the **object-language**—and, secondly, with the language in which we speak *about* the syntactical forms of the object-language—we shall call this the **syntax-language.** As we have said, we shall take as our object-languages certain symbolic languages; as our syntax language we shall at first use the English language with the help of some additional Gothic symbols.' Our own object language consists of the formulas that we are in the process of defining.

atomic formula of the form $x \in a$, we mean this formula, but also $y \in a$, and $x \in b$, and $y \in b$, and so on.

Our atomic formulas can be called more precisely **atomic $\in$-formulas** (atomic epsilon-formulas), to distinguish them from atomic formulas in other logics. The equations (2.1), (2.2), and (2.3) are in fact atomic formulas of the usual logic of $\mathbb{R}$; in that logic, *polynomials* are terms.

Arbitrary formulas

In our logic of sets, we do *not* have equations among the atomic formulas. Rather, we shall use equations as abbreviations of certain non-atomic formulas; see §3.3. Our formulas in general are defined as follows.

Definition 1. A **formula,** or more precisely an **$\in$-formula** (epsilon-formula), is a string of symbols that can be built up by application of any of the following rules, as many times as desired:

1. An atomic formula is a formula.
2. If a string φ is a formula, then the string $\neg\varphi$ is a formula.
3. If strings φ and ψ are formulas, then the string $(\varphi \Rightarrow \psi)$ is a formula.
4. If a string φ is a formula, and x is a variable, then the string $\exists x\, \varphi$ is a formula.

In rule 2 of the definition, the formula $\neg\varphi$ is the **negation** of φ. The negation of an atomic formula $t \in u$ is normally written, not as $\neg t \in u$, but as

$$t \notin u.$$

In rule 3, the formula $(\varphi \Rightarrow \psi)$ can be called an **implication,** whose **antecedent** is φ and whose **consequent** is ψ. In rule 4, the formula $\exists x\, \varphi$ is an **instantiation**[12] of φ. Note here that x serves as a syntactic variable; the formulas $\exists y\, \varphi$ and $\exists z\, \varphi$ and so on are also instantiations of φ.

Note also for example that the string $\neg\varphi$ in rule 2 is not a string of two symbols, $\neg$ and φ; it is the string that begins with $\neg$ and continues with all of the symbols that are in the string called φ.

[12] Unlike *negation* and *implication,* the term *instantiation* does not appear to be in common use, although it is found in Shoenfield [34, p. 18]. The formula $\exists x\, \varphi$ will be understood to say that φ is true in some *instance*—that is, for some value of x; so it makes sense to call the formula an instantiation.

If a formula is defined using only the first three rules of the definition, let us say that the formula is **quantifier-free.**[13] In other words, a quantifier-free formula is a formula in which the symbol $\exists$ does not occur.

Formulas and numerals

I take formulas to be fundamental objects, more fundamental than numbers. We have just described how to obtain *all* of the $\in$-formulas. However, in English at least, there is no system for naming, in words, *all* of the numbers. Number names are not customarily repeated. Two twos are *four;* ten tens are a *hundred;* a thousand thousands are a *million.* A million millions were, in France in the 16th century, given the name *billion,* although this later came to understood as the name for a thousand millions. If we must speak of a billion billions, or a billion billion billions, then we do so; but if we must refer repeatedly to these numbers, we shall probably come up with new words for these numbers.

It is clearer in writing that all numbers can be named; but still it is not easy to write down the *algorithm* whereby all numbers can be written in order, as *numerals.* We must first understand that there are ten *digits.* These may best be understood as forming a circle as in Figure 2.1. For

0
9 1
8 2
7 3
6 4
5

fig:digits

Figure 2.1. The digits

each digit, there is a *next* digit, namely the digit that comes next, in the clockwise direction, around the circle. A numeral is a string of digits.

[13] Shoenfield uses the term *open* [34, p. 36]; it is faster to say than *quantifier-free,* but the meaning is not so obvious.

Each digit of the numeral occupies a *place* in the numeral. The *first* place is the leftmost place; the last place, the rightmost.[14] A numeral may not have 0 in the first place.

The numerals themselves have an ordering, in which the first numeral is 1. Given a numeral, we obtain the next numeral as follows. We replace the digit in the last place with its next digit in the circle. If this next digit is 0, then we also replace the digit in the next-to-last place of the numeral with *its* next digit in the circle. If *this* next digit is 0, then we replace the digit in the next place to the left with *its* next digit, and so on. If the digit in the first place becomes 0 in this process, then a new first place is added to the left, and the digit 1 is placed there.

Of course the reader knows all of this implicitly; the point is that it is much harder to write down than the definition of an $\in$-formula.

Perhaps the first numerals to arise historically, and the easiest to describe, are just strings of marks: **|**, **||**, **|||**, **||||**, and so on. We can define these strings by:

1. **|** is a numeral.
2. If a string s is a numeral, then so is s**|**.

Our definition of formulas, Definition 1, is only slightly more sophisticated than this. The definition of formula *does* assume we have indefinite lists of constants and variables. We *could* assume that our constants are a, a', a'', and so on; and our variables, x, x', x'', and so on. That is, we could use the definition:

1. a is a constant, and x is a variable.
2. If s is a constant or variable, then s' is respectively a constant or variable.

But there is no need now to be this precise about defining constants and variables.[15]

2.5. Recursion and induction

Definition 1, of *formula* (more precisely, $\in$*-formula*), makes it possible to prove theorems about the collection of formulas that will be useful for us.

---

[14] It appears that Europeans learned these numerals originally from *The Short Treatise on Hindu Reckoning,* written by Muḫammad ibn Mūsā al-Khwārizmi around the year 825 [25, p. 525]. For him, the first place was the rightmost place; but he was writing left to right, in Arabic.

[15] Again, we shall be more precise in §7.1.

The definition of *formula* is the kind of definition that can be described by either of two adjectives: it is

- **recursive,** because it involves *recurrent* (repeated) application of certain rules;
- **inductive,** because theorems about the collection of things with the given definition can by proved by *induction.*

I prefer to call such a definition *recursive,* leaving the word *inductive* to describe the kind of proof that the definition allows.

For example, the natural numbers can be given a recursive definition:

1. 0 is a natural number.
2. If n is a natural number, then so is $n+1$.

Such a definition will be made officially as Definition 10 in §4.1, and then the collection of natural numbers will be called $\mathbb{N}$. Our theoretical work in this chapter and the next will give clear meanings to the expressions 0 and $n+1$. In Definition 13, we shall have a non-recursive definition of a class denoted by ω, and then the classs denoted by $\bigcup\omega$ will be called the class of *formal natural numbers.* We shall show that we *may* assume $\mathbb{N}$ is just $\bigcup\omega$; but we cannot *prove* that they are the same. After Theorem 48, we shall know that $\bigcup\omega$ and ω are the same class.

Meanwhile, we do not officially have the natural numbers. Unofficially, we may note that their recursive definition above makes possible the usual method of inductive proof, whereby every natural number has some property if

1) 0 has the property, and
2) $n+1$ has the property on the assumption that n has the property.

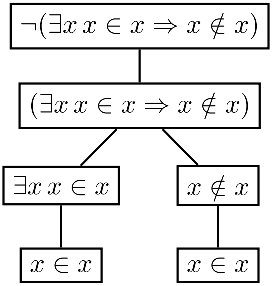
In general, an inductive proof has as many parts as the corresponding recursive definition. Our first example of an inductive proof concerning formulas will be Lemma 2 below; the proof will have four parts, like Definition 1.

When a string is a formula, then the history of its construction *as* a formula can be shown in a *tree,* called the **parsing tree** of the formula.[16] For example, the parsing tree for the formula

$$\neg(\exists x\, x \in x \Rightarrow x \notin x)$$

[16] I take the terminology from Chiswell and Hodges [6], who make much use of parsing trees. Strictly, such trees are examples of *labelled* trees. The Wikipedia article on parsing trees in the present sense has the heading *Parse tree* (on March 6, 2011), although I added the alternative form *Parsing tree.*

is as in Figure 2.2. In the definition of formula, if implications did not



tree

Figure 2.2. A parsing tree

have parentheses, then the parsing tree for a given formula might not be *unique;* there might be more than one way to construct the same formula. With the definition of formula as it is, the parsing tree *is* unique. This is a consequence of Theorem 2 below, which takes a bit of work to prove. *Part* of the proof is easy; it is the following immediate consequence of the definitions:

em:4

Lemma 1.

1. *Atomic formulas begin with terms.*
2. *Negations begin with* $\neg$.
3. *Implications begin with* (.
4. *Instantiations begin with* $\exists$.

Note well that none of the symbols $\neg$, (, and $\exists$ is a term, and none is the same as any other. So an atomic formula cannot also be a negation, an implication, or an instantiation; a negation cannot also be an implication or an instantiation; and an implication cannot be an instantiation.

There remains the question of whether an implication can be an implication in more than one way. Does an implication have a unique antecedent and consequent? We shall be able to settle this question by means of the next lemma.

We shall use the following terminology. A **proper initial segment** of a string is the string that results from deleting one or more (but not all) symbols from the end. An **initial segment** of a string is either a proper initial segment of the string, or the string itself.

lem:pis **Lemma 2.** *No proper initial segment of a formula is a formula.*

Proof. We prove the lemma in the following version: for all formulas φ,

- no proper initial segment of φ is a formula, and
- φ is not a proper initial segment of a formula.

By Lemma 1, if an initial segment of an atomic formula is itself a formula, it must be an atomic formula; if of a negation, a negation; and so on. Now we can complete the proof by means of induction. Since the recursive definition of formula has four parts, so will our inductive proof.

1. Our claim holds for atomic formulas, since each of these is exactly three symbols long.

2. If the claim holds for φ, then it holds for $\neg\varphi$. Indeed, every proper initial segment of $\neg\varphi$ is $\neg S$ for some proper initial segment S of φ. If S must not be a formula, then $\neg S$ is not a formula either. Similarly, every string of which $\neg\varphi$ is a proper initial segment is $\neg T$ for some string T of which φ is a proper initial segment; if T must not be a formula, then neither is $\neg T$ a formula.

3. Suppose the claim holds for φ and ψ. Then it holds for $(\varphi \Rightarrow \psi)$. Indeed, if an initial segment of the last formula is a formula itself, then it must be $(\theta \Rightarrow \rho)$ for some formulas θ and ρ. If θ and φ are the same formula, then ρ is an initial segment of ψ. The other possibility is that θ is an initial segment of φ, or φ is an initial segment of θ. By our inductive hypothesis, θ must be φ, and then ρ must be ψ. Similarly, if $(\varphi \Rightarrow \psi)$ is an initial segment of a formula, then that formula must be $(\varphi \Rightarrow \psi)$ itself.

4. Finally, if the claim holds for φ, then it holds for $\exists x\, \varphi$, just as it holds for $\neg\varphi$. This completes the induction. □

The following will justify various recursive definitions of *functions* on the collection of formulas.

thm:ur **Theorem 2** (Unique readability). *Each formula is of only one of the four kinds: atomic formulas, negations, implications, and instantiations. Moreover, a formula is of one of these kinds in only one way. In particular, if φ, ψ, θ, and ρ are formulas, and the two implications $(\varphi \Rightarrow \psi)$ and $(\theta \Rightarrow \rho)$ are the same formula, then φ and θ are the same formula, and so are ψ and ρ.*

Proof. The first claim follows from Lemma 1. The second claim follows from Lemma 2, since (in the notation of the claim) one of φ and θ is an initial segment of the other, so they are the same, and hence ψ and ρ are the same. □

So far we have worked out the **syntax** of formulas: the rules for their construction, and some consequences of these rules. The next job is to work out the **semantics** of formulas: what they *mean,* and which of them can be called *true.* The distinction between syntax and semantics is not always clear. In §2.7, we shall develop two notions: *logical entailment,* and *syntactic derivation.* The former notion is semantic; the latter (of course), syntactic. The two notions differ in intension; but they will turn out (in §7.4) to be the same in extension.

2.6. Sentences

:s&t

Quantifier-free formulas that have no variables are *quantifier-free sentences.* Among such formulas are the atomic formulas $a \in b$, which we may obviously call **atomic sentences.** We have just given a non-recursive definition of the quantifier-free sentences. There is also a recursive definition:

Definition 2. The **quantifier-free sentences** are given by the following rules.

1. Every atomic sentence is a quantifier-free sentence.
2. If σ is a quantifier-free sentence, then so is $\neg\sigma$.
3. If σ and τ are quantifier-free sentences, then so is $(\sigma \Rightarrow \tau)$.

We know what it means for atomic sentences to be true or false. We extend the definition as follows.

ruth **Definition 3.** A quantifier-free sentence is **false** if it is not *true;* and quantifier-free sentences are **true** under the conditions given by the following rules.

:aft 1. An atomic sentence $a \in b$ is true if $a \in b$. (That is, $a \in b$ is true if and only if the set denoted by the constant a is a member of the set denoted by the constant b.)
2. A quantifier-free sentence $\neg\sigma$ is true if σ is false.
imp2 3. If σ and τ are quantifier-free sentences, then $(\sigma \Rightarrow \tau)$ is true if either σ is false or τ is true.

This definition is **recursive.** It is not a recursive definition of a *collection.* Rather, it is a recursive definition of a *function,* namely a function on the collection of quantifier-free sentences. This collection has the recursive definition above, in three parts, and the definition of the function on this collection has three corresponding parts. But we must check that the definition of the function is valid: we must check that there really is such a function. In the definition, rule 3 assumes that σ and τ are uniquely determined by the whole formula $(\sigma \Rightarrow \tau)$. This assumption is justified by Theorem 2. Some books overlook the need for such justification; but if implications did not have parentheses, then truth could not be unambiguously defined.

If σ is true, we may write simply σ (as we did in rule 1). Then $(\sigma \Rightarrow \tau)$ is true if and only if the English sentence

If σ, then τ

is true. We can compute whether an arbitrary quantifier-free sentence is true or false by means of a *truth table.* The reader may well be familiar with truth-tables; but different writers treat them differently. I understand them as follows.

In the parsing tree for any formula, the various formulas that occur are just the **subformulas** of the original formula. Each subformula that is not atomic is obtained from one or two other subformulas by application of one of the symbols $\neg$, $\Rightarrow$, and $\exists$. (We also add parentheses when the symbol is $\Rightarrow$, and we add a variable when the symbol is $\exists$.) If the original formula is a quantifier-free sentence σ, then all of the subformulas are quantifier-free sentences. We consider one of these quantifier-free sentences to have the value 1 if it is true, 0 if it is false.[17] This value, 1 or 0, is the **truth value** of the sentence; a sentence denotes its truth value, as a constant denotes a set. We can compute the truth values of the subformulas of σ in turn, from the atomic subformulas all the way up to σ itself. Suppose, in the construction of σ, we have used letters like P, Q, and R in place of the atomic sentences: we can think of these letters as syntactic variables, either for atomic sentences, or for their possible truth values. Then each subformula corresponds to a single symbol in σ: either one of the letters just mentioned, or $\neg$, or $\Rightarrow$. We can write out the whole of σ, and write the values of its subformulas under the

[17]Some writers, as Stoll [36, Ch. 4, Exercise 3.7], use 0 and 1 in the opposite sense.

corresponding symbols. We may include all possible values of the atomic sentences that occur; then we get a **truth table.**

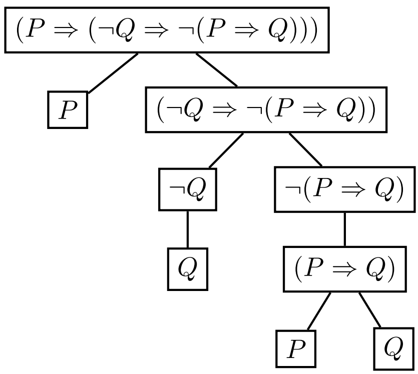
The rules of computation with truth values are shown in Table 2.1. The

$\neg$	σ
1	0
0	1

$(\sigma$	$\Rightarrow$	$\tau)$
0	1	0
1	0	0
0	1	1
1	1	1

:btt Table 2.1. The two basic truth tables

parsing tree of a particular quantifier-free sentence is shown in Figure 2.3; the truth table of this quantifier-free sentence is worked out in stages in



tree Figure 2.3. The parsing tree of a quantifier-free sentence

Table 2.2; the truth table itself is in Table 2.3. This particular quantifier-free sentence, $(P \Rightarrow (\neg Q \Rightarrow \neg(P \Rightarrow Q)))$, happens to take the value 1, no matter what values are assigned to the atomic sentences P and Q; therefore it can be called a *tautology,* or more precisely a **propositional tautology** (we shall define tautologies a bit more generally in the next section). A quantifier-free sentence that always takes the value 0 is a

$(P$	$\Rightarrow$	$(\neg$	Q	$\Rightarrow$	$\neg$	$(P$	$\Rightarrow$	$Q)))$
0			0			0		0
1			0			1		0
0			1			0		1
1			1			1		1
0		1	0			0	1	0
1		1	0			1	0	0
0		0	1			0	1	1
1		0	1			1	1	1
0		1	0		0	0	1	0
1		1	0		1	1	0	0
0		0	1		0	0	1	1
1		0	1		1	1	1	1
0		1	0	0	0	0	1	0
1		1	0	1	1	1	0	0
0		0	1	1	0	0	1	1
1		0	1	1	1	1	1	1
0	1	1	0	0	0	0	1	0
1	1	1	0	1	1	1	0	0
0	1	0	1	1	0	0	1	1
1	1	0	1	1	1	1	1	1

tab:truth

Table 2.2. The filling-out of a truth table

(propositional) contradiction.

Arbitrary sentences

Defining *sentences* in general, and their truth values, will take more work.

Definition 4. An arbitrary $\in$-formula is a **sentence,** or more precisely an **$\in$-sentence,** if it has no *free* variables. The collection of **free variables** of a formula is defined recursively:

1. The free variables of an atomic formula are just the variables that occur in the atomic formula.
2. The free variables of $\neg\varphi$ are the free variables of φ.
3. The free variables of $(\varphi \Rightarrow \psi)$ are the free variables of φ or ψ.

part:fv3

$(P$	$\Rightarrow$	$(\neg$	Q	$\Rightarrow$	$\neg$	$(P$	$\Rightarrow$	$Q)))$
0	1	1	0	0	0	0	1	0
1	1	1	0	1	1	1	0	0
0	1	0	1	1	0	0	1	1
1	1	0	1	1	1	1	1	1

b:tt

Table 2.3. A truth table

4. The free variables of $\exists x\,\varphi$ are those of φ, except x.

Here again, part 3 relies on Theorem 2. To define truth and falsity of arbitrary $\in$-sentences, we must deal with a complication. The same variable may occur several times in a formula. We therefore distinguish between:

1. A variable that occurs in a formula.
2. The particular occurrences of that variable in the formula.

So for example only one variable occurs in the formula $x \in x$, but this variable has two occurrences in the formula.

Every occurrence of a variable x in an instantiation $\exists x\,\psi$ is said to be a *bound occurrence.*[18] Evidently x occurs at least once in this formula, but it might occur more than once; each of these occurrences is bound.

The definition of bound occurrences of variables in arbitrary formulas is as follows. First note that every occurrence of a variable x in a formula φ is also an occurrence in one or more subformulas of φ. If one of these subformulas is an instantiation $\exists x\,\psi$, then the occurrence of x in φ is a **bound occurrence.** Occurrences that are not bound are **free occurrences.**

For example, the formula $\neg(\exists x\, x \in x \Rightarrow x \notin x)$ has the free variable x; but only the last two occurrences of x are free; the first three are bound. Thus it is possible that, in a formula, some *occurrences* of a free variable are bound and not free. In this case, the variable is free only because some *other* occurrences are free.

[18] A bound variable is so called because the symbol $\exists$ **binds** it—ties it down. The relevant verb is *bind, bound, bound,* whose ancestor is found in Old English (that is, English as spoken before the Norman Conquest of 1066). There is an unrelated verb *bound, bounded, bounded,* which is also used in mathematics; this verb is derived from the noun *bound,* which came to English from French (more precisely, Anglo-Norman) in the 13th century [22].

If φ is a formula in which the same variable has both free and bound occurrences, or in which the string $\exists x$ occurs twice for the same variable x, we may refer to φ as a **bad formula.** It is still a formula, just a bad one. A **good formula** then is a formula that is not bad. It is possible to prove that everything we do with formulas can be done with good formulas only. We can also define good formulas without first defining formulas. However, we shall not bother to do these things; we shall just avoid writing down any bad formulas.

Suppose φ is a **singulary formula,** namely a formula with just one free variable; and let that variable be x. (For example, the formula could be $x \in x$ or $\exists y\, y \in x$.) Then we may denote φ by

$$\varphi(x).$$

If t is a term, and if we replace every *free* occurrence of x in φ with t, we obtain the formula denoted by

$$\varphi(t).$$

We obtain $\varphi(t)$ from $\varphi(x)$ by **substitution** of t for x.

Suppose $\varphi(x)$ is a good formula. Then $\varphi(t)$ is the result of replacing *every* occurrence of x in φ with t. However, if y is a variable other than x that occurs in $\varphi(x)$, then $\varphi(y)$ need not be a good formula. Such is the case when φ is $\exists y\, y \in a \;\&\; x \in z$. In practice, we shall always avoid this situation.

We use the letters σ and τ as syntactic variables for $\in$-sentences. Now we can define truth and falsity of such sentences.

`defn:truth2` **Definition 5.** An $\in$-sentence is **false** if it is not *true.* An $\in$-sentence is **true** according to the following rules, which include the rules given above for truth of quantifier-free sentences:

1. An atomic sentence $a \in b$ is true if $a \in b$.
2. A sentence $\neg\sigma$ is true if σ is false.
3. A sentence $(\sigma \Rightarrow \tau)$ is true if either σ is false or τ is true.
4. `part:eq` A sentence $\exists x\, \varphi(x)$ is true if there exists a set a such that $\varphi(a)$ is true.

Because of rule 4, the sentence $\exists x\, \varphi(x)$ is said to result from the formula $\varphi(x)$ by **existential quantification** of the variable x.

To continue our investigations, it is convenient to have some abbreviations of formulas:

Definition 6.

1. For $(\neg\varphi \Rightarrow \psi)$, we write
$$(\varphi \vee \psi).$$
2. For $\neg(\neg\varphi \vee \neg\psi)$, we write
$$(\varphi \mathbin{\&} \psi).$$
3. For $((\varphi \Rightarrow \psi) \mathbin{\&} (\psi \Rightarrow \varphi))$, we write
$$(\varphi \Leftrightarrow \psi).$$
4. For $\neg\exists x\, \neg\varphi$, we write
$$\forall x\, \varphi.$$

The abbreviations so defined are, respectively, **disjunctions, conjunctions, equivalences,** and **generalizations.**

Let us acknowledge that these abbreviations mean what they are supposed to mean:

hm:B

Theorem 3. *Suppose σ and τ are sentences, and $\varphi(x)$ is a singulary formula.*

1. *The sentence $(\sigma \vee \tau)$ is true if and only if at least one of the two sentences σ and τ is true.*
2. *The sentence $(\sigma \mathbin{\&} \tau)$ is true if and only if both sentences σ and τ are true.*
3. *The sentence $(\sigma \Leftrightarrow \tau)$ is true if and only if either both sentences σ and τ are true or both are false.*
4. *The sentence $\forall x\, \varphi(x)$ is true if and only if, for each set a, the sentence $\varphi(a)$ is true.*

The following establishes a useful abbreviating convention for writing formulas.

hand

Definition 7.

1. We need not write the outer parentheses of a formula (if it has them).
2. We can remove internal parentheses by understanding & and $\vee$ to have priority over $\Rightarrow$ and $\Leftrightarrow$, so that for example $\varphi \mathbin{\&} \psi \Rightarrow \chi$ means $(\varphi \mathbin{\&} \psi) \Rightarrow \chi$, which in turn means $((\varphi \mathbin{\&} \psi) \Rightarrow \chi)$.
3. When the symbol $\Rightarrow$ is repeated, the occurrence on the right has priority, so $\varphi \Rightarrow \psi \Rightarrow \chi$ means $\varphi \Rightarrow (\psi \Rightarrow \chi)$.

2.7. Formal proofs

::set-theory

We now establish rules for writing formal proofs in our logic of sets.

Definition 8. A sentence is called a **tautology** if it can be obtained from a propositional tautology σ by assigning to each atomic sentence ρ that occurs in σ an arbitrary sentence $\tilde{\rho}$, then replacing each occurrence of ρ in σ with $\tilde{\rho}$. Our **logical axioms** are:

1) the tautologies,
2) the sentences $\varphi(a) \Rightarrow \exists x\, \varphi(x)$.

We name two rules of inference:

1. **Detachment:** from σ and $\sigma \Rightarrow \tau$, infer τ.
2. **$\exists$-Introduction:** from $\varphi(a) \Rightarrow \sigma$, where a does not occur in $\varphi(x)$ or σ, infer $\exists x\, \varphi(x) \Rightarrow \sigma$.

Then the **logical theorems**[19] are defined recursively so that:

1. Every logical axiom is a logical theorem.
2. Every sentence that can be inferred from two logical theorems by Detachment is a logical theorem.
3. Every sentence that can be inferred from a logical theorem by $\exists$-Introduction is a logical theorem.

If σ is a logical theorem, then as in §2.3, we may express this by writing

$$\vdash \sigma.$$

A **formal proof** of a sentence σ from no hypotheses is a string of sentences, ending with σ, in which each entry is either:

1. a logical axiom, or
2. inferrable from earlier sentences by Detachment, or
3. inferrable from an earlier sentence by $\exists$-Introduction.

Theorem 4. *A sentence is a logical theorem if and only if it has a formal proof from no hypotheses.*

Proof. By induction, every logical theorem has a formal proof from no hypotheses:

[19]Some sources, such as Shoenfield [34], will refer to logical theorems simply as *theorems;* but they should be distinguished from the sentences in ordinary language (with some symbolism) that are labelled as theorems in books of mathematics like the present one.

1. A logical axiom is is a one-line formal proof of itself from no hypotheses.
2. If σ is a logical theorem that has a formal proof S from no hypotheses, and $\sigma \Rightarrow \tau$ is a logical theorem has a formal proof T from no hypotheses, then the string

$$S$$
$$T$$
$$\tau$$

   is a formal proof of τ from no hypotheses.
3. If $\varphi(a) \Rightarrow \sigma$ is a logical theorem that has a formal proof S from no hypotheses, and a does not occur in $\varphi(x)$ or σ, then

$$S$$
$$\exists x\, \varphi(x) \Rightarrow \sigma$$

   is a formal proof of $\exists x\, \varphi(x) \Rightarrow \sigma$ from no hypotheses.

Also, the conclusion of every formal formal proof from no hypotheses is a logical theorem. To prove this, suppose S is a formal proof with conclusion σ from no hypotheses, and the conclusion of every proper initial segment of S is a logical theorem. There are three possibilities for σ, and in each case, σ must be a logical theorem. □

:vlt **Theorem 5.** *Every logical theorem is true under every interpretation of its constants.*

Proof. We use induction.

1. I take it as obvious that the tautologies are true. A sentence $\varphi(a) \Rightarrow \exists x\, \varphi(x)$ is true by definition of the truth of instantiations.

2. If σ and $\sigma \Rightarrow \tau$ are logical theorems that are true, then τ is true, by definition of the truth of implications.

3. Suppose $\varphi(a) \Rightarrow \sigma$ is a logical theorem that is true under every interpretation of a, and a does not occur in $\varphi(x)$ or σ. If $\exists x\, \varphi(x)$ is false, then $\exists x\, \varphi(x) \Rightarrow \sigma$ is true. Suppose $\exists x\, \varphi(x)$ is true. Then $\varphi(b)$ is true for some set b. Consequently $\varphi(a)$ is true when a is interpreted as b. Then σ must be true. In this case too then, $\exists x\, \varphi(x) \Rightarrow \sigma$ is true. □

Note the importance of the several conditions in the third part of the proof:

1. We require that a not occur in $\varphi(x)$, because if σ is a false sentence, and b contains a but not c, then $(a \in b \Rightarrow a \notin b) \Rightarrow \sigma$ is true, but not $\exists x\,(x \in b \Rightarrow a \notin b) \Rightarrow \sigma$: here a still occurs in $x \in b \Rightarrow a \notin b$.

2. We require that a not occur in σ, because if b contains c but not a, then $a \in b \Rightarrow a \in b$ is true, but not $\exists x\, x \in b \Rightarrow a \in b$; here a occurs in $a \in b$.

3. We require that $\varphi(a) \Rightarrow \sigma$ be true under every interpretation of a, because if b contains c but not a, then $a \in b \Rightarrow c \notin b$ is true, but not $\exists x\, x \in b \Rightarrow c \notin b$; here $a \in b \Rightarrow c \notin b$ is false when a is interpreted as c.

Definition 9. If Γ is a collection of sentences, then the sentences that are **derivable** from Γ are defined recursively as follows:

1. Every logical theorem is derivable from Γ.
2. Every element of Γ is derivable from Γ.
3. Every sentence that can be inferred from two elements of Γ by Detachment is derivable from Γ.

If σ is derivable from Γ, then as in §2.3, we may express this by writing

$$\Gamma \vdash \sigma.$$

A **formal proof** of a sentence σ from a collection Γ of sentences is a string of sentences, ending with σ, in which each entry is either:

1. a logical theorem, or
2. inferrable from earlier sentences by Detachment.

thm:der->con

Theorem 6. *A sentence is derivable from a collection of sentences if and only if the sentence has a formal proof from the collection.*

thm:dt

Theorem 7. *Every sentence that is derivable from a collection of true sentences is true.*

We have now completed the definition of our logic for sets. An **interpretation** for this logic is just a collection with a binary relation. The definition of truth in an arbitrary interpretation is analogous to Definition 3. The notion of logical consequence, and therefore of soundness, defined in §2.3 relies on consideration of all possible interpretations. Now, there is no *collection* of all collections. Nonetheless, we have the following.

Theorem 8. *Our logic for sets is sound.*

Proof. The proofs of Theorems 5 and 7 use no special properties of $(\mathbf{V}, \in)$; indeed, all we know about this structure so far is that it is a collection with a binary relation on it. But this is just what an arbitrary interpretation of our logic is. □

Our logic is minimal, in the sense that its axioms and rules of inference are chosen just so that the proof of the Completeness Theorem in §7.4 will work. Consequently, formal proofs of interesting results will be very long. However, there are various ways of showing that formal proofs exist, without actually writing out the formal proofs. We establish some of those ways now.

Theorem 9 (Contraposition).

1. *If* $\Gamma \vdash \sigma \Rightarrow \tau$*, then* $\Gamma \vdash \neg\tau \Rightarrow \neg\sigma$.
2. *If* $\Gamma \vdash \neg\sigma \Rightarrow \tau$*, then* $\Gamma \vdash \neg\tau \Rightarrow \sigma$.
3. *If* $\Gamma \vdash \sigma \Rightarrow \neg\tau$*, then* $\Gamma \vdash \tau \Rightarrow \neg\sigma$.
4. *If* $\Gamma \vdash \neg\sigma \Rightarrow \neg\tau$*, then* $\Gamma \vdash \tau \Rightarrow \sigma$.

Proof. For the first part, note that $(\sigma \Rightarrow \tau) \Rightarrow \neg\tau \Rightarrow \neg\sigma$ is a tautology and is therefore derivable from Γ. By Detachment, $\neg\tau \Rightarrow \neg\sigma$ is derivable from Γ. The remaining parts are similar. □

Theorem 10 (Syllogism). *If* $\Gamma \vdash \rho \Rightarrow \sigma$ *and* $\Gamma \vdash \sigma \Rightarrow \tau$*, then* $\Gamma \vdash \rho \Rightarrow \tau$.

Proof. Use the tautology $(\rho \Rightarrow \sigma) \Rightarrow (\sigma \Rightarrow \tau) \Rightarrow \rho \Rightarrow \tau$ and Detachment. □

Theorem 11 (Cases). *If* $\Gamma \vdash \sigma \Rightarrow \tau$ *and* $\Gamma \vdash \neg\sigma \Rightarrow \tau$*, then* $\Gamma \vdash \tau$.

Such theorems can be multiplied as needed, in order to show that familiar moves in informal proofs can be translated into formal proofs. Such a move that is made all the time is to assume the antecedent of an implication as an hypothesis, prove the consequent as a consequence of that hypothesis, then conclude the original implication itself. The formalization of this move is the following, where $\Gamma \cup \{\sigma\}$ is the collection obtained from Γ by adding σ.

Theorem 12 (Deduction). *If* $\Gamma \cup \{\sigma\} \vdash \tau$*, then*

$$\Gamma \vdash \sigma \Rightarrow \tau.$$

Proof. Suppose $\Gamma \cup \{\sigma\} \vdash \tau$. We prove $\Gamma \vdash \sigma \Rightarrow \tau$ by induction.

1. If $\vdash \tau$, then, since $\tau \Rightarrow \sigma \Rightarrow \tau$ is a tautology, we have $\vdash \sigma \Rightarrow \tau$, and therefore $\Gamma \vdash \sigma \Rightarrow \tau$.

2. Suppose τ is in $\Gamma \cup \{\sigma\}$. There are two cases to consider.

a) If τ is in Γ, then $\Gamma \vdash \tau$, but then $\Gamma \vdash \sigma \Rightarrow \tau$ as before.

b) If τ is σ, then $\sigma \Rightarrow \tau$ is a tautology, so $\Gamma \vdash \sigma \Rightarrow \tau$.

3. Suppose both ρ and $\rho \Rightarrow \tau$ are derivable from $\Gamma \cup \{\sigma\}$, and $\sigma \Rightarrow \rho$ and $\sigma \Rightarrow \rho \Rightarrow \tau$ are derivable from Γ. Since the sentence

$$(\sigma \Rightarrow \rho) \Rightarrow (\sigma \Rightarrow \rho \Rightarrow \tau) \Rightarrow \sigma \Rightarrow \tau$$

is a tautology, $\Gamma \vdash \sigma \Rightarrow \tau$. □

The converse is easy. By the Deduction Theorem, we can rewrite the theorem on cases as

$$\text{If } \Gamma \cup \{\sigma\} \vdash \tau \text{ and } \Gamma \cup \{\neg\sigma\} \vdash \tau, \text{ then } \Gamma \vdash \tau.$$

We can now establish results like the following fairly easily.

Theorem 13. *For all singulary formulas $\varphi(x)$, for all sentences τ,*

$$\vdash (\exists x\, \varphi(x) \Rightarrow \tau) \Rightarrow \exists x\, (\varphi(x) \Rightarrow \tau).$$

Proof. By the Deduction Theorem, it is enough to show

$$\{\exists x\, \varphi(x) \Rightarrow \tau\} \vdash \exists x\, (\varphi(x) \Rightarrow \tau).$$

It is therefore enough to establish the two cases,

$$\{\exists x\, \varphi(x) \Rightarrow \tau, \exists x\, \varphi(x)\} \vdash \exists x\, (\varphi(x) \Rightarrow \tau),$$
$$\{\exists x\, \varphi(x) \Rightarrow \tau, \neg\exists x\, \varphi(x)\} \vdash \exists x\, (\varphi(x) \Rightarrow \tau).$$

For the first case, it is enough to note

$$\{\exists x\, \varphi(x) \Rightarrow \tau, \exists x\, \varphi(x)\} \vdash \tau,$$
$$\vdash \tau \Rightarrow \varphi(a) \Rightarrow \tau,$$
$$\vdash (\varphi(a) \Rightarrow \tau) \Rightarrow \exists x\, (\varphi(x) \Rightarrow \tau),$$

where a is an arbitrary constant. For the second case, by Syllogism, it is enough to note

$$\vdash \neg\exists x\,\varphi(x) \Rightarrow \neg\varphi(a),$$
$$\vdash \neg\varphi(a) \Rightarrow \varphi(a) \Rightarrow \tau,$$
$$\vdash \varphi(a) \Rightarrow \tau \Rightarrow \exists x\,(\varphi(x) \Rightarrow \tau). \qquad \square$$

Compare the foregoing proof with the formal proof of the same result in Table 2.4.

More useful results about formal proofs are as follows.

Theorem 14. *If σ is a sentence in which x occurs, but y does not, and the sentence σ' is obtained from σ by replacing* each *occurrence of x with y, then*

$$\vdash (\sigma \Rightarrow \sigma').$$

If also σ is a good *sentence, then so is σ'.*

Proof. TO BE ADDED $\square$

In the theorem, note that $(\sigma \Rightarrow \sigma')$ is not a good sentence unless no bound variable other than x occurs in σ. Indeed, if z is not x, and $\exists z$ occurs in σ, then it is repeated in σ'.

Another standard tool in deriving logical consequences is the following.

Theorem 15. *These sentences are logical theorems:*

$$(\forall x\,\neg\varphi(x) \Leftrightarrow \neg\exists\varphi\,(x)), \qquad (\exists x\,\neg\varphi(x) \Leftrightarrow \neg\forall x\,\varphi(x)).$$

Proof. We have

$$\vdash (\varphi(a) \Rightarrow \exists x\,\varphi(x)),$$
$$\vdash (\neg\neg\varphi(a) \Rightarrow \varphi(a)),$$
$$\vdash ((\neg\neg\varphi(a) \Rightarrow \varphi(a)) \Rightarrow ((\varphi(a) \Rightarrow \exists x\,\varphi(x)) \Rightarrow (\neg\neg\varphi(a) \Rightarrow \exists x\,\varphi(x)))),$$
$$\vdash ((\varphi(a) \Rightarrow \exists x\,\varphi(x)) \Rightarrow (\neg\neg\varphi(a) \Rightarrow \exists x\,\varphi(x))),$$
$$\vdash (\neg\neg\varphi(a) \Rightarrow \exists x\,\varphi(x)).$$

Assuming a does not occur in $\varphi(x)$, we have then

$$\vdash (\exists x\,\neg\neg\varphi(x) \Rightarrow \exists x\,\varphi(x)).$$

Similarly,

$$\vdash (\exists x\,\varphi(x) \Rightarrow \exists x\,\neg\neg\varphi(x)).$$

By using the tautology $((P \Rightarrow Q) \Rightarrow ((Q \Rightarrow P) \Rightarrow (P \Leftrightarrow Q)))$, we obtain

$$\vdash (\exists x\,\neg\neg\varphi(x) \Leftrightarrow \exists x\,\varphi(x)).$$

Then, by means of the tautology $(P \Leftrightarrow Q) \Rightarrow (\neg P \Leftrightarrow \neg Q)$, we obtain the first claim. The second claim is established similarly. □

Theorem 16 (Contradiction)**.** *Let $\bot$ be a contradiction.*

1. *If $\Gamma \cup \{\sigma\} \vdash \bot$, then $\Gamma \vdash \neg\sigma$.*
2. *If $\Gamma \cup \{\neg\sigma\} \vdash \bot$, then $\Gamma \vdash \sigma$.*

In our theorems so far, we have made no reference to sets or the symbol $\in$. Now we do, in a version of the Russell Paradox:

thm:R **Theorem 17.** *The following is a logical theorem:*

$$\neg\exists y\,\forall x\,(x \in y \Leftrightarrow x \notin x).$$

Proof. It is enough to show

$$\vdash \exists y\,\forall x\,(x \in y \Leftrightarrow x \notin x) \Rightarrow \bot,$$

where $\bot$ is some contradiction. Suppose a is a constant not appearing in $\bot$. By $\exists$-Introduction, it is enough to show

$$\vdash \forall x\,(x \in a \Leftrightarrow x \notin x) \Rightarrow \bot.$$

For this, it is enough to show

$$\vdash \exists x\,\neg(x \in a \Leftrightarrow x \notin x).$$

This is a consequence of

$$\vdash \neg(a \in a \Leftrightarrow a \notin a).$$ □

Note then that the Russell Paradox is quite generally, saying nothing specifically about $(\mathbf{V}, \in)$, but about binary relations in general. For example, we might consider the interpretation consisting of the collection of men in a village, with the relation expressed by x *is shaved by* y (or *y shaves x*). Then we get the **Barber Paradox,** reported by Russell: There can be no man in a village who shaves precisely those men in the village that do not shave themselves. For, if there were such a man, he would shave himself if and only if he didn't. (There could however be a *woman* in the village who shaves exactly those men that do not shave themselves.)

$$\tau \Rightarrow \psi(a)$$
$$\psi(a) \Rightarrow \exists x\, \psi(x)$$
$$(\tau \Rightarrow \psi(a)) \Rightarrow (\psi(a) \Rightarrow \exists x\, \psi(x)) \Rightarrow \tau \Rightarrow \exists x\, \psi(x)$$
$$(\psi(a) \Rightarrow \exists x\, \psi(x)) \Rightarrow \tau \Rightarrow \exists x\, \psi(x)$$
$$\tau \Rightarrow \exists x\, \psi(x)$$
$$(\tau \Rightarrow \exists x\, \psi(x)) \Rightarrow \exists x\, \varphi(x) \Rightarrow \sigma \Rightarrow \exists x\, \psi(x)$$
$$\exists x\, \varphi(x) \Rightarrow \sigma \Rightarrow \exists x\, \psi(x)$$
$$\varphi(a) \Rightarrow \exists x\, \varphi(x)$$
$$(\varphi(a) \Rightarrow \exists x\, \varphi(x)) \Rightarrow \neg\exists x\, \varphi(x) \Rightarrow \neg\varphi(a)$$
$$\neg\exists x\, \varphi(x) \Rightarrow \neg\varphi(a)$$
$$\neg\varphi(a) \Rightarrow \psi(a)$$
$$(\neg\exists x\, \varphi(x) \Rightarrow \neg\varphi(a)) \Rightarrow (\neg\varphi(a) \Rightarrow \psi(a)) \Rightarrow \neg\exists x\, \varphi(x) \Rightarrow \psi(a)$$
$$(\neg\varphi(a) \Rightarrow \psi(a)) \Rightarrow \neg\exists x\, \varphi(x) \Rightarrow \psi(a)$$
$$\neg\exists x\, \varphi(x) \Rightarrow \psi(a)$$
$$\psi(a) \Rightarrow \exists x\, \psi(x)$$
$$(\neg\exists x\, \varphi(x) \Rightarrow \psi(a)) \Rightarrow (\psi(a) \Rightarrow \exists x\, \psi(x)) \Rightarrow \neg\exists x\, \varphi(x) \Rightarrow \exists x\, \psi(x)$$
$$(\psi(a) \Rightarrow \exists x\, \psi(x)) \Rightarrow \neg\exists x\, \varphi(x) \Rightarrow \exists x\, \psi(x)$$
$$\neg\exists x\, \varphi(x) \Rightarrow \exists x\, \psi(x)$$
$$(\neg\exists x\, \varphi(x) \Rightarrow \exists x\, \psi(x)) \Rightarrow \neg\exists x\, \varphi(x) \Rightarrow \sigma \Rightarrow \exists x\, \psi(x)$$
$$\neg\exists x\, \varphi(x) \Rightarrow \sigma \Rightarrow \exists x\, \psi(x)$$
$$(\exists x\, \varphi(x) \Rightarrow \sigma \Rightarrow \exists x\, \psi(x)) \Rightarrow (\neg\exists x\, \varphi(x) \Rightarrow \sigma \Rightarrow \exists x\, \psi(x)) \Rightarrow \sigma \Rightarrow \exists x\, \psi(x)$$
$$(\neg\exists x\, \varphi(x) \Rightarrow \sigma \Rightarrow \exists x\, \psi(x)) \Rightarrow \sigma \Rightarrow \exists x\, \psi(x)$$
$$\sigma \Rightarrow \exists x\, \psi(x)$$

tab:fp

Table 2.4. A formal proof: σ is $\exists x\, \varphi(x) \Rightarrow \tau$ and $\psi(x)$ is $\varphi(x) \Rightarrow \tau$

3. Classes

sses

3.1. Classes and relations

Suppose $\varphi(x)$ is a singulary formula. If $\varphi(a)$ is true for some set a, then a is said to **satisfy** φ. The collection of those sets that satisfy φ can be denoted by

$$\{x\colon \varphi(x)\}.$$

Such a collection is called a **class.** It can also be called a **singulary relation** on $\mathbf{V}$. The relation $\{x\colon \varphi(x)\}$ is said to be **defined** by the formula φ.

Theorem 17 above is that the class $\{x\colon x \notin x\}$ is not a set.[1] Indeed, we have defined classes in general only after defining the class $\mathbf{V}$ of all sets. Therefore we cannot just assume that an arbitrary class will be a set, since in that case the class must already have been a member of $\mathbf{V}$.

We may denote the class $\{x\colon \varphi(x)\}$ by a boldface[2] capital letter, such as $\boldsymbol{C}$. Then, instead of $\varphi(a)$, we may write

$$a \in \boldsymbol{C}.$$

The letter $\boldsymbol{C}$ here, like φ, is a syntactic variable. The reason for introducing it is twofold.

1. It is easier to write $\boldsymbol{C}$ than $\{x\colon \varphi(x)\}$, especially if φ is a long formula.
2. Different formulas may define the *same* class.

Indeed, we *define* two classes to be **equal,** or the **same,** if they have the same members. In other words, we consider classes only in extension. Equality is denoted by the sign

$$=,$$

[1] Frege had in effect assumed that all classes *were* sets. There is some scholarship aimed at recovering what is sound in Frege's work: see Burgess, *Fixing Frege* [4].

[2] In writing, boldface is indicated by a wavy underline.

the **equals-sign.** So we have

$$\{x\colon \varphi(x)\} = \{x\colon \psi(x)\}$$

if and only if we have

$$\forall x\,(\varphi(x) \Leftrightarrow \psi(x)).$$

If the latter sentence is indeed true, then the formulas φ and ψ can be called **equivalent.** So two formulas are equivalent if and only if they define the same class. The following is obvious, and indeed we assume it when we say that two classes are equal, rather than saying more precisely that one class is equal *to* another: the extra precision is unneeded.

thm:=

Theorem 18. *For all classes $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$,*

$$\begin{gathered}\boldsymbol{C} = \boldsymbol{C},\\ \boldsymbol{C} = \boldsymbol{D} \Rightarrow \boldsymbol{D} = \boldsymbol{C},\\ \boldsymbol{C} = \boldsymbol{D} \mathbin{\&} \boldsymbol{D} = \boldsymbol{E} \Rightarrow \boldsymbol{C} = \boldsymbol{E}.\end{gathered}$$

There are also *binary* relations on $\mathbf{V}$, such as membership itself; these relations are defined by **binary formulas,** which are formulas that have just two free variables. Suppose ψ is such a formula, and its free variables are x and y. Then we can write ψ as

$$\psi(x, y).$$

If t and u are terms, then by substituting t for each free occurrence of x, and u for each free occurrence of y, we obtain the formula denoted by

$$\psi(t, u).$$

We might obtain for example $\psi(x, x)$ or $\psi(y, x)$.[3]

If ψ is a bad formula, it might happen that, when we form $\psi(y, x)$ from $\psi(x, y)$, a new occurrence of y is bound, although (of course) the old occurrence of x at the same place was free. For example, suppose $\psi(x, y)$ is

$$\exists y\,(y \in x \mathbin{\&} y \notin y) \mathbin{\&} y \in x.$$

[3]Note that $\psi(y, x)$ will never be the formula ψ. We wrote ψ as $\psi(x, y)$ because x comes before y in the alphabet.

Then $\psi(a,b)$ is $\exists y\,(y \in a \mathbin{\&} y \notin y) \mathbin{\&} b \in a$, which will turn out be true for some a and b. However, $\psi(y,x)$ is

$$\exists y\,(y \in y \mathbin{\&} y \notin y) \mathbin{\&} x \in y,$$

which can be written as $\varphi(x,y)$; then $\varphi(a,b)$ is $\exists y\,(y \in y \mathbin{\&} y \notin y) \mathbin{\&} a \in b$, which is always false. In particular, although $\varphi(x,y)$ is $\psi(y,x)$, the formula $\varphi(a,b)$ is not $\psi(b,a)$. We shall always avoid this problem by using good formulas.

Given the binary formula $\psi(x,y)$, we may introduce a symbol such as $\boldsymbol{R}$, and then, as another way of saying that $\psi(a,b)$ is true, we may write

$$a\ \boldsymbol{R}\ b.$$

Then $\boldsymbol{R}$ can be understood to denote a **binary relation** on $\mathbf{V}$, namely the relation **defined** by $\psi(x,y)$. For the moment, $\boldsymbol{R}$ is a new kind of thing. It can be understood as the collection of *ordered pairs* (a,b) such that $a\ \boldsymbol{R}\ b$; but we do not yet *officially* know what ordered pairs are. Later we shall define ordered pairs as certain sets, and then $\boldsymbol{R}$ will indeed by a class.

If we wish, we can define *ternary* relations, *quaternary* relations, and so forth, as far as we need to go.

3.2. Relations between classes and collections

We have defined the notion of a *class* and of a binary relation on $\mathbf{V}$. More informally, we may consider the collection of all classes, along with some binary relations on this collection. Indeed, we have already defined one such relation: equality. Then of course we have *inequality:* if classes $\boldsymbol{C}$ and $\boldsymbol{D}$ are not equal, they are **unequal,** and we may write

$$\boldsymbol{C} \neq \boldsymbol{D}.$$

Now suppose $\boldsymbol{C}$ is the class $\{x\colon \varphi(x)\}$, and $\boldsymbol{D}$ is $\{x\colon \psi(x)\}$. If $\forall x\,(\varphi(x) \Rightarrow \psi(x))$, then we write

$$\boldsymbol{C} \subseteq \boldsymbol{D},$$

saying that $\boldsymbol{C}$ is a **subclass** of $\boldsymbol{D}$, and $\boldsymbol{D}$ **includes** $\boldsymbol{C}$. If $\boldsymbol{C} \subseteq \boldsymbol{D}$, but $\boldsymbol{C} \neq \boldsymbol{D}$, then $\boldsymbol{C}$ is a **proper** subclass of $\boldsymbol{D}$, and $\boldsymbol{D}$ **properly** includes $\boldsymbol{C}$, and we may write

$$\boldsymbol{C} \subset \boldsymbol{D}.$$

Theorem 19. *For all classes $\boldsymbol{C}$ and $\boldsymbol{D}$,*

$$\boldsymbol{C} = \boldsymbol{D} \Leftrightarrow \boldsymbol{C} \subseteq \boldsymbol{D} \mathbin{\&} \boldsymbol{D} \subseteq \boldsymbol{C}.$$

Proof. The claim is

$$\forall x\,(\varphi(x) \Leftrightarrow \psi(x)) \Leftrightarrow \forall x\,(\varphi(x) \Rightarrow \psi(x)) \mathbin{\&} \forall x\,(\psi(x) \Rightarrow \varphi(x)).$$

But this means $\varphi(a) \Leftrightarrow \psi(a)$ for every set a if and only if both $\varphi(a) \Rightarrow \psi(a)$ and $\psi(a) \Rightarrow \varphi(a)$ for every set a; and this is true. □

We shall have some occasion to use similar terminology and notation for collections in general. For example, the collection of $\in$-formulas includes the collection of quantifier-free $\in$-formulas.

3.3. Sets as classes

sect:sc

If a is a set, then the formula $x \in a$ defines a class. We shall consider this class to be the set a itself. Then a set is equal to a class if they have the same members, and two sets are equal if they have the same members. In particular, if $\boldsymbol{C}$ is the class $\{x\colon \varphi(x)\}$, then we can write

$$x = \boldsymbol{C}$$

as an abbreviation of the formula

$$\forall y\,(y \in x \Leftrightarrow \varphi(y)),$$

where y is a variable not occurring in $\varphi(x)$. As an abbreviation of the formula

equality

$$\forall z\,(z \in x \Leftrightarrow z \in y),$$

we can write

$$x = y.$$

Since sets are now classes, Theorem 18 applies to them. Also, a class $\boldsymbol{C}$ is a set if and only if

$$\exists y\,\forall x\,(x \in y \Leftrightarrow x \in \boldsymbol{C}).$$

For some kinds of classes, there will be easier ways to say that they are sets. Meanwhile, there is now another way to prove Theorem 17 (the

Russell Paradox): Let $\boldsymbol{C}$ be the class defined by the singulary formula $x \notin x$. If a is a set, then $a \in \boldsymbol{C} \Leftrightarrow a \notin a$, so $\boldsymbol{C}$ and a have different members, and therefore $\boldsymbol{C} \neq a$. In short, $\boldsymbol{C}$ is not a set.

Things that are equal ought to have the same behavior. We can derive this from our first axiom: it is our first true sentence that is not *logically* true.

ax:1 **Axiom 1** (Equality). *Equal sets are members of the same sets:*

$$\forall x\, \forall y\, \forall z\, (x = y \Rightarrow (x \in z \Leftrightarrow y \in z)). \tag{3.1}$$ eqn:=

The expression in (3.1) is really an abbreviation for

$$\forall x\, \forall y\, (\forall z\, (z \in x \Leftrightarrow z \in y) \Rightarrow \forall z\, (x \in z \Leftrightarrow y \in z)).$$

For all sets a, we now have

$$\forall x\, \forall y\, (x = y \Rightarrow (x \in a \Leftrightarrow y \in a)). \tag{3.2}$$ eqn:xa

By the definition of equality of sets, we have

$$\forall x\, \forall y\, (x = y \Rightarrow (a \in x \Leftrightarrow a \in y)). \tag{3.3}$$ eqn:ax

Each of the last two sentences is a part of Theorem 21 below. This is about singulary formulas, and we shall prove it by induction. Now, we did not exactly define singulary formulas recursively. We defined *formulas* recursively, and we defined the *free variables* of formulas recursively; but then we took the non-recursive step of defining singulary formulas as formulas with just one free variable. Nonetheless, our inductive proof will be justified by the following.

Theorem 20. *Suppose x is a variable, and Γ is a collection of formulas meeting the following conditions.*

1. *Every singulary atomic formula $\varphi(x)$ is in Γ.*
2. *If $\varphi(x)$ is in Γ, then so is $\neg\varphi(x)$.*
3. *If $\varphi(x)$ and $\psi(x)$ are in Γ, and σ is a sentence, then $(\varphi(x) \Rightarrow \psi(x))$ and $(\varphi(x) \Rightarrow \sigma)$ and $(\sigma \Rightarrow \psi(x))$ are in Γ.*
4. *Suppose $\varphi(x, y)$ is a binary formula such that, for each constant a, the formula $\varphi(x, a)$ is in Γ. Then $\exists y\, \varphi(x, y)$ is in Γ.*

Then Γ consists of the singulary formulas with the free variable x.

thm:set=

Theorem 21. *For all singular formulas $\varphi(x)$ in which y does not occur,*

$$\forall x\,\forall y\,(x = y \Rightarrow (\varphi(x) \Leftrightarrow \varphi(y))).$$

Proof. We prove the claim by induction, as follows.

1. The cases where $\varphi(x)$ is $x \in a$ or $a \in x$ are taken care of by (3.2) and (3.3). Now suppose $\varphi(x)$ is $x \in x$. Given arbitrary sets a and b such that $a = b$, we want to show

$$a \in a \Leftrightarrow b \in b.$$

By (3.2) and (3.3), we have $a \in a \Leftrightarrow b \in a$ and $b \in a \Leftrightarrow b \in b$.

2. If the claim is true when φ is ψ, then it is true when φ is $\neg\psi$, because of the tautology

$$(P \Leftrightarrow Q) \Rightarrow (\neg P \Leftrightarrow \neg Q).$$

3. If the claim is true when φ is ψ or χ, and σ is a sentence, then the claim is true when φ is $\psi \Rightarrow \chi$ or $\psi \Rightarrow \sigma$ or $\sigma \Rightarrow \psi$, because of the tautologies

$$\begin{gathered}(P \Leftrightarrow Q) \mathbin{\&} (R \Leftrightarrow S) \Rightarrow ((P \Rightarrow R) \Leftrightarrow (Q \Rightarrow S)),\\ (P \Leftrightarrow Q) \Rightarrow ((P \Rightarrow R) \Leftrightarrow (Q \Rightarrow R)),\\ (P \Leftrightarrow Q) \Rightarrow ((R \Rightarrow P) \Leftrightarrow (R \Rightarrow Q)).\end{gathered}$$

4. Finally, suppose that, for some binary formula $\psi(x, z)$, for all sets a, the claim is true when $\varphi(x)$ is $\psi(x, a)$. We want to show

$$x = y \Rightarrow (\exists z\,\psi(x, z) \Leftrightarrow \exists z\,\psi(y, z))$$

(where y does not occur in $\psi(x, z)$). But if $b = c$, and $\exists y\,\psi(b, y)$, then $\psi(b, a)$ for some set a, and then $\psi(c, a)$ by inductive hypothesis, so $\exists z\,\psi(c, z)$. This establishes what is desired. □

In another version of the logic of set theory, equality is accepted, along with membership, as a fundamental notion. This means making the following adjustments:

1. Equations $t = u$ (where t and u are terms) are counted as atomic formulas.
2. The equation $a = b$ is defined to be true if $a = b$.
3. Theorem 21 is counted as being logically true: it is a logical axiom.

4. In particular, Axiom 1 is counted as a logical axiom.
5. A nonlogical axiom is then needed, namely

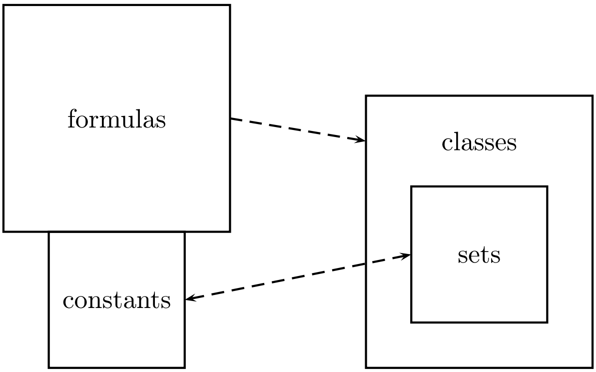
$$\forall x \,\forall y \,(\forall z \,(z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$$

   (which for us is true by definition); this axiom is called something like the **Axiom of Extension.**

Either way, we get to where we are now.

3.4. The logic of sets

Formulas and their symbols are not originally considered as sets themselves. Our constants *denote* sets, and formulas *define* classes. Now we know that sets are classes, but not every class is a set. The situation can be depicted as in Figure 3.1. The arrows point as they do, because again:



:scf

Figure 3.1. The logic of sets

1. Every constant will denote a particular set.
2. Every set can be denoted by some constant.
3. Every formula will denote a class.

Conversely, every class is denoted by a formula; but the formula is not unique. In the terminology introduced in §2.1, a class can be understood as the *extension* of a formula; but different formulas can have the same extension.[4]

[4] One can consider classes as *equivalence-classes* of formulas.

More precisely, the correspondences in Figure 3.1 occur *in a particular context:* they are not fixed once for all. This is why the arrows in the diagram are dashed. The letter a does not denote a set right now; but whenever we happen to have a particular set, we may denote it by some letter, and this letter can be a, unless a has already been used to denote another set. (We can however use different letters for the same set.)

The logic of sets can be constrasted with other logics, such as the logic of $(\mathbb{R}, +, \cdot)$. The equations (2.1), (2.2), and (2.3) can be considered to denote their *solution-sets.*[5] These solution-sets are, respectively, a certain op set of real numbers, a certain set of *ordered pairs* of real numbers, and a certain set of *ordered triples* of real numbers. A real number itself cannot be any of these sets. The situation is as in Figure 3.2. A remarkable

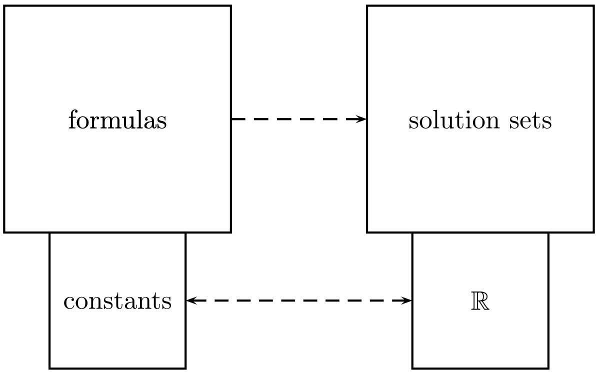


fig:R Figure 3.2. The logic of $\mathbb{R}$

point about the logic of sets is seen in the difference between figures 3.1 and 3.2: in the logic of sets, constants denote some of the *same things* that formulas denote.

3.5. Operations on classes

There are many ways to combine two singulary formulas into a new singulary formula. These correspond to ways of combining classes. Some of

[5]The solution-sets of (2.2) and (2.3) are usually called *graphs:* a parabola and an hyperboloid of one sheet, respectively.

these ways are given special names and symbols:

$$\begin{aligned}
\boldsymbol{C} \smallsetminus \boldsymbol{D} &= \{x\colon x \in \boldsymbol{C} \ \&\ x \notin \boldsymbol{D}\},\\
\boldsymbol{C} \cap \boldsymbol{D} &= \{x\colon x \in \boldsymbol{C} \ \&\ x \in \boldsymbol{D}\},\\
\boldsymbol{C} \cup \boldsymbol{D} &= \{x\colon x \in \boldsymbol{C} \lor x \in \boldsymbol{D}\};
\end{aligned}$$

these are the **complement** of $\boldsymbol{D}$ in $\boldsymbol{C}$, the **intersection** of $\boldsymbol{C}$ and $\boldsymbol{D}$, and the **union** of $\boldsymbol{C}$ and $\boldsymbol{D}$. The complement of $\boldsymbol{D}$ in $\mathbf{V}$ is simply the **complement** of $\boldsymbol{D}$ and can be denoted by

$$\boldsymbol{D}^{\mathrm{c}}.$$

None of these combinations of $\boldsymbol{C}$ and $\boldsymbol{D}$ makes special use of the relation of membership of *sets* symbolized by $\in$. We used the symbol $\in$, but we could have done without this. If $\boldsymbol{C}$ and $\boldsymbol{D}$ are defined by $\varphi(x)$ and $\psi(x)$ respectively, then, for example, $\boldsymbol{C} \smallsetminus \boldsymbol{D}$ is defined by $\varphi(x)\ \&\ \neg\psi(x)$.

By making use of membership of sets, we can obtain new classes from a single class as follows:

$$\begin{aligned}
\bigcup \boldsymbol{C} &= \{x\colon \exists y\,(y \in \boldsymbol{C} \ \&\ x \in y)\},\\
\bigcap \boldsymbol{C} &= \{x\colon \forall y\,(y \in \boldsymbol{C} \Rightarrow x \in y)\},\\
\mathscr{P}(\boldsymbol{C}) &= \{x\colon \forall y\,(y \in x \Rightarrow y \in \boldsymbol{C}\};
\end{aligned}$$

these are the **union, intersection,** and **power class** of $\boldsymbol{C}$. We have then

$$\mathscr{P}(\boldsymbol{C}) = \{x\colon x \subseteq \boldsymbol{C}\}.$$

Finally, classes can be formed from no, one, and two sets:

$$\begin{aligned}
0 &= \{x\colon x \neq x\},\\
\{a\} &= \{x\colon x = a\},\\
\{a, b\} &= \{x\colon x = a \lor x = b\};
\end{aligned}$$

these are the **null class,** the **singleton** of a, and the **pair** of a and b. If $a = b$, then the pair of a and b is the singleton of a. If $\boldsymbol{C} \cap \boldsymbol{D} = 0$, then $\boldsymbol{C}$ is **disjoint** from $\boldsymbol{D}$. In this case, since $\boldsymbol{C} \cap \boldsymbol{D} = \boldsymbol{D} \cap \boldsymbol{C}$, the two classes themselves, considered together, are simply **disjoint.**

4. Numbers

`ch:N`

4.1. The collection of natural numbers

`sect:N`

Having constants in our language commits us to the existence of sets. Let us now say something about *which* sets exist. Since all sets are classes, we shall generally try to say which classes are sets.

We have to be careful. If $\boldsymbol{C}$ is the class $\{x\colon x \notin x\}$, then we know by the Russell Paradox that $\boldsymbol{C}$ is not a set. However, if $\boldsymbol{C}$ were a set, it would be a member of itself. In particular, we cannot know which sets belong to $\boldsymbol{C}$ unless we know whether $\boldsymbol{C}$ is a set.[1] Our next axioms do not appear to have this ambiguity.

`ax:null` **Axiom 2** (Null set)**.** 0 *is a set:*

$$\exists x\,\forall y\,(y \notin x).$$

`ax:adj` **Axiom 3** (Adjunction)**.** $a \cup \{b\}$ *is always a set:*

$$\forall x\,\forall y\,\exists z\,\forall w\,(w \in z \Leftrightarrow w \in x \lor w = y).$$

Here we might refer to $a \cup \{b\}$ as the **adjunction of** b **to** a**.** We can immediately derive:

`thm:s&p` **Theorem 22** (Singling and Pairing)**.** $\{a\}$ *and* $\{a, b\}$ *are always sets.*

Proof. $\{a\} = 0 \cup \{a\}$ and $\{a, b\} = \{a\} \cup \{b\}$. □

As a special case, we have the sets 0, $\{0\}$, $\{\{0\}\}$, $\{\{\{0\}\}\}$, $\{\{\{\{0\}\}\}\}$, and so on. These sets *could* serve as definitions of the natural numbers 0, 1, 2, 3, 4, and so on.[2] An inconvenience is that the sets all have one element each. However, given a set a, we also have that $a \cup \{a\}$ is a set. Let us write

$$a' = a \cup \{a\}.$$

Then we have the sets 0, $0'$, $0''$, $0'''$, and so on. We shall take *these* as the official natural numbers:

[1] The term is that the definition of $\boldsymbol{C}$ is *impredicative.*

[2] Zermelo [40] defines the natural numbers this way.

fn:N **Definition 10.** The **natural numbers** are given recursively by two rules:

1. 0 is a natural number.
2. If n is a natural number, then so is n'.

Let us denote the collection of natural numbers by

$$\mathbb{N}.$$

Then we may write

$$\mathbb{N} = \{0, 0', 0'', \dots\}.$$

There are standard names for some elements of $\mathbb{N}$:

$$\begin{aligned}
1 &= 0' = \{0\},\\
2 &= 1' = \{0, 1\},\\
3 &= 2' = \{0, 1, 2\},\\
4 &= 3' = \{0, 1, 2, 3\},
\end{aligned}$$

and so on. Note that 1 is now a set with just one element, 2 has just two elements, 3 has just three elements, and so forth. We may write

$$\mathbb{N} = \{0, 1, 2, \dots\}.$$

It is not clear whether $\mathbb{N}$ is a *class,* much less a set. The definition gives us a way to confirm that a particular set a is in $\mathbb{N}$: we just compare a with 0, 1, 2, and so on until we find a number that is equal to a. However, if $a \notin \mathbb{N}$, the definition does not show us a way to prove this. We shall investigate $\mathbb{N}$ further after looking at another consequence of our axioms; the existence of the *ordered pair* as a set.

4.2. Relations and functions

l&fn

By Theorem 22, given sets a and b we can define

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

This set is the **ordered pair** of a and b. In case $a = b$, we have $(a, b) = (a, a) = \{\{a\}\}$. The sole purpose of the definition of an ordered pair is to make the following true.

Theorem 23. $(a, b) = (c, d) \Leftrightarrow a = c \; \& \; b = d.$

A binary formula $\varphi(x, y)$ can now be understood to define the class

$$\{z\colon \exists x\, \exists y\, (z = (x, y) \; \& \; \varphi(x, y))\}.$$

We may write this class also as

$$\{(x, y)\colon \varphi(x, y)\}. \tag{4.1}$$

eqn:

A binary relation is now such a class. If $\boldsymbol{C}$ and $\boldsymbol{D}$ are classes, then the class $\{(x, y)\colon x \in \boldsymbol{C} \; \& \; y \in \boldsymbol{D}\}$ is denoted by

$$\boldsymbol{C} \times \boldsymbol{D};$$

this is the **Cartesian product** of $\boldsymbol{C}$ and $\boldsymbol{D}$.

The notation in (4.1) is similar to the notation for the *image* of a class under a *function.* A binary relation $\boldsymbol{F}$ is a **function** if

$$\forall x\, \forall y\, \forall z\, (x \; \boldsymbol{F} \; y \; \& \; x \; \boldsymbol{F} \; z \Rightarrow y = z).$$

When $\boldsymbol{F}$ is such, and $a \; \boldsymbol{F} \; b$, we can use $\boldsymbol{F}(a)$ as a name for b. Then we can use for $\boldsymbol{F}$ itself the notation

$$x \mapsto \boldsymbol{F}(x).$$

For example, if $\boldsymbol{F}$ is the function $\{(x, y)\colon y = x \cup \{x\}\}$, that is, $\{(x, y)\colon y = x'\}$, then we can write this function as $x \mapsto x'$. In general, the **domain** of a function $\boldsymbol{F}$ is the class $\{x\colon \exists y\, x \; \boldsymbol{F} \; y\}$; this can be denoted by

$$\operatorname{dom}(\boldsymbol{F}).$$

If $\boldsymbol{C} \subseteq \operatorname{dom}(\boldsymbol{F})$, then the class

$$\{y\colon \exists x\, (x \in \boldsymbol{C} \; \& \; x \; \boldsymbol{F} \; y)\}$$

can be denoted by either of

$$\{\boldsymbol{F}(x)\colon x \in \boldsymbol{C}\}, \qquad \qquad \boldsymbol{F}[\boldsymbol{C}];$$

this is the **image** of $\boldsymbol{C}$ under $\boldsymbol{F}$. If $\boldsymbol{C} = \operatorname{dom}(\boldsymbol{F})$, then $\boldsymbol{F}[\boldsymbol{C}]$ is the **range** of $\boldsymbol{F}$ and can be denoted by

$$\operatorname{rng}(\boldsymbol{F}).$$

If also $\mathrm{rng}(\boldsymbol{F}) \subseteq \boldsymbol{D}$, then we may say that $\boldsymbol{F}$ is a function **from $\boldsymbol{C}$ to $\boldsymbol{D}$**. More generally, if $\boldsymbol{C} \subseteq \mathrm{dom}(\boldsymbol{F})$, we may want to consider the **restriction** of $\boldsymbol{F}$ to $\boldsymbol{C}$, namely the function $\{(x, y) \colon x \in \boldsymbol{C} \mathbin{\&} \boldsymbol{F}(x) = y\}$, which can be denoted by

$$\boldsymbol{F} \restriction \boldsymbol{C}.$$

For example, we may have two functions $\boldsymbol{F}$ and $\boldsymbol{G}$ whose domains include $\boldsymbol{C}$; if $\boldsymbol{F} \restriction \boldsymbol{C} = \boldsymbol{G} \restriction \boldsymbol{C}$, we may say that $\boldsymbol{F}$ and $\boldsymbol{G}$ **agree on $\boldsymbol{C}$**.

We may consider restrictions in a more general sense. If $\boldsymbol{R}$ is an arbitrary relation, and the relation $\{(x, y) \colon x \in \boldsymbol{C} \mathbin{\&} x \mathrel{\boldsymbol{R}} y\}$ is a function whose domain is $\boldsymbol{C}$, then $\boldsymbol{R}$ may be described as being a function **on $\boldsymbol{C}$** (even though $\boldsymbol{R}$ itself is not a function, simply).

If we have two classes, $\boldsymbol{F}$ and $\boldsymbol{C}$, such that

1) $\boldsymbol{F}$ is a function on $\boldsymbol{C}$, and
2) $\boldsymbol{F}[C] \subseteq \boldsymbol{C}$,

then $\boldsymbol{C}$ is **closed** under $\boldsymbol{F}$, and $\boldsymbol{F}$ is a **singulary operation** on $\boldsymbol{C}$. If one of the two conditions is not met, then we may say that $\boldsymbol{F}$ is not a **well-defined** operation on $\boldsymbol{C}$.

If $\boldsymbol{R}$ is a binary relation, then the **converse** of $\boldsymbol{R}$ is the binary relation

$$\{(y, x) \colon x \mathrel{\boldsymbol{R}} y\};$$

this can be denoted by

$$\breve{\boldsymbol{R}}.$$

A function $\boldsymbol{F}$ is **injective** if

$$\forall x \, \forall y \, (x \in \mathrm{dom}(\boldsymbol{F}) \mathbin{\&} y \in \mathrm{dom}(\boldsymbol{F}) \mathbin{\&} \boldsymbol{F}(x) = \boldsymbol{F}(y) \Rightarrow x = y).$$

If $\boldsymbol{F}$ is a function with domain $\boldsymbol{C}$, and $\breve{\boldsymbol{F}}$ is a function with domain $\boldsymbol{D}$, then $\boldsymbol{F}$ is a **bijection** from $\boldsymbol{C}$ to $\boldsymbol{D}$, and $\boldsymbol{C}$ is **equipollent** to $\boldsymbol{D}$, and we may write

$$\boldsymbol{C} \approx \boldsymbol{D}.$$

Theorem 24. *If $\boldsymbol{F}$ is a bijection from $\boldsymbol{C}$ to $\boldsymbol{D}$, then both $\boldsymbol{F}$ and $\breve{\boldsymbol{F}}$ are injective.*

Given two binary relations $\boldsymbol{R}$ and $\boldsymbol{S}$, we can **compose** them to get the relation

$$\{(x, z) \colon \exists y \, (x \mathrel{\boldsymbol{R}} y \mathbin{\&} y \mathrel{\boldsymbol{S}} z\}.$$

This relation can be denoted by

$$\boldsymbol{R}/\boldsymbol{S},$$

although some people will write

$$\boldsymbol{S} \circ \boldsymbol{R}.$$

The latter notation is standard when $\boldsymbol{R}$ and $\boldsymbol{S}$ are functions such that the range of $\boldsymbol{R}$ is included in the domain of $\boldsymbol{S}$. In this case, $\boldsymbol{R}/\boldsymbol{S}$ or $\boldsymbol{S} \circ \boldsymbol{R}$ is a function with the same domain as $\boldsymbol{R}$. For example, if $\boldsymbol{F}$ is a bijection from $\boldsymbol{C}$ to $\boldsymbol{D}$, then $\boldsymbol{F} \circ \breve{\boldsymbol{F}} = \{(y, y) \colon y \in \boldsymbol{D}\}$ and $\breve{\boldsymbol{F}} \circ \boldsymbol{F} = \{(x, x) \colon x \in \boldsymbol{C}\}$.

Theorem 25. *For all classes $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$,*

$$\boldsymbol{C} \approx \boldsymbol{C},$$
$$\boldsymbol{C} \approx \boldsymbol{D} \Rightarrow \boldsymbol{D} \approx \boldsymbol{C},$$
$$\boldsymbol{C} \approx \boldsymbol{D} \mathbin{\&} \boldsymbol{D} \approx \boldsymbol{E} \Rightarrow \boldsymbol{C} \approx \boldsymbol{E}.$$

Instead of saying that $\boldsymbol{C}$ is equipollent to $\boldsymbol{D}$, we are allowed by the theorem to say simply that $\boldsymbol{C}$ and $\boldsymbol{D}$ are equipollent (or $\boldsymbol{D}$ and $\boldsymbol{C}$ are equipollent).

If all we know is that $\boldsymbol{F}$ is an injective function with domain $\boldsymbol{C}$, and $\boldsymbol{F}[\boldsymbol{C}] \subseteq \boldsymbol{D}$, then $\boldsymbol{C}$ **embeds** in $\boldsymbol{D}$, and we may write

$$\boldsymbol{C} \preccurlyeq \boldsymbol{D}.$$

Immediately, we have

Theorem 26. *For all classes $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$,*

$$\boldsymbol{C} \approx \boldsymbol{D} \Rightarrow \boldsymbol{C} \preccurlyeq \boldsymbol{D},$$
$$\boldsymbol{C} \preccurlyeq \boldsymbol{C},$$
$$\boldsymbol{C} \preccurlyeq \boldsymbol{D} \mathbin{\&} \boldsymbol{D} \preccurlyeq \boldsymbol{E} \Rightarrow \boldsymbol{C} \preccurlyeq \boldsymbol{E}.$$

The question of what happens when $\boldsymbol{C} \preccurlyeq \boldsymbol{D}$ and $\boldsymbol{D} \preccurlyeq \boldsymbol{C}$ will be dealt with in Chapter 6. Meanwhile, if $\boldsymbol{C}$ and $\boldsymbol{D}$ are not equipollent, we may write

$$\boldsymbol{C} \not\approx \boldsymbol{D}.$$

If $\boldsymbol{C} \preccurlyeq \boldsymbol{D}$, but $\boldsymbol{C} \not\approx \boldsymbol{D}$, we write

$$\boldsymbol{C} \prec \boldsymbol{D}.$$

Theorem 27. *For all classes $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$,*

$$\boldsymbol{C} \prec \boldsymbol{D} \Rightarrow \boldsymbol{C} \not\approx \boldsymbol{D},$$
$$\boldsymbol{C} \prec \boldsymbol{D} \mathbin{\&} \boldsymbol{D} \prec \boldsymbol{E} \Rightarrow \boldsymbol{C} \prec \boldsymbol{E}.$$

4.3. The class of formal natural numbers

:fnn

We have that $x \mapsto x'$ is a function on $\mathbf{V}$; let us refer to this function as **succession,** or **set-theoretic succession** if we need to be more precise. The recursive definition of $\mathbb{N}$, Definition 10, means simply that every collection of sets that contains 0 and is closed under succession includes $\mathbb{N}$. In short, the definition means that a certain kind of proof by induction is possible. Let us call this **finite induction** (because later there will be *transfinite induction*). Perhaps the most basic application of finite induction is the following:

down

Theorem 28. *Let $\boldsymbol{D}$ be the class of all sets a such that*
1) $0 \in a$ and
2) a is closed under $x \mapsto x'$.
Then

$$\mathbb{N} \subseteq \bigcap \boldsymbol{D}.$$

Proof. If $a \in \boldsymbol{D}$, then immediately by finite induction, $\mathbb{N} \subseteq a$. Therefore $\mathbb{N} \subseteq \bigcap \boldsymbol{D}$. □

In the notation of the theorem, if $a \in \boldsymbol{D}$, then $\bigcap \boldsymbol{D} \subseteq a$. This means the class $\bigcap \boldsymbol{D}$ allows proof by finite induction in a restricted sense: if a meets the conditions of being in $\boldsymbol{D}$, then all elements of $\bigcap \boldsymbol{D}$ are in a. This is a restricted sense of finite induction, because a must be a set, not an arbitrary collection. If $\mathbb{N}$ should be a set, then it would meet the conditions, so $\bigcap \boldsymbol{D} \subseteq \mathbb{N}$; by the theorem itself then, $\mathbb{N} = \bigcap \boldsymbol{D}$. But perhaps $\mathbb{N}$ is not a set. Indeed, for all we know so far, $\boldsymbol{D}$ may be empty, so that $\bigcap \boldsymbol{D} = \mathbf{V}$. In this case, there may still a proper class $\boldsymbol{C}$ that contains 0 and is closed under succession, although $\boldsymbol{C} \neq \mathbf{V}$; then $\boldsymbol{C} \subset \bigcap \boldsymbol{D}$, and therefore $\mathbb{N} \subset \bigcap \boldsymbol{D}$.

In many expositions of set theory, there is an *Axiom of Infinity,* which is that the class $\boldsymbol{D}$ is nonempty.[3] This axiom is a radical assumption,

[3] This is one of Zermelo's axioms [40].

and it would be premature to make it now; so we do not assume this axiom yet.

Even if we do not have $\mathbb{N}$ as a set, we know that some collections of its elements are sets. Indeed, the subclasses 0, $\{0\}$, $\{0,1\}$, $\{0,1,2\}$, and so on are sets. In fact they are *elements* of $\mathbb{N}$ too, but let us ignore this for the moment. They are all members of the class $\boldsymbol{C}$ described in the following:

thm:up

Theorem 29. *Let $\boldsymbol{C}$ be the class of all sets a such that, for all sets b in a, either*

1) $b = 0$, or

2) there is a set d in a such that $b = d'$.

That is, $\boldsymbol{C}$ is defined by $\forall y\,(y \in x \Rightarrow y = 0 \lor \exists z\,(z \in x \;\&\; y = z'))$. Then

$$\mathbb{N} \subseteq \bigcup \boldsymbol{C}.$$

Proof. We can prove the claim by finite induction.

1. Since $\{0\} \in \boldsymbol{C}$, we have $0 \in \bigcup \boldsymbol{C}$.

2. Suppose $n \in \bigcup \boldsymbol{C}$. Then $n \in a$ for some a in $\boldsymbol{C}$. Then $a \cup \{n'\} \in \boldsymbol{C}$, so $n' \in \bigcup \boldsymbol{C}$. Thus $\bigcup \boldsymbol{C}$ is closed under succession. □

We did not prove $\mathbb{N} = \bigcup \boldsymbol{C}$. Indeed, in the notation of the theorem, possibly $\boldsymbol{C}$ has an element a such that *every* element of a is b' for some element b of a. Such a set a is *ill-founded.* If such sets are allowed in $\boldsymbol{C}$, then $\bigcup \boldsymbol{C}$ may have elements that are definitely not in $\mathbb{N}$.

efn:wf-class

Definition 11. A class is **well-founded** if every nonempty subset has an element that is disjoint from that subset. That is, $\boldsymbol{C}$ is well-founded if and only if

$$\forall y\,(y \subseteq \boldsymbol{C} \;\&\; y \neq 0 \Rightarrow \exists z\,(z \in y \;\&\; z \cap y = 0)).$$

A class is **ill-founded** if it is not well-founded, that is, if it has a nonempty subset whose every element is not disjoint from it.

Some examples of ill-foundedness are as follows.

1. If $a \in a$, then $a \in a \cap \{a\}$, so $\{a\}$ is ill-founded. Since $\{a\} \subseteq a$, also a is ill-founded. Note here $a' = a$.
2. If $a \in b$ and $b \in a$, then $a \in b \cap \{a,b\}$ and $b \in a \cap \{a,b\}$, so $\{a,b\}$ is ill-founded. If also $a' = b$ and $b' = a$, then $a = b$.

3. If $a \in b$ and $b \in c$ and $c \in a$, then $\{a, b, c\}$ is ill-founded.
4. If there is an infinite *set* $\{a_0, a_1, a_2, \dots\}$, where $a_1 \in a_0$, and $a_2 \in a_1$, and so on, then the set is ill-founded.[4] Possibly $a_0 = a_1{}'$, and $a_1 = a_2{}'$, and so on.[5]

In the last theorem, we could require the elements of $\boldsymbol{C}$ to be well-founded. However, in the last example, it may be that $\{a_0, a_1, a_2, \dots\}$ is a proper class with no 'infinite' subsets.[6] Then the class is well-founded. This situation can arise when $\{a_0, a_1, a_2, \dots\} \cup \{0\}$ is itself a set, but every 'infinite' subset contains 0. This is actually not a problem in trying to obtain $\mathbb{N}$ as $\bigcup \boldsymbol{C}$ as in the theorem. At least, it is not a problem we can do anything about. I shall say more about this later in the section.

Meanwhile, another problem may arise. If $a_0 = a_1{}'$, and $a_1 = a_2{}'$, and so on, then

$$a_0 = a_1 \cup \{a_1\} = a_2 \cup \{a_2, a_1\} = a_3 \cup \{a_3, a_2, a_1\} = \cdots$$

For all we know, there may be some set that belongs to each of the sets a_0, a_1, a_2, and so on, but is not equal to any of them. This common element could be $a_0{}'$. Then $\{a_0, a_1, a_2, \dots\} \cup \{0\}$ could be a well-founded set as before, although $\{a_0{}'\} \cup \{a_0, a_1, a_2, \dots\} \cup \{0\}$ would be an ill-founded set, since neither element of the pair $\{a_0{}', a_0\}$ would be disjoint from the pair. Thus, in the last theorem, even if the elements of $\boldsymbol{C}$ are well-founded, maybe $\bigcup \boldsymbol{C}$ contains a_0, but not $a_0{}'$. To avoid this problem, we shall need another notion:

Definition 12. A class is called **transitive** if it *includes* each of its elements. That is, $\boldsymbol{C}$ is transitive if and only if

$$\forall y\, (y \in \boldsymbol{C} \Rightarrow y \subseteq \boldsymbol{C}),$$

or more suggestively (see Definition 18 below),

$$\forall x\, \forall y\, (x \in y \;\&\; y \in \boldsymbol{C} \Rightarrow x \in \boldsymbol{C}).$$

Now we define a subclass of the class defined in Theorem 29.

[4]One could write $\cdots \in a_2 \in a_1 \in a_0$, or $a_0 \ni a_1 \ni a_2 \ni \cdots$; but it must not be assumed that this implies, for example, $a_2 \in a_0$.
[5]In this case, $a_2 \in a_0$.
[6]We must speak informally here. We have no definition of infinite set.

defn:omega

Definition 13. We denote by

$$\omega$$

the class of all transitive, well-founded sets a that meet each of the following two conditions:

1. For all sets b in a, either
   a) $b = 0$, or
   b) there is a set c in a such that $b = c'$.
2. There is an element b of a such that $b' \notin a$.

Then the **formal natural numbers** compose the class

$$\bigcup \omega.$$

Evidently ω contains 0, and $\{0\}$, and $\{0, 0'\}$, and $\{0, 0', 0''\}$, and so on; but these are just the natural numbers themselves. Indeed, we shall be able to show

$$\bigcup \omega = \omega, \tag{4.2}$$

eqn:

but this will take some work. Without the second condition on elements of ω, (4.2) might be false. Indeed, if this second condition were not imposed, and $\bigcup \omega$ were a set, then $\bigcup \omega$ would be an element of ω, but not of ω.

lem:uo-wf

Lemma 3. *Every element of $\bigcup \omega$ is well-founded.*

Proof. If $n \in \bigcup \omega$, then $n \in a$ for some a in ω. But then a is transitive and well-founded, so $n \subseteq a$, and hence also n is well-founded. $\square$

lem:n-a-eo

Lemma 4. *If $a \in \omega$ and $n \in a$, then $a \cup \{n'\} \in \omega$.*

Proof. The conclusion is trivially true if $n' \in a$; so we may assume $n' \notin a$. By transitivity of a, we have $n \subseteq a$. Since also $\{n\} \subseteq a$, we have $n' \subseteq a$. Thus $a \cup \{n'\}$ is transitive. Next, we show it is well-founded. We have $n' \notin n'$ (since $n' \notin a$). Then $\{n'\} \cap n' = 0$. Suppose $b \subseteq a \cup \{n'\}$ and $b \cap a \neq 0$. Then $b \cap a$ has an element c such that $c \cap b \cap a = 0$. But $c \subseteq a$, so c does not contain n', and therefore $c \cap b = 0$. Thus $a \cup \{n'\}$ is well-founded. Finally, since $n' \notin n'$, we have $n'' \neq n'$; therefore, if $n'' \in a \cup \{n'\}$, then $n'' \in a$, so $n'' \subseteq a$ and therefore $n' \in a$, which is assumed to be false. So $n'' \notin a \cup \{n'\}$. Therefore $a \cup \{n'\} \in \omega$. $\square$

To prove any more, we shall need:

:sep **Axiom 4** (Separation)**.** *Every subclass of a set is a set: For all classes* $\boldsymbol{C}$*,*

$$\forall x\,\exists y\; y = \boldsymbol{C} \cap x. \tag{4.3}$$ eqn:sep

Note that this axioms is really a *scheme* of axioms, one for each class. More precisely, a class $\boldsymbol{C}$ may be given in terms of **parameters:** these are the constants appearing in a formula that defines $\boldsymbol{C}$. If a is such a constant, then the sentence (4.3) above is of the form $\varphi(a)$ for some singulary formula φ; and then the sentence should be understood as an abbreviation of the sentence $\forall z\,\varphi(z)$ (where z is a variable not appearing in φ). If there are more parameters, these too should be universally quantified in turn.

gst The collection of axioms that we have so far—Equality, Null Set, Adjunction, and Separation—together with the sentences derivable from it, can be called **General Set Theory,**[7] or

$$\mathrm{GST}\,.$$

Since, as noted at the beginning of the chapter, some set does exist, the Null Set Axiom is a logical consequence of the Separation Axiom.

We can now make the following refinement of Theorem 29.

-ind **Theorem 30** (Finite Induction)**.** *The class* $\bigcup \omega$ *is the smallest of the classes* $\boldsymbol{D}$ *such that*

1) $0 \in \boldsymbol{D}$*,*

2) for all sets b *in* $\boldsymbol{D}$*, also* $b' \in \boldsymbol{D}$*.*

That is, $\bigcup \omega$ *is such a class, and it is included in* every *such class. In particular,* $\mathbb{N} \subseteq \bigcup \omega$*.*

Proof. We have $\{0\} \in \omega$, so $0 \in \bigcup \omega$. Suppose $n \in \bigcup \omega$. Then $n \in a$ for some a in ω, so $a \cup \{n'\} \in \omega$ by the last lemma, and therefore $n' \in \bigcup \omega$. We have now shown that $\bigcup \omega$ is one of the classes $\boldsymbol{D}$.

Considering any one of these classes $\boldsymbol{D}$, suppose if possible $a \in \bigcup \omega \smallsetminus \boldsymbol{D}$. Then $a \in b$ for some b in ω. The class $b \smallsetminus \boldsymbol{D}$ is a set, by the Separation Axiom. Since $0 \in \boldsymbol{D}$, every element of $b \smallsetminus \boldsymbol{D}$ is c' for some element c of b, and in fact then $c \in b \smallsetminus \boldsymbol{D}$ (since otherwise $c' \in \boldsymbol{D}$). But $c \in c'$. Thus every element of $b \smallsetminus \boldsymbol{D}$ has nonempty intersection with this set. Since b

[7] The theory is so called by Boolos [2, p. 196], but is called STZ by Burgess [4, p. 223], for **S**zmielev and **T**arski with **Z**ermelo's Axiom of Separation.

is well-founded, $b \smallsetminus \boldsymbol{D}$ must be empty. Therefore $b \subseteq \boldsymbol{D}$. Consequently, $\bigcup \omega \subseteq \boldsymbol{D}$. $\square$

So $\bigcup \omega$ admits finite induction for classes. Since, as far as our formal set theory is concerned, classes are the only collections of sets that we can talk about, we may assume $\mathbb{N} = \bigcup \omega$; that is, the formal natural numbers are just the natural numbers. We have not *proved* that $\mathbb{N}$ and $\bigcup \omega$ are the same, only that nothing in our theory will enable us to distinguish them. We *cannot* prove that $\mathbb{N}$ and $\bigcup \omega$ are the same: this will be a consequence of the Compactness Theorem in §7.4. Nonetheless, henceforth **natural number** will mean an element of $\bigcup \omega$.

The easiest use of finite induction is perhaps:

Lemma 5. *Every element of $\bigcup \omega$ is either 0 or n' for some n in $\bigcup \omega$.*

lem:uo-tr

Lemma 6. *Every element of $\bigcup \omega$ is transitive.*

Proof. Trivially, 0 is transitive. If n in $\bigcup \omega$ is transitive, and $a \in n'$, then $a \in n$ or $a = n$, so $a \subseteq n$ and therefore $a \subseteq n'$. By finite induction, all elements of $\bigcup \omega$ are transitive. $\square$

Theorem 31. *$\bigcup \omega$ is transitive.*

Proof. Trivially, $0 \subseteq \bigcup \omega$. Suppose $n \in \bigcup \omega$ and $n \subseteq \bigcup \omega$. Then $n' \subseteq \bigcup \omega$. By finite induction, $\bigcup \omega$ includes each of its elements. $\square$

thm:uo-eo

Theorem 32. *$\bigcup \omega \subseteq \omega$.*

Proof. Let $n \in \bigcup \omega$. Then n is transitive and well-founded, by Lemmas 3 and 6. Also, $n \subseteq \bigcup \omega$ by the last theorem, so every element of n is either 0 or m' for some m in $\bigcup \omega$. In the latter case, $m \in m'$ and $m' \in n$; but also $m' \subseteq n$, so $m \in n$. Finally, if $n \neq 0$, then $n = m'$ for some m, and then $m \in n$, but $m' \notin n$ (since n is well-founded). This shows $n \in \omega$. $\square$

The reverse inclusion is in Theorem 48.

4.4. Arithmetic

sect:arith

By an **iterative structure,**[8] I mean a nonempty class, considered together with

[8]This is my terminology; it is not standard.

1) a distinguished element of the class, and
2) a distinguished singulary operation on the class.

If the class is $\boldsymbol{C}$; the element, e; and the operation, $\boldsymbol{F}$; then we can write $\boldsymbol{C}$ as

$$(\boldsymbol{C}, e, \boldsymbol{F}).$$

Possibly $\boldsymbol{C}$ is a proper subclass of $\operatorname{dom}(\boldsymbol{F})$; but we shall not distinguish between $(\boldsymbol{C}, e, \boldsymbol{F})$ and $(\boldsymbol{C}, e, \boldsymbol{F} \upharpoonright \boldsymbol{C})$. For example, $\bigcup \omega$ is an iterative structure, when considered with 0 and $x \mapsto x'$; in this situation, we may write $\bigcup \omega$ as

$$(\bigcup \omega, 0, {}').$$

If both $(\boldsymbol{C}, e, \boldsymbol{F})$ and $(\boldsymbol{D}, e, \boldsymbol{F})$ are iterative structures, and $\boldsymbol{D} \subseteq \boldsymbol{C}$, then $\boldsymbol{D}$ (or more precisely $(\boldsymbol{D}, e, \boldsymbol{F})$) is an **iterative substructure** of $\boldsymbol{C}$. For example, $(\bigcup \omega, 0, {}')$ is an iterative substructure of $(\mathbf{V}, 0, {}')$.

Generalizing some earlier terminology, we may say that an iterative structure admits **finite induction** if it has no proper iterative substructure. Theorem 30 is that $(\bigcup \omega, 0, {}')$ admits (formal) finite induction.

-inj **Theorem 33.** *Succession on $\bigcup \omega$ is injective.*

Proof. Suppose $a \neq b$, but $a' = b'$. Then $a \cup \{a\} = b \cup \{b\}$, so in particular $a \in b \cup \{b\}$, and therefore $a \in b$; similarly $b \in a$. Then $\{a, b\}$ is ill-founded. But it is a subset of every transitive set that contains a. Therefore $a \notin \bigcup \omega$. □

In sum, we now have:

eano **Theorem 34.**
1. $0 \in \bigcup \omega$.
2. $\bigcup \omega$ *is closed under* $x \mapsto x'$.
3. $0 \neq n'$ *for any* n *in* $\bigcup \omega$.
4. *Succession is injective on* $\bigcup \omega$.
5. $(\bigcup \omega, 0, {}')$ *admits finite induction.*

Proof. The claim is a summary of Theorems 30 and 33, along with the observation that n' is never empty, since it contains n. □

The five conditions in the theorem are called the **Peano Axioms** [31], although Dedekind [9, II: §§ 71, 132] recognized them a bit earlier and understood them better.[9] In any case, for us they are not axioms, but follow from the *definition* of $\bigcup\omega$. A fundamental consequence of the Peano Axioms is the Theorem of Finite Recursion below.

A **homomorphism** from an iterative structure $(\boldsymbol{C}, e, \boldsymbol{F})$ to an iterative structure $(\boldsymbol{D}, f, \boldsymbol{G})$ is a function $\boldsymbol{H}$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ such that

$$\boldsymbol{H}(e) = f, \qquad\qquad \boldsymbol{H} \circ (\boldsymbol{F} \upharpoonright \boldsymbol{C}) = \boldsymbol{G} \circ \boldsymbol{H}.$$

This situation is depicted in Figure 4.1. Another way to write the second

$$\begin{tikzcd}
\{e\} \arrow[r] \arrow[d, "\boldsymbol{H}"'] & \boldsymbol{C} \arrow[r, "\boldsymbol{F}"] \arrow[d, "\boldsymbol{H}"'] & \boldsymbol{C} \arrow[d, "\boldsymbol{H}"] \\
\{f\} \arrow[r] & \boldsymbol{D} \arrow[r, "\boldsymbol{G}"'] & \boldsymbol{G}
\end{tikzcd}$$

fig:hom

Figure 4.1. A homomorphism of iterative structures

equation is

$$\forall x\,(x \in \boldsymbol{C} \Rightarrow \boldsymbol{H}(\boldsymbol{F}(x)) = \boldsymbol{G}(\boldsymbol{H}(x))).$$

We use the previous theorem to establish the following.

thm:rec

Theorem 35 (Finite Recursion). *For every iterative structure* $(\boldsymbol{D}, e, \boldsymbol{F})$, *there is a unique homomorphism from* $(\bigcup\omega, 0, {}')$ *to* $(\boldsymbol{D}, e, \boldsymbol{F})$.

Proof. Let $\boldsymbol{C}$ be the class of all sets h such that, for some a in ω,

1) h is a function from a to $\boldsymbol{D}$,
2) $h(0) = e$, that is, $(0, e) \in h$, and
3) if $k' \in a$, so that $k \in a$, then $h(k') = \boldsymbol{F}(h(k))$, that is, if $(k, x) \in h$, then
$$(k', \boldsymbol{F}(x)) \in h.$$

Let $\boldsymbol{R} = \bigcup \boldsymbol{C}$. We first prove that, for each n in $\bigcup\omega$, there is b in $\boldsymbol{D}$ such that $n\,\boldsymbol{R}\,b$.

1. Since $\{0\} \in \omega$, and $0 \neq n'$ for any n, we have $\{(0, e)\} \in \boldsymbol{C}$, so $0\,\boldsymbol{R}\,e$.

[9]I say this because, unlike Peano, Dedekind stated clearly that induction was not enough for proving the Theorem of Finite Recursion, Theorem 35 below.

2. Suppose $k \boldsymbol{R} a$. Then $a = h(k)$ for some h in $\boldsymbol{C}$. By Lemma 4, $\operatorname{dom}(h) \cup \{k'\} \in \omega$. If $k' = \ell'$, then $k = \ell$, by Theorem 33. Hence $h \cup \{k', \boldsymbol{F}(h(k))\} \in \boldsymbol{C}$, so $k' \boldsymbol{R} (\boldsymbol{F}(a))$.

Thus, by finite induction, for each n in $\bigcup \omega$ there is b in $\boldsymbol{D}$ such that $n \boldsymbol{R} b$.

We next prove that there is only one such b.

1. Suppose $0 \boldsymbol{R} a$. Then $h(0) = a$ for some h in $\boldsymbol{C}$, but then also $h(0) = e$, so $e = a$.
2. Suppose, for some k in $\bigcup \omega$, there is just one set b such that $k \boldsymbol{R} b$. Say $k' \boldsymbol{R} c$. Then $h(k') = c$ for some h in $\boldsymbol{C}$. But also $h(k') = \boldsymbol{F}(h(k))$, and by our assumption $h(k)$ must be b, so $c = \boldsymbol{F}(b)$.

By finite induction again, $\boldsymbol{R}$ is a function on $\bigcup \omega$ with the desired properties.

By induction yet again, this function is unique. □

If the homomorphism guaranteed by the theorem is called $\boldsymbol{H}$, we may say that it is determined by the requirements

$$\boldsymbol{H}(0) = e, \qquad \boldsymbol{H}(x') = \boldsymbol{F}(\boldsymbol{H}(x)).$$

Now we can obtain the usual operations on $\bigcup \omega$.

:add **Definition 14** (Addition)**.** For each m in $\bigcup \omega$, the operation $x \mapsto m + x$ on $\bigcup \omega$ is the homomorphism from $(\bigcup \omega, 0, ')$ to $(\bigcup \omega, m, ')$ determined by

$$m + 0 = m, \qquad m + n' = (m + n)'.$$

In particular, we have

$$m + 1 = m + 0' = (m + 0)' = m',$$

so we may write $m + 1$ for m'.

Lemma 7. *For all n and m in $\bigcup \omega$,*

1) $0 + n = n$;
2) $m' + n = m + n'$.

Theorem 36. *For all n, m, and k in $\bigcup \omega$,*

1) $n + m = m + n$;

2) $(n+m)+k = n+(m+k)$;

In fact an operation of addition satisfying the theorem can be defined on *any* iterative structure that admits induction.

`defn:mult` **Definition 15** (Multiplication)**.** For each m in $\bigcup\omega$, the operation $x \mapsto m{\cdot}x$ on $\bigcup\omega$ is the homomorphism from $(\bigcup\omega, 0, {}')$ to $(\bigcup\omega, 0, x \mapsto x{+}m)$. That is,

$$m \cdot 0 = 0, \qquad\qquad m \cdot (n+1) = m \cdot n + m.$$

Lemma 8. *For all n and m in $\bigcup\omega$,*
1) $0 \cdot n = 0$;
2) $(m+1) \cdot n = m \cdot n + n$.

Theorem 37. *For all n, m, and k in $\bigcup\omega$,*
1) $n \cdot m = m \cdot n$;
2) $n \cdot (m+k) = n \cdot m + n \cdot k$;
3) $(n \cdot m) \cdot k = n \cdot (m \cdot k)$;

As with addition, so with multiplication: an operation satisfying the theorem can be defined on any iterative structure that admits induction. However, the next theorem needs all of the Peano Axioms.

Theorem 38 (Cancellation)**.** *For all n, m, and k in $\bigcup\omega$,*
1) if $n+k = m+k$, *then* $n = m$;
2) if $n \cdot k = m \cdot k$, *then* $n = m$.

`defn:exp` **Definition 16** (Exponentiation)**.** For each m in $\bigcup\omega$, the operation $x \mapsto m^x$ on $\bigcup\omega$ is the homomorphism from $(\bigcup\omega, 0, {}')$ to $(\bigcup\omega, 1, x \mapsto x \cdot m)$ determined by

$$m^0 = 1, \qquad\qquad m^{n+1} = m^n \cdot m.$$

Theorem 39. *For all n, m, and k in $\bigcup\omega$,*
1) $n^{m+k} = n^m \cdot n^k$;
2) $(n \cdot m)^k = n^k \cdot m^k$;
3) $(n^m)^k = n^{m \cdot k}$.

In contrast with addition and multiplication, exponentiation requires more than induction for its existence.

For some operations on $\bigcup\omega$, Theorem 35 as stated is not enough to establish their existence. One needs:

Theorem 40 (Finite Recursion with Parameter)**.** *Suppose $e \in \boldsymbol{D}$, and $\boldsymbol{F}$ is a function from $\bigcup \omega \times \boldsymbol{D}$ to $\boldsymbol{D}$. Then there is a* unique *function $\boldsymbol{G}$ from $\bigcup \omega$ to $\boldsymbol{D}$ such that*

1) *$\boldsymbol{G}(0) = e$, and*
2) *$\boldsymbol{G}(n+1) = \boldsymbol{F}(n, \boldsymbol{G}(n))$ for all n in $\bigcup \omega$.*

Proof. The function $\boldsymbol{G}$ will be such that $x \mapsto (x, \boldsymbol{G}(x))$ is the homomorphism from $(\bigcup \omega, 0, ')$ to $(\bigcup \omega \times \boldsymbol{D}, (0, e), (x, y) \mapsto (x, \boldsymbol{F}(x, y)))$. □

Definition 17 (Factorial)**.** The operation $x \mapsto x!$ on $\bigcup \omega$ is the function $\boldsymbol{G}$ guaranteed by the theorem when $\boldsymbol{D}$ is $\bigcup \omega$ and e is 1 and $\boldsymbol{F}$ is $(x, y) \mapsto (x+1) \cdot y$. That is,

$$0! = 1, \qquad\qquad (n+1)! = (n+1) \cdot n!$$

4.5. Orderings

The relations $\subset$ and $\prec$ on $\mathbf{V}$ are examples of *orderings.*

rder

Definition 18. A binary relation $\boldsymbol{R}$ on a class $\boldsymbol{C}$ is **irreflexive** if, for all a and b in $\boldsymbol{C}$,

$$a \mathrel{\boldsymbol{R}} b \Rightarrow a \neq b;$$

transitive, if

$$a \mathrel{\boldsymbol{R}} b \;\&\; b \mathrel{\boldsymbol{R}} c \Rightarrow a \mathrel{\boldsymbol{R}} c.$$

If $\boldsymbol{R}$ is both irreflexive and transitive on $\boldsymbol{C}$, it is an **ordering** of $\boldsymbol{C}$, and $\boldsymbol{C}$ is an **order** with respect to $\boldsymbol{R}$. So considered, $\boldsymbol{C}$ can be written as

$$(\boldsymbol{C}, \boldsymbol{R}).$$

If in addition $\boldsymbol{R}$ **connects** $\boldsymbol{C}$, that is, for all a and b in $\boldsymbol{C}$,

$$a \neq b \Rightarrow a \mathrel{\boldsymbol{R}} b \vee b \mathrel{\boldsymbol{R}} a,$$

then $\boldsymbol{R}$ **linearly orders** $\boldsymbol{C}$, and $\boldsymbol{R}$ is a **linear ordering** of $\boldsymbol{C}$, and $(\boldsymbol{C}, \boldsymbol{R})$ is a **linear order.** In this case, if $a \mathrel{\boldsymbol{R}} b$, we say that a is **less** than b with respect to $\boldsymbol{R}$. Also a is the **least** element of $\boldsymbol{C}$ if it is less than all other elements.

Note well that a transitive *relation* is not the same thing as a transitive *class* (even though a relation is technically a class). Membership may be transitive on a class that is not transitive: a trivial example is $\{1\}$, a more interesting example is $\{1, 2, 3\}$. Membership may fail to be transitive on a transitive class, for example, $\{0, \{0\}, \{\{0\}\}\}$.

By the definition, $(\mathbf{V}, \subset)$ and $(\mathbf{V}, \prec)$ are orders. They are not linear orders, because neither relation contains either of $(\{0\}, \{1\})$ and $(\{1\}, \{0\})$.

The converse of an ordering of a class is also an ordering of that class; of a linear ordering, a linear ordering. Often a linear ordering is denoted by a symbol like $<$, and then its converse is $>$. Also the relation defined by $x < y \vee x = y$ is denoted by $\leqslant$.[10]

To understand how $\bigcup \omega$ is ordered, we observe:

Theorem 41. *On $\bigcup \omega$, membership is the same as proper inclusion.*

Proof. Since elements of $\bigcup \omega$ are transitive and well-founded by Lemmas 3 and 6, for all k and n in $\bigcup \omega$ we have

$$k \in n \Rightarrow k \subset n.$$

We show the converse, namely

$$k \subset n \Rightarrow k \in n.$$

This is vacuously true when $n = 0$. Suppose it is true when $n = m$. If $m \in k$, then $m \subseteq k$ and hence $m + 1 \subseteq k$. So, supposing $k \subset m + 1$, we have $m \notin k$ and therefore $k \subseteq m$. Either $k = m$, or by inductive hypothesis, $k \in m$; in either case, $k \in m + 1$. □

[10] I have chosen terminology so that the relations that we are interested in will have the simplest possible descriptions. We have a standard symbol, namely $\in$, for the relation defined by $x \in y$, but not for the relation defined by $x \in y \vee x = y$. Therefore, even though both $\subset$ and $\subseteq$ are standard symbols, I treat $\subset$ as more basic: it is *this* that I call an ordering (and $\in$ will be an ordering of $\bigcup \omega$). In many references, it is $\subseteq$ that is called an ordering; in that case, $\subset$ would be a *strict* ordering. In fact, $\subseteq$ is often called a *partial* ordering; but then an ordering is still called *linear* or *total* if it connects the class it orders. I have chosen not to require the use of an adjective in every case, but to let $\subset$ be an *ordering,* simply. Note that, while $\prec$ is now also an ordering, the relation $\preccurlyeq$ is not defined by $x \prec y \vee x = y$. Finally, since the words *order* and *ordering* are both available, I have decided to use the latter for the relation, and the former for the class that the relation orders.

On $\bigcup \omega$ therefore, we can denote membership and proper inclusion by the same symbol,

$$<;$$

this orders $\bigcup \omega$, since partial inclusion orders all classes. If $m < n$, we may say that m is a **predecessor** of n.

near **Theorem 42.** *$(\bigcup \omega, <)$ is a linear order.*

Proof. We show, for all m and n in $\bigcup \omega$,

$$m \subseteq n \vee n \subseteq m.$$

This is trivially true when $n = 0$. Suppose it is true when $n = k$. If we do not have $m \subseteq k + 1$, then *a fortiori* we do not have $m \subseteq k$, so, by inductive hypothesis, we have $k \subset m$, that is, $k \in m$, so $k + 1 \subseteq m$. $\square$

The connection between the ordering of $\bigcup \omega$ and the algebraic structure of $\bigcup \omega$ is given by:

defn **Theorem 43.** *If n and m are in $\bigcup \omega$, then*

$$m \leqslant n \Leftrightarrow \exists x \, (x \in \bigcup \omega \;\&\; m + x = n).$$

The theorem can be taken as a *definition* of $\leqslant$ on $\bigcup \omega$. Using this definition, one can prove the next two theorems.

Theorem 44. *For all n, m, and k in $\bigcup \omega$,*
1) $0 \leqslant n$;
2) $m \leqslant n$ if and only if $m + k \leqslant n + k$;
3) $m \leqslant n$ if and only if $m \cdot (k + 1) \leqslant n \cdot (k + 1)$.

m:<1 **Theorem 45.** *For all m and n in $\bigcup \omega$,*
1) $m < n$ if and only if $m + 1 \leqslant n$;
:leq *2) $m \leqslant n$ if and only if $m < n + 1$.*

Theorems 43 and 45 can be used to prove Theorem 42. The next theorem introduces a new proof-technique, admitted by certain orders.

m:SI **Theorem 46** (Transfinite Induction)**.** *Suppose $\boldsymbol{C}$ is a class such that, for all n in $\bigcup \omega$,*

$$n \subseteq \boldsymbol{C} \Rightarrow n \in \boldsymbol{C}. \tag{4.4}$$ eqn:tr-ind

Then $\bigcup \omega \subseteq \boldsymbol{C}$.

Proof. Another way to write (4.4) is

$$n \subseteq \boldsymbol{C} \Rightarrow n + 1 \subseteq \boldsymbol{C}.$$

Then by induction, $n \subseteq \boldsymbol{C}$ for all n in $\bigcup \omega$. In particular, for all n in $\bigcup \omega$, we have $n + 1 \subseteq \boldsymbol{C}$, and therefore $n \in \boldsymbol{C}$. □

An application of transfinite induction is the following.

thm:omega-wo **Theorem 47.** *Every non-empty subclass of $\bigcup \omega$ has a least element with respect to $\leqslant$.*

Proof. Suppose $\boldsymbol{C}$ is a subclass of $\bigcup \omega$ with no least element. We show $\boldsymbol{C} = 0$, that is, $\bigcup \omega \smallsetminus \boldsymbol{C} = \bigcup \omega$. We use transfinite induction. Suppose $n \subseteq \bigcup \omega \smallsetminus \boldsymbol{C}$. Then $\boldsymbol{C} \subseteq \bigcup \omega \smallsetminus n$. Since n is the least element of $\bigcup \omega \smallsetminus n$, we must have $n \notin \boldsymbol{C}$, so $n \in \bigcup \omega \smallsetminus \boldsymbol{C}$. □

Finally, we can simplify notation with the following, which complements Theorem 32.

thm:eo-uo **Theorem 48.** $\omega = \bigcup \omega$.

Proof. It is enough to show $\omega \subseteq \bigcup \omega$. Suppose if possible $\omega \smallsetminus \bigcup \omega$ contains a. Then $a \in \omega$, so $a \subseteq \bigcup \omega$, and a is transitive, but $a \neq 0$. Hence a has an element n, which is in $\bigcup \omega$. Then $n+1 \subseteq a$, but $a \neq n+1$ (since $n + 1 \in \bigcup \omega$), so a has an element r that is greater than n. Then $n + 1 \leqslant r$ by Theorem 45, and therefore $n + 1 \in a$ by transitivity of a. In short, a is nonempty, but closed under succession. This violates the last condition (in Definition 13) of being in ω. □

Now we have $\bigcup \omega = \omega$, so can write ω instead of $\bigcup \omega$.

4.6. Finite sets

Presumably a set is *finite* if and only if it is equipollent with some natural number. But we can define finite sets without referring to natural numbers as such, just by following the pattern of the *definition* of $\mathbb{N}$ (Definition 10):

Definition 19. The **finite** sets are given recursively by two rules:

1. 0 is finite.

2. If a is finite, then so is $a \cup \{b\}$ (for all sets b).
Sets that are not finite are **infinite.**

As with $\mathbb{N}$, so with the collection of finite sets: we should like to understand it as a class. We can try using an analogue of Definition 13, the definition of $\bigcup \omega$, which turns out to be ω itself. This does not work. Indeed, suppose $\boldsymbol{C}$ is the class of all well-founded sets a such that, for all b in a, either $b = 0$, or $b = c \cup \{d\}$ for some set c in a and some set d. If ω is a set, then $\boldsymbol{C}$ contains $\{0\} \cup \{\omega \smallsetminus x \colon x \in \omega\}$, and therefore $\omega \in \bigcup \boldsymbol{C}$, although presumably ω is not finite. Well-foundedness does not prevent this problem; something else is needed.

Definition 20. A subset of $\mathscr{P}(a)$ it **inductive** if it contains 0 and is closed under each operation $x \mapsto x \cup \{b\}$, where $b \in a$. A set a is **formally finite** if it belongs to each inductive subset of $\mathscr{P}(a)$.

We aim to prove an analogue of Theorem 30; but for this, we need a characterization of the formally finite sets that involves natural numbers. We develop this now.

Theorem 49. *Every function whose domain is a natural number is a set, and then the range of the function is also a set.*

Proof. Since a function as such is a kind of class, we cannot speak of the 'class of functions with domain n' until we actually prove this theorem. In particular, we can prove this theorem by induction only for one theorem at a time. Given a function $\boldsymbol{F}$ whose domain is a natural number n, we can embed $\boldsymbol{F}$ in $\boldsymbol{F} \cup \{(x, 0) \colon x \in \bigcup \omega \smallsetminus n\}$, which is a function whose domain is $\bigcup \omega$. Suppose $\boldsymbol{G}$ is a function on $\bigcup \omega$. By induction, for each n in $\bigcup \omega$, the restriction of $\boldsymbol{G}$ to n is a set, and the image of n under $\boldsymbol{G}$ is a set. $\square$

In particular, every class that is equipollent to a natural number is a set.

Lemma 9. *For all m and n in ω, if $m + 1 \approx n + 1$, then $m \approx n$.*

Proof. Suppose f is a bijection from $m + 1$ to $n + 1$. Say $f(k) = n$. Let

$$g = (f \smallsetminus \{(k, n), (m, f(m))\}) \cup \{(k, f(m)), (m, n)\}.$$

(See Figure 4.2. If $k = m$, then $g = f$.) Then $g \restriction m$ is a bijection from m to n. $\square$

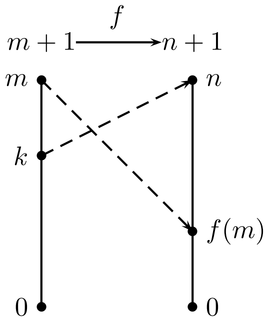


fig:mn

Figure 4.2. A bijection from a natural number to another

Theorem 50. *Every set is equipollent to at most one natural number.*

Proof. By finite induction, we show that equipollent natural numbers are equal. If $n \in \omega$, and $0 \approx n$, then $n = 0$. Suppose m is a natural number that is equipollent only to itself among natural numbers. If $m + 1$ is equipollent to some natural number, then that number must be $n + 1$ for some n, and therefore $m \approx n$ by the lemma, so $m = n$ and therefore $m + 1 = n + 1$. □

If $n \in \omega$ and $a \approx n$, we can now call n the **size** of a; we denote this size by

$$|a|.$$

Theorem 51. *If $|a| = n$ and $|b| = m$ and $a \cap b = 0$, then $a \cup b$ is a finite set, and*

$$|a \cup b| = n + m.$$

Proof. Use finite induction. Assume $|a| = n$. Then $|a \cup 0| = n = n + 0$. Suppose $|b| = m$ and $|a \cup b| = n + m$. If $c \notin a \cup b$, then $|a \cup b \cup \{c\}| = (n + m) + 1 = n + (m + 1)$. □

thm:fin-pow

Theorem 52. *If $a \approx n$, then $\mathscr{P}(a)$ is a set, and*

$$|\mathscr{P}(a)| = 2^n.$$

Proof. If $|a| = 0$, then $\mathscr{P}(a) = \{0\} = 1 = 2^0$. Suppose $|a| = n$ and $|\mathscr{P}(a)| = 2^n$. If $b \notin a$, then

$$\mathscr{P}(a \cup \{b\}) = \mathscr{P}(a) \cup \{x \cup \{b\} \colon x \in \mathscr{P}(a)\},$$

and the two sets are disjoint and equipollent, so

$$|\mathscr{P}(a \cup \{b\})| = |\mathscr{P}(a)| + |\mathscr{P}(a)| = 2^n + 2^n = 2^n \cdot 2 = 2^{n+1}. \qquad \square$$

Lemma 10. *If $n \in \omega$ and $a \approx n$, the only inductive subset of $\mathscr{P}(a)$ is itself.*

Proof. Since $\mathscr{P}(0) = \{0\}$, and all inductive subsets of power sets contain 0, the claim is true for sets of size 0. Suppose the claim is true for sets of size n. Say $|a| = n$ and $b \notin a$. If c is an inductive subset of $\mathscr{P}(a \cup \{b\})$, then $c \cap \mathscr{P}(a)$ is an inductive subset of $\mathscr{P}(a)$, so by hypothesis $\mathscr{P}(a) \subseteq c$, and therefore $d \cup \{b\} \in c$ for all d in $\mathscr{P}(a)$,—that is, $\mathscr{P}(a \cup \{b\}) \subseteq c$. □

Theorem 53. *A set is formally finite if and only if it is equipollent to a natural number.*

Proof. If $|a| = n$, then we now know that $\mathscr{P}(a)$ is a set and is the only inductive subset of itself; and it contains a; so a is formally finite.

Conversely, let $\boldsymbol{F}$ be the class of all sets that are equipollent to natural numbers. Say a is formally finite. Let b be an inductive subset of $\mathscr{P}(a)$. Then $b \cap \boldsymbol{F}$ is also an inductive subset of $\mathscr{P}(a)$, so it contains a. In particular, $a \in \boldsymbol{F}$. □

Now finally we have an analogue of Theorem 30:

Theorem 54. *The class of formally finite sets is the smallest of the classes $\boldsymbol{D}$ that contain 0 and are closed under each operation $x \mapsto x \cup \{a\}$.*

Proof. Trivially, 0 is formally finite. Suppose a is formally finite, so that $a \approx n$ for some n in ω. Then $a \cup \{b\}$ is equipollent to n or $n + 1$, so it is formally finite. Therefore the class of formally finite sets is one of the classes $\boldsymbol{D}$.

Now let $\boldsymbol{D}$ be any one of those classes, and suppose a is formally finite. Then $\boldsymbol{D} \cap \mathscr{P}(a)$ is an inductive subset of $\mathscr{P}(a)$, so it contains a, and in particular $a \in \boldsymbol{D}$. □

Therefore finite sets are formally finite, and we may assume that all formally finite sets are finite.

Theorem 55. *Subsets of finite sets are finite. Moreover, if $|b| = n$, and $a \subset b$, then $|a| < n$.*

Proof. It is enough to consider subsets of natural numbers. The only subset of 0 is itself. Suppose every subset of n is finite, and every proper subset has less size. Say $a \subset n+1$. If $a = n$, then $|a| < n+1$. If $a \neq n$, then $a \smallsetminus \{n\} \subset n$, so $|a \smallsetminus \{n\}| < n$, and therefore $|a| < n+1$. □

Corollary. *If $\omega \preccurlyeq a$, then a is infinite.*

Proof. By the theorem, if $|a| = n$, then there is not even an injection from $n+1$ into a, much less from ω. □

We cannot now prove the converse of the corollary. We do however have an alternative formulation of the condition $\omega \preccurlyeq a$.

Theorem 56. *Let a be a set with an element b. Then $a \preccurlyeq a \smallsetminus \{b\}$ if and only if $\omega \preccurlyeq a$.*

Proof. Suppose $\boldsymbol{F}$ is an injection from a into $a \smallsetminus \{b\}$. Then $(a, b, \boldsymbol{F})$ is an iterative structure. Let $\boldsymbol{H}$ be the unique homomorphism from $(\omega, 0, {}')$ to $(a, b, \boldsymbol{F})$. Then $\boldsymbol{H}$ is injective. Indeed, if $0 < n$, then $\boldsymbol{H}(n) \neq b$, that is, $\boldsymbol{H}(n) \neq \boldsymbol{H}(0)$. Suppose $m \in \omega$, and for all n in ω, if $m \neq n$, then $\boldsymbol{H}(m) \neq \boldsymbol{H}(n)$. Then $\boldsymbol{F}(\boldsymbol{H}(m)) \neq \boldsymbol{F}(\boldsymbol{H}(n))$, that is, $\boldsymbol{H}(m+1) \neq \boldsymbol{H}(n+1)$. This is enough to show $\boldsymbol{H}$ is injective, and therefore $\omega \preccurlyeq a$.

Suppose conversely $\boldsymbol{H}$ is an injection from ω into a. Let $b = \boldsymbol{H}(0)$. Then $\{(\boldsymbol{H}(x), \boldsymbol{H}(x+1)) \colon x \in \omega\} \cup \{(x, x) \colon x \in a \smallsetminus \boldsymbol{H}[\omega]\}$ is an injection from a into $a \smallsetminus \{b\}$. □

5. Ordinality

h:ON

5.1. Well-ordered classes

:woc

By Theorem 47 (and the equality of $\bigcup \omega$ and ω guaranteed by Theorems 32 and 48), the class ω is *well-ordered* by membership.

n:ln

Definition 21. Suppose $(\boldsymbol{C}, <)$ is a linear order. If $b \in \boldsymbol{C}$, we define

$$\operatorname{pred}_{(\boldsymbol{C},<)}(b) = \{x \colon x \in \boldsymbol{C} \;\&\; x < b\};$$

this class is called a **section** of $(\boldsymbol{C}, <)$, and its elements are the **predecessors** of b in $(\boldsymbol{C}, <)$. We may denote the section simply by

$$\operatorname{pred}(b),$$

if the linear order is understood. Suppose all sections of $(\boldsymbol{C}, <)$ are sets. Then let us say that $(\boldsymbol{C}, <)$ is a **left-narrow** linear order.[1] Two possibilities are distinguished by name:

1. Suppose every nonempty subclass of $\boldsymbol{C}$ has a least element with respect to $<$. Then $(\boldsymbol{C}, <)$ is a **good order,**[2] and $\boldsymbol{C}$ is **well-ordered** by $<$.
2. Suppose $\boldsymbol{C}$ is the only subclass $\boldsymbol{D}$ of $\boldsymbol{C}$ such that, for every element b of $\boldsymbol{C}$,

   $$\operatorname{pred}(b) \subseteq \boldsymbol{D} \Rightarrow b \in \boldsymbol{D}.$$

   Then $(\boldsymbol{C}, <)$ admits **transfinite induction.**

When the linear order in question is $(\omega, <)$, the situation is simple: If $n \in \omega$, then

$$\operatorname{pred}_{(\omega,\in)}(n) = n,$$

[1] This terminology is used by Levy [28, p. 33].

[2] One of the irregularities of English is that *well* is the adverb corresponding to the adjective *good.* One does not want technical terminology to have to conform to linguistic irregularities; therefore a *good order* as defined here is often called a *well-order.*

which is of course a set; so $(\omega, <)$ is left-narrow. Again, by Theorem 47, $(\omega, <)$ is well-ordered; it admits transfinite induction, by Theorem 46. Indeed, any left-narrow linear order that has one property has the other:

Theorem 57. *A left-narrow linear order is good if and only if it admits transfinite induction.*

Proof. Suppose $(\boldsymbol{C}, <)$ is a left-narrow linear order, $\boldsymbol{D} \subset \boldsymbol{C}$, and $b \in \boldsymbol{C}$. Then b is the least element of $\boldsymbol{C} \smallsetminus \boldsymbol{D}$ if and only if $\operatorname{pred}(b) \subseteq \boldsymbol{D}$, but $b \notin \boldsymbol{D}$. In other words, $\boldsymbol{C} \smallsetminus \boldsymbol{D}$ has no least member if and only if $\forall x\,(\operatorname{pred}(x) \subseteq \boldsymbol{D} \Rightarrow x \in \boldsymbol{D})$. So $\boldsymbol{C} \smallsetminus \boldsymbol{D}$ is a counterexample, showing that $(\boldsymbol{C}, <)$ is not a good order, if and only if $\boldsymbol{D}$ is a counterexample, showing that $(\boldsymbol{C}, <)$ does not admit transfinite induction. □

In the definitions of linear orders that are good and that admit transfinite induction, only sub*sets* and complements of subsets are considered, respectively. This ensures that each of these properties is expressed by a single sentence. In fact, subclasses and their complements can be considered (as in Theorems 46 and 47):

thmsch:go

Theorem 58. *Suppose $(\boldsymbol{C}, <)$ is a left-narrow linear order. For it to be a good order, either of the following conditions is sufficient.*

1. *Every nonempty subset of $\boldsymbol{C}$ has a least element.*
2. *The empty set is the only subset a of $\boldsymbol{C}$ such that, for all b in $\boldsymbol{C}$,*

$$\operatorname{pred}(b) \subseteq \boldsymbol{C} \smallsetminus a \Rightarrow b \in \boldsymbol{C} \smallsetminus a.$$

Proof. 1. Suppose $\boldsymbol{D}$ is a subclass of $\boldsymbol{C}$, and $a \in \boldsymbol{D}$. Then $\boldsymbol{D} \cap \operatorname{pred}(a)$ is a set. If this set is empty, then a is the least element of $\boldsymbol{D}$. If the set is not empty, but has a least element, then this is the least element of $\boldsymbol{D}$.

2. Under the given condition, if a is a nonempty subset of $\boldsymbol{C}$, then a has an element b such that $\operatorname{pred}(b) \subseteq \boldsymbol{C} \smallsetminus a$, so that b is the least element of a. Thus the first condition is met. □

We shall want to know that subclasses of well-ordered classes are well-ordered.

Theorem 59. *If $(\boldsymbol{C}, <)$ is a good order, and $\boldsymbol{D} \subseteq \boldsymbol{C}$, then $(\boldsymbol{D}, <)$ is a good order.*

Proof. All that needs to be checked is that sections of $(\boldsymbol{D}, <)$ are sets; but every such section is $\boldsymbol{D} \cap \mathrm{pred}_{(\boldsymbol{C},<)}(a)$ for some a in $\boldsymbol{D}$, so the section is a set by the Separation Axiom. □

In particular, as ω is well-ordered by membership, so are its subclasses; and among these subclasses are its elements, because ω is transitive. A further connection with what we already know is made by the following.

Theorem 60. *A class is well-ordered by membership if and only if it is well-founded and linearly ordered by membership.*

Proof. Suppose $\boldsymbol{C}$ is linearly ordered by membership. If $a \subseteq \boldsymbol{C}$, and $b \in a$, then $b \cap a = 0$ if and only if b is the least element of a with respect to membership. □

5.2. Ordinals

nals

We now know that ω and its elements are transitive and well-ordered by membership; equivalently, by the last theorem, they are transitive, well-founded, and linearly ordered by membership.

Definition 22. A *set* that is transitive and well-ordered by membership is called an **ordinal.**[3] The class of all ordinals is denoted by

$$\mathbf{ON}.$$

nals

The Greek letters α, β, γ, ... will invariably denote ordinals.

In particular, ω is a class of ordinals, and ω itself is either an ordinal or a proper class of ordinals.

Theorem 61. **ON** *contains* 0 *and is closed under* $x \mapsto x'$*; so* $(\mathbf{ON}, 0, ')$ *is an iterative structure with substructure* $(\omega, 0, ')$.

Lemma 11. **ON** *is transitive, that is, every element of an ordinal is an ordinal. Also every ordinal* properly *includes its elements.*

[3]The definition is due to von Neumann [39].

Proof. Suppose $\alpha \in \mathbf{ON}$ and $b \in \alpha$. We want to show b is an ordinal, that is, b is transitive and well-ordered by $\in$. But $b \subseteq \alpha$ by transitivity of α, so b, like α, is well-ordered by membership.

Suppose $c \in b$; we want to show $c \subseteq b$. That is, suppose $d \in c$; we want to show $d \in b$. But $b \subseteq \alpha$, so $c \in \alpha$. Then also $c \subseteq \alpha$, and hence $d \in \alpha$. So b, c, and d are all in α, and $d \in c$, and $c \in b$. Since membership is a transitive relation on α, we have $d \in b$. Thus $c \subseteq b$, so b is transitive. Now we know b is an ordinal. Therefore $\alpha \subseteq \mathbf{ON}$. So $\mathbf{ON}$ is transitive.

Finally, $\alpha \notin \alpha$, since membership, being a linear ordering of α, is irreflexive. But $b \in \alpha$, so $b \neq \alpha$, and therefore $b \subset \alpha$. □

So every element of an element of an ordinal α is an ordinal; and every element of an element of an element of an ordinal is an ordinal, and so on; moreover, all of these elements are elements of α.

n:pinc->cont

Lemma 12. *Every ordinal contains every ordinal that it properly includes.*

Proof. Suppose $\beta \subset \alpha$. Then $\alpha \smallsetminus \beta$ contains some γ, which is an ordinal by the last lemma. We first show $\beta \subseteq \gamma$. Suppose $\delta \in \beta$; we show $\delta \in \gamma$. Since $\gamma \notin \beta$, we have $\gamma \neq \delta$. But also $\delta \subseteq \beta$, so $\gamma \notin \delta$. Since membership on β is a linear ordering, we must have $\delta \in \gamma$.

We now show that, if γ is the *least* member of $\alpha \smallsetminus \beta$, then $\gamma = \beta$. Suppose on the contrary $\beta \subset \gamma$. Then $\gamma \smallsetminus \beta$ contains some δ. In particular, since $\gamma \subseteq \alpha$, we have $\delta \in \alpha \smallsetminus \beta$. So $\alpha \smallsetminus \beta$ contains γ and δ, and $\delta \in \gamma$. In particular, γ is not the least element of $\alpha \smallsetminus \beta$. □

thm:on

Theorem 62 (Burali-Forti Paradox [3])**.** $\mathbf{ON}$ *is transitive and well-ordered by membership; so it is not a set.*

Proof. Let α and β be two ordinals such that $\beta \notin \alpha$. By transfinite induction in α, we show $\alpha \subseteq \beta$. Indeed, say $\gamma \in \alpha$ and $\operatorname{pred}_{(\alpha,\in)}(\gamma) \subseteq \beta$, that is, $\gamma \subseteq \beta$. Then $\gamma \neq \beta$, so $\gamma \subset \beta$ and therefore, by the last lemma, $\gamma \in \beta$. By transfinite induction, $\alpha \subseteq \beta$. In particular, if $\alpha \neq \beta$, then $\alpha \subset \beta$, so $\alpha \in \beta$. Therefore $(\mathbf{ON}, \in)$ is a left-narrow linear order.

In particular, if $\alpha \in \mathbf{ON}$, then $\alpha \neq \mathbf{ON}$. So $\mathbf{ON}$ is not a member of itself, even if it is a set; in particular, $\mathbf{ON}$ is not an ordinal.

If a is a set of ordinals with an element β, then the least element of a is the least element of $a \cap \beta$, if this set is nonempty; otherwise it is β.

Thus **ON** is well-ordered by membership. Since however **ON** is not an ordinal, it must not be a set. □

As a consequence of the last two lemmas, we have

$$\alpha \in \beta \Leftrightarrow \alpha \subset \beta.$$

Implicitly, α and β are ordinals; an ordinal β may have a proper subset that is not an ordinal and is not an element of β. But on **ON**, we may use $<$ to denote either membership or proper inclusion. Then

$$\mathrm{pred}_{(\mathbf{ON},<)}(\alpha) = \alpha.$$

Again, if ω is a set, then it is an ordinal. There is only one alternative:

Theorem 63. *If ω is a proper class, then it is* **ON**.

Proof. We know $\omega \subseteq \mathbf{ON}$. If $\omega \subset \mathbf{ON}$, and $\alpha \in \mathbf{ON} \setminus \omega$, then, by the first part of the proof of Lemma 12, we have $\omega \subseteq \alpha$, so ω is a set. □

5.3. Limits

Suppose ω is indeed a set. Then $\omega \in \mathbf{ON}$, and $\omega \neq 0$, but for all ordinals α, if $\alpha < \omega$, then $\alpha' < \omega$. In a word, if it is a set, then ω is a *limit* of $(\mathbf{ON}, <)$.

Definition 23. An element b of an arbitrary well-ordered class $(\boldsymbol{C}, <)$
ssor is a **successor,** and in particular is the successor of the element a, if b is the least element of $\{x\colon x \in \boldsymbol{C}\ \&\ a < x\}$ (which is $\boldsymbol{C} \setminus (\mathrm{pred}(a) \cup \{a\})$). In this case, we may write

$$b = a^+.$$

An element of $\boldsymbol{C}$ is a **limit** if it is neither a successor nor the least element of $\boldsymbol{C}$.

succ **Theorem 64.** *In* **ON**, *the successor of α is α'.*

Proof. If $\alpha < \beta$, this means $\alpha \subset \beta$ and $\alpha \in \beta$, hence also $\{\alpha\} \subseteq \beta$; therefore $\alpha \cup \{\alpha\} \subseteq \beta$, that is, $\alpha' \leqslant \beta$. □

In general, a good order $(\boldsymbol{C}, <)$ has at most three distinct kinds of elements:

1) the least element,
2) successors,
3) limits.

If $\boldsymbol{C}$ has a greatest element, then this has no successor; every other element does have a successor, which is unique. An element of $\boldsymbol{C}$ that is not a successor is just an element a such that

$$\forall x\,(x \in \boldsymbol{C} \;\&\; x < a \Rightarrow x^{+} < a).$$

Such an element is either the least element of $\boldsymbol{C}$, or a limit. The least element of $\boldsymbol{C}$ might be thought of as a 'degenerate' limit. Still, by the official definition, 0 is *not* a limit. If it were, we should have to change the wording of the following.

Theorem 65. *ω is the class of ordinals that neither* are *limits nor* contain *limits.*[4]

Now we can make an alternative formulation of transfinite induction:

thm:si **Theorem 66** (Transfinite induction in two parts)**.** *Suppose $(\boldsymbol{C},<)$ is a well-ordered class, and $\boldsymbol{D}$ is a subclass meeting the following two conditions.*

1. If a is not the greatest element of $\boldsymbol{C}$, then

$$a \in \boldsymbol{D} \Rightarrow a^{+} \in \boldsymbol{D}.$$

2. If b is not a successor of $\boldsymbol{C}$, then

$$\operatorname{pred}(b) \subseteq \boldsymbol{D} \Rightarrow b \in \boldsymbol{D}. \tag{5.1}$$ eqn:

[4] One could say, *The class of ordinals that neither* are *nor* contain *limits is denoted by ω;* but this would violate the grammatical principles laid down by Fowler in [15, Cases] and reaffirmed by his editor Gowers in [14]. In the original sentence of the theorem, the second occurrence of *limits* is the direct object of *contain,* so it is notionally in the 'objective case'; but the first instance of *limits* is is not an object of *are* (which does not take objects), but is in the 'subjective case', like the subject, *that,* of the relative clause, *that neither are limits nor contain limits.* On similar grounds, the common expression, *x is less than or equal to y,* is objectionable, unless *than,* like *to,* is construed as a preposition. However, allowing *than* to be used as a preposition can cause ambiguity: does *She likes tea better than me* mean *She likes tea better than she likes me,* or *She likes tea better than I do*? Therefore it is recommended in [15, Than 6] and (less strongly) in [14] that *than* not be used as a preposition. If we were to follow this recommendation thoroughly, then we should read the inequality $x \leqslant y$ as, *x is less than y or equal to y,* rather than simply, *x is less than or equal to y.* I do not actually propose to make this change.

Then $\boldsymbol{D} = \boldsymbol{C}$.

Proof. Suppose $(\boldsymbol{C}, <)$ is a good order and $\boldsymbol{D} \subset \boldsymbol{C}$, but (5.1) holds whenever b is an element of $\boldsymbol{C}$ that is not a successor. By transfinite induction, there is some b in $\boldsymbol{C}$ such that $\operatorname{pred}(b) \subseteq \boldsymbol{D}$, but $b \notin \boldsymbol{D}$. Then b must be a successor, so $b = a^+$ for some b in $\boldsymbol{C}$. Then $a \in \operatorname{pred}(b)$, so $a \in \boldsymbol{D}$, but $a^+ \notin \boldsymbol{D}$. □

It is sometimes useful to distinguish the least element of a good order from the other non-successors, that is, the limits.

Corollary (Transfinite induction in three parts)**.** *Suppose* $(\boldsymbol{C}, <)$ *is a well-ordered class with least element* ℓ*, and* $\boldsymbol{D}$ *is a subclass meeting the following three conditions.*

1. $\ell \in \boldsymbol{D}$.
2. *If* a *is not the greatest element of* $\boldsymbol{C}$*, then*
$$a \in \boldsymbol{D} \Rightarrow a^+ \in \boldsymbol{D}.$$
3. *If* b *is a limit of* $\boldsymbol{C}$*, then*
$$\operatorname{pred}(b) \subseteq \boldsymbol{D} \Rightarrow b \in \boldsymbol{D}.$$

Then $\boldsymbol{D} = \boldsymbol{C}$.

5.4. Transfinite recursion

t:wo

The Separation Axiom could be formulated as follows. Given a class $\boldsymbol{C}$, if a is a set a such that $a \cap \boldsymbol{C}$ has an element b, we can let $\boldsymbol{F}$ be the class $\{(x, x) \colon x \in a \cap \boldsymbol{C}\} \cup \{(x, b) \colon x \in a \smallsetminus \boldsymbol{C}\}$. Then $\boldsymbol{F}$ is a function given by

$$\boldsymbol{F}(x) = \begin{cases} x, & \text{if } x \in a \cap \boldsymbol{C}, \\ b, & \text{if } x \in a \smallsetminus \boldsymbol{C}. \end{cases}$$

In particular, $\operatorname{rng}(\boldsymbol{F}) = a \cap \boldsymbol{C}$. The axiom is that this range is a set. We might say that, because a is a set, so is $\boldsymbol{F}[a]$. The latter set is obtained by, so to speak, *replacing* each element x of a with $\boldsymbol{F}(x)$. The Separation Axiom is thus a special case of the following:

ax:rep **Axiom 5** (Replacement). *The image of a set under a function is a set: for all functions* $\boldsymbol{F}$, *if* $a \subseteq \operatorname{dom}(\boldsymbol{F})$, *then* $\boldsymbol{F}[a]$ *is a set:*

$$\forall x\, \exists y\, (x \subseteq \operatorname{dom}(\boldsymbol{F}) \Rightarrow y = \boldsymbol{F}[x]).$$

Like the Separation Axiom, the Replacement Axiom is really a scheme of axioms, one for each function; and any parameters of the functions should be universally quantified. We need the scheme, to ensure that the following makes sense:

defn:t-rec **Definition 24.** A left-narrow linear order $(\boldsymbol{C}, <)$ admits **transfinite recursion** if, for every class $\boldsymbol{D}$, for every function $\boldsymbol{F}$ from $\mathscr{P}(\boldsymbol{D})$ to $\boldsymbol{D}$, there is a unique function $\boldsymbol{G}$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ such that

$$\forall x\, (x \in \boldsymbol{C} \Rightarrow \boldsymbol{G}(x) = \boldsymbol{F}(\boldsymbol{G}[\operatorname{pred}(x)])). \tag{5.2}$$ eqn:

Note that this property of a given linear order is not *obviously* expressible with a single formula. It *is* so expressible though, by the following.

thm:o-eq **Theorem 67** (Transfinite Recursion). *A left-narrow linear order admits transfinite recursion if and only if it is well-ordered.*

Proof. Let $(\boldsymbol{C}, <)$ be a left-narrow linear order. Suppose first that $(\boldsymbol{C}, <)$ is good, $\boldsymbol{D}$ is a class, and $\boldsymbol{F}$ is a function from $\mathscr{P}(\boldsymbol{D})$ to $\boldsymbol{D}$. We show by transfinite induction that, for all a in $\boldsymbol{C}$, there is a unique function g_a with domain $\operatorname{pred}(a) \cup \{a\}$ such that, whenever $c \leqslant a$, then

$$g_a(c) = \boldsymbol{F}(g_a[\,\operatorname{pred}(c)\,]).$$

Suppose the claim holds whenever $a < b$. If $a < d < b$, then $g_d \restriction (\operatorname{pred}(a) \cup \{a\})$ has the defining property of g_a, so it is equal to g_a; in particular, $g_d(a) = g_a(a)$. Therefore we can define g_b by

$$g_b(x) = \begin{cases} g_x(x), & \text{if } x < b; \\ \boldsymbol{F}(\{g_y(y) \colon y < b\}), & \text{if } x = b. \end{cases} \tag{5.3}$$ eqn:

Moreover, as before, any g_b as desired must agree with g_a on $\operatorname{pred}(a) \cup \{a\}$ when $a < b$, and then $g_b(b)$ must be as in (5.3). By transfinite induction, we have a function g_a as desired for all a in $\boldsymbol{C}$. Then we have (5.2) if and only if $\boldsymbol{G}$ is $x \mapsto g_x(x)$.

Now suppose $(\boldsymbol{C}, <)$ is not good, but $\boldsymbol{D}$ is a nonempty subclass of $\boldsymbol{C}$ with no least element. Let

$$\boldsymbol{E} = \{x \colon x \in \boldsymbol{C} \mathbin{\&} \exists y\,(y \in \boldsymbol{D} \mathbin{\&} y \leqslant x)\},$$

and let $\boldsymbol{F}$ be the function from $\mathscr{P}(2)$ to 2 such that

$$\forall x\,(x \subseteq 2 \Rightarrow (\boldsymbol{F}(x) = 1 \Leftrightarrow 1 \in x)).$$

Then there are two functions $\boldsymbol{G}$ from $\boldsymbol{C}$ to 2 such that (5.2) holds. Indeed, let

$$\boldsymbol{G}_0 = \{(x, 0) \colon x \in \boldsymbol{C}\}, \quad \boldsymbol{G}_1 = \{(x, 0) \colon x \in \boldsymbol{C} \smallsetminus \boldsymbol{E}\} \cup \{(x, 1) \colon x \in \boldsymbol{E}\};$$

that is, if $e \in 2$, let $\boldsymbol{G}_e$ be the function from $\boldsymbol{C}$ into 2 given by

$$\boldsymbol{G}_e(x) = \begin{cases} 0, & \text{if } x \in \boldsymbol{C} \smallsetminus \boldsymbol{E}; \\ e, & \text{if } x \in \boldsymbol{E}. \end{cases}$$

Then $\boldsymbol{G}_e(a) = \boldsymbol{F}(\boldsymbol{G}_e[\,\mathrm{pred}(a)\,])$. □

In the notation of Definition 24, if a is an element of $\boldsymbol{C}$ with a successor, then $\boldsymbol{G}(a^+)$ depends on $\{\boldsymbol{G}(x) \colon x \in \boldsymbol{C} \mathbin{\&} x \leqslant a\}$, not just on $\boldsymbol{G}(a)$. We cannot generally recover $\boldsymbol{G}(a)$ from $\{\boldsymbol{G}(x) \colon x \in \boldsymbol{C} \mathbin{\&} x \leqslant a\}$. However, in our applications, we shall *want* to define $\boldsymbol{G}(a^+)$ in terms of $\boldsymbol{G}(a)$ alone. We can do this as follows.

-alt

Theorem 68 (Transfinite recursion in two parts)**.** *Suppose $(\boldsymbol{C}, <)$ is a well-ordered class, $\boldsymbol{D}$ is a class, $\boldsymbol{F}$ is a function from $\boldsymbol{D}$ to $\boldsymbol{D}$, and $\boldsymbol{G}$ is a function from $\mathscr{P}(\boldsymbol{D})$ to $\boldsymbol{D}$. Then there is a unique function $\boldsymbol{H}$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ such that*

1) *$\boldsymbol{H}(a^+) = \boldsymbol{F}(\boldsymbol{H}(a))$, if a is not the greatest element of $\boldsymbol{C}$;*
2) *$\boldsymbol{H}(d) = \boldsymbol{G}(\boldsymbol{H}[\,\mathrm{pred}(d)\,])$, if d is not a successor.*

Proof. By transfinite induction in two parts, as in Theorem 66, there is at most one such function $\boldsymbol{H}$. Indeed, suppose $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$ are two such functions.

1. If $\boldsymbol{H}_0(a) = \boldsymbol{H}_1(a)$, then $\boldsymbol{H}_0(a^+) = \boldsymbol{F}(\boldsymbol{H}_0(a)) = \boldsymbol{F}(\boldsymbol{H}_1(a)) = \boldsymbol{H}_1(a^+)$.

2. If a is not a successor, and $\boldsymbol{H}_0 \upharpoonright \operatorname{pred}(a) = \boldsymbol{H}_1 \upharpoonright \operatorname{pred}(a)$, then

$$\boldsymbol{H}_0(a) = \boldsymbol{G}(\boldsymbol{H}_0[\operatorname{pred}(a)]) = \boldsymbol{G}(\boldsymbol{H}_1[\operatorname{pred}(a)]) = \boldsymbol{H}_1(a).$$

Therefore $\boldsymbol{H}_0 = \boldsymbol{H}_1$.

In case $\boldsymbol{C}$ has a greatest element a, so that $\boldsymbol{C} = \operatorname{pred}(a) \cup \{a\}$, then the desired function $\boldsymbol{H}$ is a set, which we may denote by h_a. As in the proof of Theorem 67, but this time using transfinite induction as given by Theorem 66, we have that h_a exists as desired for all a in $\boldsymbol{C}$. Indeed, if a is not the greatest element of $\boldsymbol{C}$, then

$$h_{a^+} = h_a \cup \{(a, \boldsymbol{F}(h_a(a)))\},$$

while if a is not a successor, then

$$h_a = \{(x, h_x(x)) \colon x \in \operatorname{pred}(a)\} \cup \{(a, \boldsymbol{G}(\{h_x(x) \colon x \in \operatorname{pred}(a)\}))\}.$$

Then the desired function $\boldsymbol{H}$ on $\boldsymbol{C}$ is $x \mapsto h_x(x)$. □

In applications of the theorem, the function $\boldsymbol{G}$ may be defined by one formula at the empty set, and by another at the non-empty subsets of $\boldsymbol{D}$. That is, we may apply the theorem in the following form:

Corollary (Transfinite recursion in three parts)**.** *Suppose* $(\boldsymbol{C}, <)$ *is a well-ordered class with least element* ℓ, $\boldsymbol{D}$ *is a class with element* m, $\boldsymbol{F}$ *is a function from* $\boldsymbol{D}$ *to* $\boldsymbol{D}$, *and* $\boldsymbol{G}$ *is a function from* $\mathscr{P}(\boldsymbol{D}) \smallsetminus \{0\}$ *to* $\boldsymbol{D}$. *Then there is a unique function* $\boldsymbol{H}$ *from* $\boldsymbol{C}$ *to* $\boldsymbol{D}$ *such that*

1) $\boldsymbol{H}(\ell) = m$,
2) $\boldsymbol{H}(a^+) = \boldsymbol{F}(\boldsymbol{H}(a))$, *if* a *is not the greatest element of* $\boldsymbol{C}$;
3) $\boldsymbol{H}(d) = \boldsymbol{G}(\boldsymbol{H}[\operatorname{pred}(d)])$, *if* d *is a limit.*

An **initial segment** of an order $(\boldsymbol{C}, <)$ is a subclass $\boldsymbol{D}$ of $\boldsymbol{C}$ such that, for all a in $\boldsymbol{D}$,

$$\forall x\, (x \in \boldsymbol{C} \;\&\; x < a \Rightarrow x \in \boldsymbol{D}).$$

thm:is-cs

Theorem 69. *Every initial segment of a well-ordered class is either the class itself or a section of it.*

Definition 25. An **embedding** of a linearly ordered class $(\boldsymbol{C}, <)$ in another one, $(\boldsymbol{D}, \boldsymbol{R})$, is an injection $\boldsymbol{F}$ of the class $\boldsymbol{C}$ in $\boldsymbol{D}$ that is also **order-preserving** or **increasing** in the sense that, for all a and b in $\boldsymbol{C}$,

$$a < b \Rightarrow \boldsymbol{F}(a) \; \boldsymbol{R} \; \boldsymbol{F}(b).$$

Since the orders in question are linear, this condition implies its converse, so that

$$a < b \Leftrightarrow \boldsymbol{F}(a)\ \boldsymbol{R}\ \boldsymbol{F}(b).$$

If the range of $\boldsymbol{F}$ is $\boldsymbol{D}$, then $\boldsymbol{F}$ is an **isomorphism** from $(\boldsymbol{C}, <)$ to $(\boldsymbol{D}, \boldsymbol{R})$. The least element (if it exists) of a subclass $\boldsymbol{E}$ of $\boldsymbol{C}$ can be called the **minimum** element of $\boldsymbol{E}$ and can accordingly be denoted by

$$\min(\boldsymbol{E}).$$

m:wo **Theorem 70.** *Of any two well-ordered classes, one is uniquely isomorphic to a unique initial segment of the other.*

Proof. Let $(\boldsymbol{C}, <)$ and $(\boldsymbol{D}, \boldsymbol{R})$ be well-ordered classes. Suppose first that there is an isomorphism $\boldsymbol{H}$ from the former to an initial segment of the latter. Then for all a in $\boldsymbol{C}$, we have, by the definitions,

$$\boldsymbol{H}[\operatorname{pred}(a)] \subseteq \operatorname{pred}(\boldsymbol{H}(a)).$$

Also, since $\boldsymbol{H}[\boldsymbol{C}]$ is an initial segment of $\boldsymbol{D}$, we have $\operatorname{pred}(\boldsymbol{H}(a)) \subseteq \boldsymbol{H}[\boldsymbol{C}]$. Since $\boldsymbol{H}$ is order-preserving, we therefore have

$$\boldsymbol{H}[\operatorname{pred}(a)] = \operatorname{pred}(\boldsymbol{H}(a)),$$

and consequently

$$\boldsymbol{H}(a) = \min(\boldsymbol{D} \smallsetminus \boldsymbol{H}[\operatorname{pred}(a)]).$$

Thus $\boldsymbol{H}$ is defined recursively and is therefore unique.

However, the definition fails if, for some a in $\boldsymbol{C}$, it should happen that $\boldsymbol{H}[\operatorname{pred}(a)] = \boldsymbol{D}$. We can then adjust the definition so that, for all a in $\boldsymbol{C}$,

$$\boldsymbol{H}(a) = \begin{cases} \min(\boldsymbol{D} \smallsetminus \boldsymbol{H}[\operatorname{pred}(a)]), & \text{if } \boldsymbol{H}[\operatorname{pred}(a)] \subset \boldsymbol{D}, \\ \min(\boldsymbol{D}), & \text{otherwise.} \end{cases}$$

Then $\operatorname{rng}(\boldsymbol{H})$ is indeed an initial segment of $\boldsymbol{D}$. Indeed, suppose $\boldsymbol{H}(a) = b$, but $c\ \boldsymbol{R}\ b$. Then $\boldsymbol{H}(a) = \min(\boldsymbol{D} \smallsetminus \boldsymbol{H}[\operatorname{pred}(a)])$, so $c \notin \boldsymbol{D} \smallsetminus \boldsymbol{H}[\operatorname{pred}(a)]$, and therefore $c \in \boldsymbol{H}[\operatorname{pred}(a)]$.

If $\boldsymbol{H}$ is injective, then it is an isomorphism from $\boldsymbol{C}$ to its range. If $\boldsymbol{H}$ is not injective, let a be the least of those b in $\boldsymbol{C}$ such that $\boldsymbol{H}(b) = \min(\boldsymbol{D})$, but $b \neq \min(\boldsymbol{C})$. Then $\breve{\boldsymbol{H}}$ is an isomorphism from $\boldsymbol{D}$ to $\operatorname{pred}_{(\boldsymbol{C},<)}(a)$. $\square$

Corollary. *Every well-ordered set is isomorphic to a unique ordinal. Every well-ordered proper class is isomorphic to* **ON**.

Definition 26. If $(a, <)$ is a well-ordered set, then the unique ordinal to which it is isomorphic is its **order-type** or **ordinality;** this can be denoted by

$$\mathrm{ord}(a, <)$$

or simply $\mathrm{ord}(a)$.

5.5. Suprema

Definition 27. Suppose $(\boldsymbol{C}, <)$ is a linear order, $\boldsymbol{D} \subseteq \boldsymbol{C}$, and $a \in \boldsymbol{C}$. Then a is an **upper bound** of $\boldsymbol{D}$ (with respect to $<$) if

$$\forall x\,(x \in \boldsymbol{D} \Rightarrow x \leqslant a);$$

and a is a **strict upper bound** if

$$\forall x\,(x \in \boldsymbol{D} \Rightarrow x < a).$$

If $\boldsymbol{D}$ has a *least* upper bound, then this is unique and is the **supremum** of $\boldsymbol{D}$; it is denoted by

$$\sup(\boldsymbol{D}).$$

If $\boldsymbol{D} = \{\boldsymbol{F}(x) \colon \varphi(x)\}$ for some function $\boldsymbol{F}$ and formula φ, then we may write $\sup(\boldsymbol{D})$ as

$$\sup_{\varphi(x)} \boldsymbol{F}(x).$$

Note in particular that $\sup(0)$ is the least element of $\boldsymbol{C}$, if there is one.

We shall make use of these notions on **ON**.

thm:sup-ord

Theorem 71. *For all ordinals α,*

$$\sup(\alpha') = \alpha.$$

If α is not a successor itself, then

$$\sup(\alpha) = \alpha.$$

Theorem 72. *The union of a set of transitive sets is transitive. In particular, the union of a set of ordinals is either an ordinal or* **ON** *itself.*

To preclude the possibility that the union of a set of ordinals might be **ON**, we have:

nion **Axiom 6** (Union)**.** *The union of a set is a set:*

$$\forall x\, \exists y\, y = \bigcup x.$$

Theorem 73. *For all sets a and b, the union $a \cup b$ is the set $\bigcup\{a, b\}$.*

Theorem 74. *The union of a set of ordinals is an ordinal, which is the supremum of the set:*

$$b \subset \mathbf{ON} \Rightarrow \bigcup b = \sup(b).$$

Proof. Suppose $b \subseteq \mathbf{ON}$. If $\alpha \in b$, then $\alpha \subseteq \bigcup b$; so $\bigcup b$ is an upper bound of b. If $\beta < \bigcup b$, then β belongs to an element α of b; that is, $\beta < \alpha$, so β is not an upper bound of b. □

east **Theorem 75.** *If b is a set of ordinals, then $\bigcup\{x' : x \in b\}$ is the least strict upper bound of b.*

5.6. Ordinal addition

We can now extend Definition 14, of addition on ω, to **ON**, using transfinite recursion in three parts (the corollary to Theorem 68).

:on+ **Definition 28** (Ordinal addition)**.** For each ordinal α, the operation $x \mapsto \alpha + x$ on **ON** is given by

$$\alpha + 0 = \alpha, \qquad \alpha + \beta' = (\alpha + \beta)', \qquad \alpha + \gamma = \sup_{x \in \gamma}(\alpha + x),$$

where γ is a limit. In particular,

$$\alpha + 1 = \alpha',$$

so the second part of the definition can be written as

$$\alpha + (\beta + 1) = (\alpha + \beta) + 1.$$

The operation $(x, y) \mapsto x + y$ on **ON** is **ordinal addition,** and $\alpha + \beta$ is the **ordinal sum** of α and β.

Some of the properties of addition on **ON** are just as on ω. To begin with, we have:

Theorem 76. *For all ordinals α,*

$$0 + \alpha = \alpha.$$

Proof. We use transfinite induction in three parts, as in the corollary to Theorem 66.

1. $0 + 0 = 0$ by definition.
2. If $0 + \alpha = \alpha$, then $0 + (\alpha + 1) = (0 + \alpha) + 1 = \alpha + 1$.
3. If β is a limit, and $0 + \alpha = \alpha$ whenever $\alpha < \beta$, then

$$0 + \beta = \sup_{x \in \beta}(0 + x) = \sup_{x \in \beta} x = \sup(\beta) = \beta$$

by Theorem 71. □

thm:ord-add

Theorem 77. *For all ordinals α, β, and γ,*

$$\alpha < \beta \Rightarrow \gamma + \alpha < \gamma + \beta.$$

In particular, ordinal addition admits left cancellation:

$$\gamma + \alpha = \gamma + \beta \Rightarrow \alpha = \beta.$$

Proof. We prove the first claim by transfinite induction in three parts on β.

1. The claim is vacuous when $\beta = 0$.
2. If the claim holds when $\beta = \delta$, and now $\alpha < \delta + 1$, then $\alpha \leqslant \delta$, and therefore

$$\gamma + \alpha \leqslant \gamma + \delta < (\gamma + \delta) + 1 = \gamma + (\delta + 1).$$

3. If δ is a limit, and the claim holds when $\beta < \delta$, and now $\alpha < \delta$, then $\alpha < \alpha + 1 < \delta$, and therefore

$$\gamma + \alpha < \gamma + \alpha + 1 \leqslant \sup_{x \in \delta}(\gamma + x) = \gamma + \delta.$$

The second claim is nearly the contrapositive: If $\alpha \neq \beta$, then we may assume $\alpha < \beta$, so $\gamma + \alpha < \gamma + \beta$, and in particular $\gamma + \alpha \neq \gamma + \beta$. □

Now we can establish an alternative definition of ordinal addition, using transfinite recursion in the original, one-part form (Definition 24).

Theorem 78. *For all ordinals α and β,*

$$\alpha + \beta = \sup(\{\alpha\} \cup \{\alpha + (x + 1) \colon x \in \beta\}).$$

Proof. We consider the three parts of the original definition (Definition 28).

1. $\sup(\{\alpha\} \cup \{\alpha + (x + 1) \colon x \in 0\}) = \sup(\{\alpha\}) = \alpha = \alpha + 0$.
2. If $\delta < \gamma + 1$, then $\delta \leqslant \gamma$, so $\delta + 1 \leqslant \gamma + 1$, and hence

   $$\alpha + (\delta + 1) \leqslant \alpha + (\gamma + 1).$$

   Also $\alpha < \alpha + (\gamma + 1)$. Therefore

   $$\sup(\{\alpha\} \cup \{\alpha + (x + 1) \colon x \in \gamma + 1\}) \leqslant \alpha + (\gamma + 1).$$

   The reverse inequality also holds, because $\gamma < \gamma + 1$, so

   $$\alpha + (\gamma + 1) \in \{\alpha + (x + 1) \colon x \in \gamma + 1\}.$$

3. Suppose β is a limit. Then $0 < \beta$, and if $\delta < \beta$, then $\delta + 1 < \beta$. Therefore

   $$\{\alpha\} \cup \{\alpha + (x + 1) \colon x \in \beta\} \subseteq \{\alpha + x \colon x \in \beta\},$$

   so

   $$\sup(\{\alpha\} \cup \{\alpha + (x + 1) \colon x \in \beta\}) \leqslant \sup_{x \in \beta}(\alpha + x) = \alpha + \beta.$$

   The reverse inequality also holds, since $\alpha + \delta < \alpha + (\delta + 1)$. □

To establish some additional properties, yet another understanding of ordinal addition will be useful. We develop this now.

Theorem 79. *For all sets a and b, the class $a \times b$ is a set.*

Proof. The class $a \times \{c\}$ is the image of a under the function $x \mapsto (x, c)$, so it is a set. Then $a \times b$ is the set $\bigcup\{a \times \{x\} : x \in b\}$. $\square$

Definition 29. Suppose $(\boldsymbol{C}, \boldsymbol{R})$ and $(\boldsymbol{D}, \boldsymbol{S})$ are linear orders. The **(right) lexicographic ordering** of $\boldsymbol{C} \times \boldsymbol{D}$ is the relation $<$ such that, for all a and b in $\boldsymbol{C}$, and all c and d in $\boldsymbol{D}$,

$$(a, c) < (b, d) \Leftrightarrow c \mathrel{\boldsymbol{S}} d \lor (c = d \mathrel{\&} a \mathrel{\boldsymbol{R}} b).$$

The lexicographic ordering of 4×6 is given in Table 5.1.

$$\begin{tikzcd}
(0,0) \arrow[d] & (0,1) \arrow[d] & (0,2) \arrow[d] & (0,3) \arrow[d] & (0,4) \arrow[d] & (0,5) \arrow[d] \\
(1,0) \arrow[d] & (1,1) \arrow[d] & (1,2) \arrow[d] & (1,3) \arrow[d] & (1,4) \arrow[d] & (1,5) \arrow[d] \\
(2,0) \arrow[d] & (2,1) \arrow[d] & (2,2) \arrow[d] & (2,3) \arrow[d] & (2,4) \arrow[d] & (2,5) \arrow[d] \\
(3,0) \arrow[uuur] & (3,1) \arrow[uuur] & (3,2) \arrow[uuur] & (3,3) \arrow[uuur] & (3,4) \arrow[uuur] & (3,5)
\end{tikzcd}$$

tab:lex

Table 5.1. The lexicographic ordering of 4×6

Theorem 80. *If $\boldsymbol{C}$ and $\boldsymbol{D}$ are linear orders, then the lexicographic ordering of $\boldsymbol{C} \times \boldsymbol{D}$ is a linear ordering. If $\boldsymbol{C}$ and $\boldsymbol{D}$ are* well-ordered, *then $\boldsymbol{C} \times \boldsymbol{D}$ is well-ordered by the lexicographic ordering.*

Proof. In the good case, if $\boldsymbol{E} \subseteq \boldsymbol{C} \times \boldsymbol{D}$ and is nonempty, then its least element is (a, b), where b is the least element of $\{y \colon \exists x\, (x, y) \in \boldsymbol{E}\}$, and a is the least element of $\{x \colon (x, b) \in \boldsymbol{C}\}$. □

s-wo **Theorem 81.** *If $\mathfrak{a}$ is a section of the well-ordered set $\mathfrak{b}$, then*

$$\operatorname{ord}(\mathfrak{a}) < \operatorname{ord}(\mathfrak{b}).$$

Proof. If f is the isomorphism from $\mathfrak{b}$ to $\operatorname{ord}(\mathfrak{b})$ guaranteed by Theorem 70, then, by the proof of that theorem, $f[\mathfrak{a}]$ is a section of $\operatorname{ord}(\mathfrak{b})$, so

$$\operatorname{ord}(\mathfrak{a}) = f[\mathfrak{a}] < \operatorname{ord}(\mathfrak{b}).$$ □

+ord **Theorem 82.** *For all ordinals α and β,*

$$\alpha + \beta = \operatorname{ord}((\alpha \times \{0\}) \cup (\beta \times \{1\})),$$

where the union has the lexicographic ordering of $\mathbf{ON} \times 2$.

Proof. Fixing α, let us use the notation

$$(\alpha \times \{0\}) \cup (\beta \times \{1\}) = \boldsymbol{F}(\beta).$$

We want to show $\alpha + \beta = \operatorname{ord}(\boldsymbol{F}(\beta))$. We use transfinite induction in three parts.

1. We have $\operatorname{ord}(\boldsymbol{F}(0)) = \operatorname{ord}(\alpha \times \{0\}) = \alpha = \alpha + 0$.
2. Suppose the claim holds when $\beta = \gamma$. Then there is an isomorphism h from $\alpha + \gamma$ to $\boldsymbol{F}(\gamma)$. Then $h \cup \{(\alpha + \gamma, (\gamma, 1))\}$ is an isomorphism from $(\alpha + \gamma) + 1$—which is $\alpha + (\gamma + 1)$—to $\boldsymbol{F}(\gamma + 1)$. So the claim holds when $\beta = \gamma + 1$.
3. Suppose γ is a limit, and the claim holds when $\beta < \gamma$. When $\beta < \gamma$, then $\boldsymbol{F}(\beta)$ is a section of $\boldsymbol{F}(\gamma)$, so by the last theorem,

   $$\alpha + \beta = \operatorname{ord}(\boldsymbol{F}(\beta)) < \operatorname{ord}(\boldsymbol{F}(\gamma)),$$

   and therefore

   $$\alpha + \gamma = \sup_{x \in \gamma}(\alpha + x) \leqslant \operatorname{ord}(\boldsymbol{F}(\gamma)).$$

   For the reverse inequality, suppose $\zeta < \operatorname{ord}(\boldsymbol{F}(\gamma))$. Then ζ is the order type of some section of $\boldsymbol{F}(\gamma)$. This section is either $\boldsymbol{F}(\beta)$ for some β in γ, or $\beta \times \{0\}$ for some β such that $\beta \leqslant \alpha$. In either case, $\zeta \leqslant \alpha + \beta$ for some β in γ. Therefore $\zeta < \alpha + \gamma$. We also have $\beta + 1 < \gamma$, so $\zeta + 1 \leqslant \operatorname{ord}(\boldsymbol{F}(\beta + 1)) < \operatorname{ord}(\boldsymbol{F}(\gamma))$. Therefore $\operatorname{ord}(\boldsymbol{F}(\gamma))$ is not a successor, and so

$$\operatorname{ord}(\boldsymbol{F}(\gamma)) = \sup(\{x \colon x \in \operatorname{ord}(\boldsymbol{F}(\gamma))\}) \leqslant \alpha + \gamma. \qquad \square$$

thm:ord-sub **Theorem 83** (Subtraction). *If $\alpha \leqslant \beta$, then the equation*

$$\alpha + x = \beta$$

has a unique ordinal solution, namely $\operatorname{ord}(\beta \smallsetminus \alpha)$.

Proof. Let $\operatorname{ord}(\beta \smallsetminus \alpha) = \gamma$, and let f be the isomorphism from $\beta \smallsetminus \alpha$ to γ. Then there is an isomorphism g from β to $(\alpha \times \{0\}) \cup (\gamma \times \{1\})$ given by

$$g(x) = \begin{cases} (x, 0), & \text{if } x \in \alpha, \\ (f(x), 1), & \text{if } \alpha \in x \in \beta. \end{cases}$$

Therefore $\beta = \alpha + \gamma$, by the last theorem. If also $\beta = \alpha + \delta$, then $\gamma = \delta$ by Theorem 77. $\square$

The theorem can be proved by transfinite induction. However, this method does not give insight into what the solution of the equation *is;* and we can use that insight for the following.

Theorem 84. *For all ordinals α, β, and γ,*

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

Proof. By the last theorem,

$$\operatorname{ord}((\alpha + \beta) \smallsetminus \alpha) = \beta, \qquad \operatorname{ord}(((\alpha + \beta) + \gamma) \smallsetminus (\alpha + \beta)) = \gamma,$$

so by Theorem 82,

$$\operatorname{ord}(((\alpha + \beta) + \gamma) \smallsetminus \alpha) = \beta + \gamma.$$

By the last theorem again, the claim follows. $\square$

In Theorem 81, it is important that $\mathfrak{a}$ is a *section* of $\mathfrak{b}$. In a more general situation, we have the following.

ord< **Lemma 13.** *Suppose* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are well-ordered sets, and* $\mathfrak{a}$ *embeds in* $\mathfrak{b}$. *Then*

$$\operatorname{ord}(\mathfrak{a}) \leqslant \operatorname{ord}(\mathfrak{b}).$$

Proof. An embedding of $\mathfrak{a}$ in $\mathfrak{b}$ induces an embedding g of $\operatorname{ord}(\mathfrak{a})$ in $\operatorname{ord}(\mathfrak{b})$. Then

$$\alpha < \operatorname{ord}(\mathfrak{a}) \Rightarrow g(\alpha) < \operatorname{ord}(\mathfrak{b}).$$

We shall show by transfinite induction (in one part) that

$$\alpha < \operatorname{ord}(\mathfrak{a}) \Rightarrow \alpha \leqslant g(\alpha).$$

Suppose this is so when $\alpha < \beta$, and suppose $\beta < \operatorname{ord}(\mathfrak{a})$. If $\alpha < \beta$, then $\alpha \leqslant g(\alpha) < g(\beta)$, so $\alpha < g(\beta)$. Briefly, $\alpha < \beta \Rightarrow \alpha < g(\beta)$. Therefore $\beta \leqslant g(\beta)$. This completes the induction. We conclude

$$\alpha < \operatorname{ord}(\mathfrak{a}) \Rightarrow \alpha < \operatorname{ord}(\mathfrak{b}),$$

and hence $\operatorname{ord}(\mathfrak{a}) \leqslant \operatorname{ord}(\mathfrak{b})$. □

The following should be contrasted with Theorem 77.

ord< **Theorem 85.** *For all ordinals* α *and* β,

$$\alpha \leqslant \beta \Rightarrow \alpha + \gamma \leqslant \beta + \gamma.$$

Proof. If $\alpha \leqslant \beta$, then $(\alpha \times \{0\}) \cup (\gamma \times \{1\}) \subseteq (\beta \times \{0\}) \cup (\gamma \times \{1\})$; this inclusion is an embedding of the well-ordered sets, so $\alpha + \gamma \leqslant \beta + \gamma$ by the lemma. □

We already knew the foregoing in case $\omega = \mathbf{ON}$. Moreover, we cannot now *prove* that $\omega \neq \mathbf{ON}$. Indeed, by finite induction, and Theorem 52,—ultimately, by GST alone (p. 63)—, we can define a function $\boldsymbol{F}$ on ω by

$$\boldsymbol{F}(0) = 0, \qquad \boldsymbol{F}(n+1) = \mathscr{P}(\boldsymbol{F}(n)).$$

Let $\boldsymbol{C}$ be the class $\{x\colon \exists y\,(y \in \omega \mathbin{\&} x \in \boldsymbol{F}(y)\}$. Then all of our axioms so far are true in $\boldsymbol{C}$; that is, they are true when we assume $\mathbf{V} = \boldsymbol{C}$ and
-ref $\in$ is the relation $\{(x, y)\colon y \in \boldsymbol{C} \mathbin{\&} x \in y\}$. Even the Power Set Axiom (Axiom 8, p. 113) is true in $\boldsymbol{C}$. However, all elements of $\boldsymbol{C}$ are finite.

Nonetheless, there is no obvious way to prove that $\mathbf{V} \smallsetminus \boldsymbol{C}$ is empty. For the sake of developing some interesting possibilities, we assume $\omega \in \mathbf{ON}$:

ax:infinity **Axiom 7** (Infinity). *The class of natural numbers is a set:*

$$\exists x\, x = \omega.$$

m:+not-comm **Theorem 86.** *If $n < \omega \leqslant \alpha$, then*

$$n + \alpha = \alpha.$$

Proof. We have $\alpha = \omega + \beta$ for some β, so $n + \alpha = n + \omega + \beta$. By Theorem 43, there is an isomorphism f from ω into $(n \times \{0\}) \cup (\omega \times \{1\})$ given by

$$f(x) = \begin{cases} (x, 0), & \text{if } x < n; \\ (y, 1), & \text{if } x = n + y. \end{cases}$$

So $\omega = n + \omega$, and therefore $n + \alpha = \omega + \beta = \alpha$. □

For example, we have

$$1 + \omega = \omega \neq \omega + 1;$$

so ordinal addition is not commutative. Also,

$$0 < 1, \qquad\qquad 0 + \omega = \omega = 1 + \omega,$$

so the ordering in Theorem 85 cannot be made strict.

By Theorem 77, along with the Axiom of Infinity, we have the following initial segment of **ON**:

$$\{0, 1, 2, \dots; \omega, \omega + 1, \omega + 2, \dots; \omega + \omega, \omega + \omega + 1, \dots; \omega + \omega + \omega, \dots\}.$$

Here the ordinals following the semicolons (;) are limits.

5.7. Ordinal multiplication

Following the pattern of the previous section, we can extend Definition 15, of multiplication on ω, to **ON**. This time, the recursion needs only two parts.

:on. **Definition 30** (Ordinal multiplication)**.** For each ordinal α, the operation $x \mapsto \alpha \cdot x$ on **ON** is given by

$$\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha, \qquad \alpha \cdot \gamma = \sup(\{\alpha \cdot x \colon x \in \gamma\}),$$

where γ is not a successor. In particular,

$$\alpha \cdot 0 = 0, \qquad \alpha \cdot 1 = \alpha.$$

The operation $(x, y) \mapsto x \cdot y$ on **ON** is **ordinal multiplication,** and $\alpha \cdot \beta$ (or simply $\alpha\beta$) is the **ordinal product** of α and β. Notationally, multiplication is more binding than addition, so that $\alpha \cdot \beta + \gamma$ means $(\alpha \cdot \beta) + \gamma$.

Theorem 87. *For all ordinals α,*

$$0 \cdot \alpha = 0, \qquad 1 \cdot \alpha = \alpha.$$

rict **Theorem 88.** *For all ordinals α, β, and γ, where $\gamma > 0$,*

$$\alpha < \beta \Rightarrow \gamma \cdot \alpha < \gamma \cdot \beta.$$

Theorem 89. *For all ordinals α and β,*

$$\alpha \cdot \beta = \sup_{x \in \beta}(\alpha \cdot x + \alpha).$$

.ord **Theorem 90.** *For all ordinals α and β,*

$$\alpha \cdot \beta = \operatorname{ord}(\alpha \times \beta),$$

where $\alpha \times \beta$ has the lexicographic ordering.

Lemma 14. *For all well-ordered classes $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$, the classes $(\boldsymbol{C} \times \boldsymbol{D}) \times \boldsymbol{E}$ and $\boldsymbol{C} \times (\boldsymbol{D} \times \boldsymbol{E})$, with the lexicographic orderings, are isomorphic.*

mult **Theorem 91.** *For all ordinals α, β, and γ,*

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma, \qquad (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

Proof. Since isomorphic good orders have the same order-type, we have

$$\begin{aligned}\alpha\cdot(\beta+\gamma) &= \operatorname{ord}(\alpha\times(\beta+\gamma))\\ &= \operatorname{ord}(\alpha\times((\beta\times\{0\})\cup(\gamma\times\{1\})))\\ &= \operatorname{ord}((\alpha\times\beta\times\{0\})\cup(\alpha\times\gamma\times\{1\}))\\ &= \operatorname{ord}((\alpha\cdot\beta\times\{0\})\cup(\alpha\cdot\gamma\times\{1\}))\\ &= \alpha\cdot\beta+\alpha\cdot\gamma,\end{aligned}$$

and also $(\alpha\cdot\beta)\cdot\gamma = \operatorname{ord}((\alpha\times\beta)\times\gamma) = \operatorname{ord}(\alpha\times(\beta\times\gamma)) = \alpha\cdot(\beta\cdot\gamma)$. □

:mult-loose **Theorem 92.** *For all ordinals α, β, and γ,*

$$\alpha\leqslant\beta\Rightarrow\alpha\cdot\gamma\leqslant\beta\cdot\gamma.$$

Theorem 93 (Division)**.** *If $0<\alpha$, then the system*

$$\alpha\cdot x+y=\beta,\qquad\qquad y<\alpha$$

has a unique ordinal solution, namely (γ,δ), where

$$\gamma=\sup(\{x\colon x\in\mathbf{ON}\ \&\ \alpha\cdot x\leqslant\beta\}),\qquad \delta=\operatorname{ord}(\beta\smallsetminus(\alpha\cdot\gamma)).$$

Proof. We have $\beta=1\cdot\beta\leqslant\alpha\cdot\beta$, so γ does exist, and $\gamma\leqslant\beta$. Then

$$\alpha\cdot\gamma\leqslant\beta<\alpha\cdot(\gamma+1)=\alpha\cdot\gamma+\alpha.$$

Then $\alpha\cdot\gamma+\delta=\beta$, by Theorem 83, and $\delta<\alpha$. If $\alpha\cdot\zeta+\eta=\beta$ and $\eta<\alpha$, then $\alpha\cdot\zeta\leqslant\beta<\alpha\cdot(\zeta+1)$, so $\zeta=\gamma$, and then $\eta=\delta$. □

An alternative way to solve the system is to note that, if $\beta<\alpha\cdot\beta$, then β is isomorphic to a section of $\alpha\times\beta$, by Theorems 69 and 70. This section is $\operatorname{pred}((\delta,\gamma))$ for some γ and δ; but

$$\operatorname{pred}((\delta,\gamma))=(\alpha\times\gamma)\cup(\delta\times\{\gamma\}),$$

whose order type is $\alpha\cdot\gamma+\delta$.

There is a partial analogue of Theorem 86:

thm:nw=w **Theorem 94.** *If $0<n<\omega$, then $n\cdot\omega=\omega$.*

Proof. The function $(x, y) \mapsto n \cdot y + x$ from $n \times \omega$ to ω is an isomorphism, by the last theorem. $\square$

We cannot replace ω here with an arbitrary infinite ordinal, since we have

$$n \cdot (\omega + 1) = n \cdot \omega + n \cdot 1 = \omega + n.$$

We do have

$$2 \cdot \omega = \omega \neq \omega \cdot 2,$$

so ordinal multiplication is not commutative. Also,

$$1 < 2, \qquad\qquad 1 \cdot \omega = \omega = 2 \cdot \omega,$$

so the ordering in Theorem 92 cannot be made strict. Finally,

$$(1 + 1) \cdot \omega = 2 \cdot \omega = \omega \neq \omega + \omega = 1 \cdot \omega + 1 \cdot \omega,$$

so ordinal multiplication does not distribute from the right over addition.

We can extend the initial segment of **ON** given at the end of the last section:

$$\{0, 1, \dots; \omega, \omega + 1, \dots; \omega \cdot 2, \dots; \omega \cdot 3, \dots; \omega \cdot \omega, \dots; \omega \cdot \omega \cdot \omega, \dots\}.$$

5.8. Ordinal exponentiation

We now extend Definition 16, of exponentiation on ω.

-exp **Definition 31** (Ordinal exponentiation)**.** For each ordinal α, where $\alpha > 0$, the operation $x \mapsto \alpha^x$ on **ON** is given by

$$\alpha^0 = 1, \qquad \alpha^{\beta+1} = \alpha^\beta \cdot \alpha, \qquad \alpha^\gamma = \sup_{x \in \gamma} \alpha^x,$$

where γ is a limit. In particular,

$$\alpha^1 = \alpha.$$

We also define

$$0^0 = 1, \qquad\qquad 0^\alpha = 0,$$

where again $\alpha > 0$. The binary operation $(x, y) \mapsto x^y$ on **ON** is **ordinal exponentiation,** and α^β is the β-th **ordinal power** of α.

Theorem 95. *For all ordinals* α,

$$1^{\alpha} = 1.$$

n:exp-strict

Theorem 96. *If* $\alpha > 1$ *and* $\beta < \gamma$, *then*

$$\alpha^{\beta} < \alpha^{\gamma}.$$

Theorem 97. *If* $\alpha > 0$ *and* $\beta > 0$, *then*

$$\alpha^{\beta} = \sup_{x \in \beta}(\alpha^{x} \cdot \alpha).$$

It is possible to understand α^{β} as the ordinality of a certain well-ordered set obtained directly from α and β. Meanwhile, we *can* obtain the basic properties of exponentiation directly from Definition 31. The process is simplified by the following notions.

Definition 32. An embedding of a linear order in itself is an **endomorphism.** An endomorphism $\boldsymbol{F}$ of **ON** is called **normal** if

$$\boldsymbol{F}(\alpha) = \sup(\boldsymbol{F}[\,\alpha\,])$$

whenever α is a limit. (There is no requirement on $\boldsymbol{F}(0)$.)

Theorem 98. *The following operations on* **ON** *are normal:*

1) $x \mapsto \alpha + x$;

2) $x \mapsto \alpha \cdot x$, *when* $\alpha > 0$;

3) $x \mapsto \alpha^{x}$, *when* $\alpha > 1$.

Proof. They are endomorphisms of **ON**, by Theorems 77, 88, and 96. Then they are normal by the original definitions (Definitions 28, 30, and 31). □

lem:normal

Lemma 15. *If* $\boldsymbol{F}$ *is normal and* $0 \subset c \subset$ **ON**, *then*

$$\boldsymbol{F}(\sup(c)) = \sup(\boldsymbol{F}[\,c\,]).$$

Proof. Let $\alpha = \sup(c)$. There are two cases to consider.

1. If $\alpha \in c$, then α is the greatest element of c, so $\sup(\boldsymbol{F}[\,c\,]) = \boldsymbol{F}(\alpha)$ since $\boldsymbol{F}$ preserves order.

2. Suppose $\alpha \notin c$. Then $c \subseteq \alpha$, so $\sup(\boldsymbol{F}[\,c\,]) \leqslant \sup(\boldsymbol{F}[\,\alpha\,])$. Also, if $\beta < \alpha$, then $\beta < \gamma < \alpha$ for some γ in c, so $\sup(\boldsymbol{F}[\,\alpha\,]) \leqslant \sup(\boldsymbol{F}[\,c\,])$. Therefore $\sup(\boldsymbol{F}[\,\alpha\,]) = \sup(\boldsymbol{F}[\,c\,])$. But α must be a limit, and hence $\sup(\boldsymbol{F}[\,\alpha\,]) = \boldsymbol{F}(\alpha)$ by normality of $\boldsymbol{F}$. □

prop **Theorem 99.** *For all ordinals α, β, and γ,*

$$\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}, \qquad\qquad \alpha^{\beta\cdot\gamma} = (\alpha^{\beta})^{\gamma}.$$

Proof. By the lemma, if $1 < \alpha$ and δ is a limit, if the first equation holds when $\gamma < \delta$, then

$$\alpha^{\beta} \cdot \alpha^{\delta} = \alpha^{\beta} \cdot \sup_{x\in\delta} \alpha^{x} = \sup_{x\in\delta}(\alpha^{\beta} \cdot \alpha^{x}) = \sup_{x\in\delta} \alpha^{\beta+x} = \alpha^{\sup_{x\in\delta}(\beta+x)} = \alpha^{\beta+\delta}.$$

□

Just as natural numbers can be written in base ten, so the next theorem below allows ordinals to be written in base α whenever $\alpha > 1$. We shall work out the details for base ω in the next section. Meanwhile, the theorem needs the following:

Lemma 16. *If $\alpha > 1$, then for all ordinals β,*

$$\alpha^{\beta} \geqslant \beta.$$

Proof. Since

$$\beta > \alpha^{\beta} \Rightarrow \alpha^{\beta} > \alpha^{\alpha^{\beta}},$$

the class $\{x \colon x \in \mathbf{ON} \mathrel{\&} \alpha^{x} < x\}$ has no least element, so it is empty. □

olve **Theorem 100.** *For all ordinals α and β, where $\alpha > 1$ and $\beta > 0$, the system*

$$\alpha^{x} \cdot y + z = \beta, \qquad 0 < y < \alpha, \qquad z < \alpha^{x}$$

has a unique ordinal solution.

Proof. The system implies

$$\alpha^{x} \cdot y \leqslant \beta < \alpha^{x} \cdot (y+1), \qquad\qquad \alpha^{x} \leqslant \beta < \alpha^{x+1}.$$

Since the functions $y \mapsto \alpha^{x} \cdot y$ and $x \mapsto \alpha^{x}$ are increasing, the original system has at most one solution. The class $\{x \colon x \in \mathbf{ON} \mathrel{\&} \alpha^{x} \leqslant \beta\}$ has the upper bound β, so it has a supremum, γ, which belongs to the class by normality of $x \mapsto \alpha^x$. Then the class $\{y\colon \alpha^\gamma \cdot y \leqslant \beta\}$ has the strict upper bound α, so the supremum δ of the class is less than α, but it belongs to the class. Then $(\gamma, \delta, \operatorname{ord}(\beta \smallsetminus \alpha^\gamma \cdot \delta))$ is the desired solution. □

We now have the following initial segment of **ON**:

$$\{0, 1, \dots; \omega, \omega+1, \dots, \omega \cdot 2, \dots; \omega^2, \omega^2+1, \dots; \omega^2+\omega, \dots;$$
$$\omega^2 \cdot 2, \dots; \omega^3, \dots; \omega^\omega, \dots; \omega^{\omega \cdot 2}, \dots; \omega^{\omega^2}, \dots; \omega^{\omega^\omega}, \dots; \omega^{\omega^{\omega^\omega}}, \dots\}.$$

This set is $\sup_{x \in \omega} \boldsymbol{F}(x)$, where

$$\boldsymbol{F}(0) = \omega, \qquad \boldsymbol{F}(n+1) = \omega^{\boldsymbol{F}(n)}.$$

We may use for $\sup_{x \in \omega} \boldsymbol{F}(x)$ the notation

$$\epsilon_0.$$

This set is closed under the operations that we have defined so far. The reason for the subscript 0 in ϵ_0 is the following.

Theorem 101. *ϵ_0 is the least solution of the equation*

$$\omega^x = x.$$

We end the section by developing an alternative definition of exponentiation, parallel to Theorems 82 and 90.

defn:fs

Definition 33. The set of functions from a set b to a set a is denoted by

$${}^b a.$$

It is indeed a set, since it is included in $\mathscr{P}(b \times a)$. Suppose a is an ordinal α. If $f \in {}^b\alpha$, the **support** of f is the set $\{x\colon x \in b \mathbin{\&} f(x) \neq 0\}$; this can be denoted by

$$\operatorname{supp}(f).$$

Let $\operatorname{fs}({}^b\alpha)$ be the set of elements of ${}^b\alpha$ with finite support, that is,

$$\operatorname{fs}({}^b\alpha) = \{x\colon x \in {}^b\alpha \mathbin{\&} |\operatorname{supp}(x)| < \omega\}.$$

Suppose now b is an ordinal β. Then $\operatorname{fs}({}^\beta\alpha)$ can be given the **right lexicographic ordering,** whereby $f < g$, provided $f(\gamma) < g(\gamma)$, where $\gamma = \sup(\{x\colon f(x) \neq g(x)\})$. See Table 5.2.

$$
\begin{array}{l}
(0,0,0,0,\dots) \\
(1,0,0,0,\dots) \\
(2,0,0,0,\dots) \\
\dots\dots\dots\dots\dots \\
(0,1,0,0,\dots) \\
(1,1,0,0,\dots) \\
(2,1,0,0,\dots) \\
\dots\dots\dots\dots\dots \\
(0,0,1,0,\dots) \\
(1,0,1,0,\dots) \\
\dots\dots\dots\dots\dots \\
(0,1,1,0,\dots) \\
\dots\dots\dots\dots\dots
\end{array}
$$

:a^b Table 5.2. The right lexicographic ordering of $\mathrm{fs}({}^{\beta}\alpha)$.

Theorem 102. *The right lexicographic ordering well-orders* $\mathrm{fs}({}^{\beta}\alpha)$.

exp2 **Theorem 103.** *For all ordinals α and β,*

$$\alpha^{\beta} = \mathrm{ord}(\mathrm{fs}({}^{\beta}\alpha)),$$

where $\mathrm{fs}({}^{\beta}\alpha)$ *has the right lexicographic ordering,*

Proof. Suppose $\alpha > 0$. Then

$$0^{\alpha} = 0 = \mathrm{ord}(0) = \mathrm{ord}(\mathrm{fs}({}^{\alpha}0)), \quad \alpha^{0} = \{0\} = \mathrm{ord}(\{0\}) = \mathrm{ord}(\mathrm{fs}({}^{0}\alpha)).$$

Suppose $\alpha^{\beta} = \mathrm{ord}(\mathrm{fs}({}^{\beta}\alpha))$. Then

$$\alpha^{\beta+1} = \alpha^{\beta}\cdot\alpha = \mathrm{ord}(\alpha^{\beta}\times\alpha) = \mathrm{ord}(\mathrm{fs}({}^{\beta}\alpha)\times\alpha) = \mathrm{ord}(\mathrm{fs}({}^{\beta+1}\alpha)),$$

since the function $(x,y)\mapsto x\cup\{(\beta,y)\}$ is an order-preserving bijection from $\mathrm{fs}({}^{\beta}\alpha)\times\alpha$ to $\mathrm{fs}({}^{\beta+1}\alpha)$.

Suppose finally γ is a limit, and $\alpha^\beta = \operatorname{ord}(\operatorname{fs}({}^\beta\alpha))$ whenever $\beta < \gamma$. We have

$$\operatorname{fs}({}^\gamma\alpha) = \bigcup\{\operatorname{fs}({}^x\alpha) \colon x \in \gamma\}.$$

Also, if $\beta < \gamma$, then $\operatorname{fs}({}^\beta\alpha)$ is a section of $\operatorname{fs}({}^\gamma\alpha)$, so the isomorphism from $\operatorname{fs}({}^\gamma\alpha)$ to $\operatorname{ord}(\operatorname{fs}({}^\gamma\alpha))$ restricts to an isomorphism from $\operatorname{fs}({}^\beta\alpha)$ to $\operatorname{ord}(\operatorname{fs}({}^\beta\alpha))$. Therefore

$$\operatorname{ord}(\operatorname{fs}({}^\gamma\alpha)) = \bigcup\{\operatorname{ord}(\operatorname{fs}({}^x\alpha)) \colon x \in \gamma\} = \sup\{\alpha^x \colon x \in \gamma\} = \alpha^\gamma. \qquad \square$$

5.9. Base omega

If $1 < b < \omega$, then every element n of ω can be written uniquely as a sum

$$b^m \cdot a_0 + b^{m-1} \cdot a_1 + \cdots + b \cdot a_{m-1} + a_m, \tag{5.4}$$

where $m \in \omega$ and $a_k \in b$, and $a_0 > 0$ unless $n = 0$. The sum is the *base-b representation* of n. The notation in (5.4) can be defined precisely as follows.

Definition 34. Given a function $x \mapsto \alpha_x$ from ω (or one of its elements) into **ON**, and an element c of ω, we define the function $x \mapsto \sum_{i=c}^{x} \alpha_i$ on ω recursively by

$$k < c \Rightarrow \sum_{i=c}^{k} \alpha_i = 0, \qquad \sum_{i=c}^{c} \alpha_i = \alpha_c, \qquad \sum_{i=c}^{c+n+1} \alpha_i = \sum_{i=c}^{c+n} \alpha_i + \alpha_{c+n+1}.$$

We may write

$$\sum_{i=c}^{c+n} \alpha_i = \alpha_c + \cdots + \alpha_{c+n}.$$

An ordinal is **positive** if it is greater than 0. By Theorem 100, for every positive ordinal α, there is a unique ordinal β such that

$$\omega^\beta \leqslant \alpha < \omega^{\beta+1};$$

we may refer to β as the **degree** of α, writing

$$\deg(\alpha) = \beta.$$

In particular, the ordinals of degree 0 are just the natural numbers that are successors. We may suppose

$$\deg(0) < 0.$$

Lemma 17. *For all functions $x \mapsto \alpha_x$ from ω to* **ON**, *for all c, k, and n in ω,*

$$\sum_{i=c}^{k+n} \alpha_i = \sum_{i=c}^{k} \alpha_i + \sum_{i=k+1}^{k+n} \alpha_i.$$

Theorem 104. *For every positive ordinal α, there are, uniquely,*

1) a function $x \mapsto (\alpha_x, a_x)$ from ω to $\mathbf{ON} \times \omega$,

2) an element $\ell(\alpha)$ of ω,

such that

$$\alpha_0 > \cdots > \alpha_{\ell(\alpha)}, \qquad \ell(\alpha) < i \Rightarrow \alpha_i = 0, \qquad a_i > 0 \Leftrightarrow i \leqslant \ell(\alpha),$$

and

$$\alpha = \sum_{k=0}^{\ell(\alpha)} \omega^{\alpha_k} \cdot a_k = \omega^{\alpha_0} \cdot a_0 + \cdots + \omega^{\alpha_{\ell(\alpha)}} \cdot a_{\ell(\alpha)}. \tag{5.5}$$

eqn:w-sum

Here

$$\alpha_0 = \deg(\alpha).$$

Proof. Given α, and using Theorems 83 and 100, by finite recursion on ω we define $x \mapsto (\alpha_x, a_x)$ by requiring $a_k \in \omega$, and $a_k > 0$ unless $\alpha_k = 0$, and

$$\omega^{\alpha_0} \cdot a_0 \leqslant \alpha < \omega^{\alpha_0} \cdot (a_0 + 1),$$

$$\omega^{\alpha_{k+1}} \cdot a_{k+1} \leqslant \operatorname{ord}(\alpha \smallsetminus \sum_{i=0}^{k} \omega^{\alpha_i} \cdot a_i) < \omega^{\alpha_{k+1}} \cdot (a_{k+1} + 1).$$

Then $\alpha_0 = \deg(\alpha)$. Also, for all k in ω,

$$\operatorname{ord}(\alpha \smallsetminus \sum_{i=0}^{k} \omega^{\alpha_i} \cdot a_i) < \omega^{\alpha_k},$$

so we have $\alpha_{k+1} < \alpha_k$, unless $a_{k+1} = 0$. Therefore $a_{k+1} = 0$ for some k. The least such k is $\ell(\alpha)$, and then we have (5.5).

Uniqueness is by the lemma and Theorem 100. In detail: Suppose we have also

$$\alpha = \sum_{i=0}^{\ell(\beta)} \omega^{\beta_i} \cdot b_i,$$

where

$$\beta_0 > \dots > \beta_{\ell(\beta)}, \qquad \ell(\alpha) < i \Rightarrow \beta_i = 0, \qquad a_i > 0 \Leftrightarrow i \leqslant \ell(\beta).$$

If $(\alpha_i, a_i) = (\beta_i, b_i)$ when $i < n$, then, by the lemma, we have

$$\begin{aligned} \omega^{\alpha_n} \cdot a_n + \sum_{i=n+1}^{\ell(\alpha)} \omega^{\alpha_i} \cdot a_i &= \sum_{i=n}^{\ell(\alpha)} \omega^{\alpha_i} \cdot a_i \\ &= \sum_{i=n}^{\ell(\beta)} \omega^{\beta_i} \cdot b_i = \omega^{\beta_n} \cdot b_n + \sum_{i=n+1}^{\ell(\beta)} \omega^{\beta_i} \cdot b_i. \end{aligned}$$

If either side is 0, then $a_n = 0 = b_n$, so $\alpha_n = 0 = \beta_n$ by definition. If one side is not 0, then $(\alpha_n, a_n) = (\beta_n, b_n)$ by Theorem 100. □

When α is written as in (5.5), it is said to be in **normal form.**[5] We can add ordinals in normal form by using:

thm:b<a

Theorem 105. *If* $\beta < \alpha$, *and* $n \in \omega$, *and* $0 < m < \omega$, *then*

$$\omega^\beta \cdot n + \omega^\alpha \cdot m = \omega^\alpha \cdot m.$$

Proof. For some positive γ we have $\beta + \gamma = \alpha$, and then

$$\omega^\beta \cdot n + \omega^\alpha \cdot m = \omega^\beta \cdot (n + \omega^\gamma \cdot m) = \omega^\beta \cdot \omega^\gamma \cdot m = \omega^\alpha \cdot m$$

by Theorem 86. □

For example,

$$\omega^3 + \omega \cdot 8 + \omega^2 \cdot 5 = \omega^3 + \omega^2 \cdot 5.$$

[5] The terminology is due to Cantor, as are most of the results of this section and, indeed, this chapter.

Corollary. *If* $\deg(\alpha) > \deg(\beta)$, *then*

$$\beta + \alpha = \alpha.$$

To *multiply,* we need a generalization of Theorem 94:

Theorem 106. *For all* n *such that* $1 < n < \omega$, *for all positive ordinals* α,

$$n \cdot \omega^{\alpha} = \omega^{\alpha}.$$

Theorem 107. *For all infinite ordinals* α *and positive natural numbers* n,

$$\alpha \cdot n = \omega^{\alpha_0} \cdot a_0 \cdot n + \sum_{i=1}^{\ell(\alpha)} \omega^{\alpha_i} \cdot a_i.$$

nf-. **Theorem 108.** *For all infinite ordinals* α *and positive ordinals* β,

$$\alpha \cdot \omega^{\beta} = \omega^{\alpha_0 + \beta}.$$

Proof. For the case $\beta = 1$, we note that, by the last theorem,

$$\alpha \cdot \omega = \sup_{x \in \omega} \alpha \cdot x = \sup_{x \in \omega} \omega^{\alpha_0} \cdot a_0 \cdot x = \sup_{x \in \omega} \omega^{\alpha_0} \cdot x = \omega^{\alpha_0} \cdot \omega = \omega^{\alpha_0 + 1}.$$

Similarly, if the claim holds for some β, then

$$\alpha \cdot \omega^{\beta+1} = \sup_{x \in \omega} \alpha \cdot \omega^{\beta} \cdot x = \sup_{x \in \omega} \omega^{\alpha_0 + \beta} \cdot x = \omega^{\alpha_0 + \beta + 1}.$$

Finally, if γ is a limit, and the claim holds when $\beta < \gamma$, then

$$\alpha \cdot \omega^{\gamma} = \sup_{x \in \gamma} \alpha \cdot \omega^{x} = \sup_{x \in \gamma} \omega^{\alpha_0 + x} = \omega^{\alpha_0 + \gamma}.$$ □

For example, we can now compute:

$$\begin{aligned}
&(\omega^{\omega+1} \cdot 3 + \omega^{6} \cdot 4 + 1) \cdot (\omega^{\omega^2} \cdot 2 + 3) \\
&= (\omega^{\omega+1} \cdot 3 + \omega^{6} \cdot 4 + 1) \cdot (\omega^{\omega^2} \cdot 2) + (\omega^{\omega+1} \cdot 3 + \omega^{6} \cdot 4 + 1) \cdot 3 \\
&= \omega^{\omega+1+\omega^2} \cdot 2 + \omega^{\omega+1} \cdot 3 \cdot 3 + \omega^{6} \cdot 4 + 1 \\
&= \omega^{\omega^2} \cdot 2 + \omega^{\omega+1} \cdot 9 + \omega^{6} \cdot 4 + 1.
\end{aligned}$$

Finally, for *exponentiation,* we have:

Theorem 109. *For all positive natural numbers n and positive ordinals α,*

$$n^{\omega^\alpha} = \begin{cases} \omega^{\omega^{\alpha-1}}, & \text{if } 0 < \alpha < \omega, \\ \omega^{\omega^{\alpha}}, & \text{if } \omega < \alpha. \end{cases}$$

Theorem 110. *For all infinite ordinals α and natural numbers n,*

1) if α is a limit, that is, $\alpha_{\ell(\alpha)} > 0$, then

$$\alpha^{n+1} = \omega^{\alpha_0 \cdot n} \cdot \alpha,$$

2) if α is a successor, that is, $\alpha_{\ell(\alpha)} = 0$, then

$$\alpha^{n+1} = \omega^{\alpha_0 \cdot (n+1)} \cdot a_0 + \sum_{i=1}^{n} \omega^{\alpha_0 \cdot (n+1-i)} \cdot \left(\beta + \alpha_0 \cdot \alpha_{\ell(\alpha)}\right) + \beta + \alpha_{\ell(\alpha)},$$

where

$$\alpha = \sum_{i=0}^{\ell(\alpha)} \omega^{\alpha_i} \cdot a_i = \omega^{\alpha_0} \cdot a_0 + \beta + \alpha_{\ell(\alpha)}.$$

Proof. 1. The claim holds trivially when $n = 0$. If it holds when $n = m$, then it holds when $n = m + 1$ by Theorem 108.

2. The claim holds trivially when $n = 0$. If it holds when $n = m$, then

$$\begin{aligned} \alpha^{m+2} &= \alpha^{m+1} \cdot (\omega^{\alpha_0} \cdot a_0 + \beta + a_{\ell(\alpha)}) \\ &= \omega^{\alpha_0 \cdot (m+2)} \cdot a_0 + \omega^{\alpha_0 \cdot (m+1)} \cdot \beta + \omega^{\alpha_0 \cdot (m+1)} \cdot a_0 \cdot a_{\ell(\alpha)} \\ &\qquad + \sum_{i=1}^{m} \omega^{\alpha_0 \cdot (m+1-i)} \cdot \left(\beta + \alpha_0 \cdot \alpha_{\ell(\alpha)}\right) + \beta + \alpha_{\ell(\alpha)} \\ &= \omega^{\alpha_0 \cdot (m+2)} \cdot a_0 + \sum_{i=1}^{m+1} \omega^{\alpha_0 \cdot (m+2-i)} \cdot \left(\beta + \alpha_0 \cdot \alpha_{\ell(\alpha)}\right) + \beta + \alpha_{\ell(\alpha)}, \end{aligned}$$

so it holds when $n = m + 1$. $\square$

Theorem 111. *For all infinite ordinals α and positive ordinals β,*

$$\alpha^{\omega^\beta} = \omega^{\alpha_0 \cdot \omega^\beta}.$$

Proof. By the previous theorem, we have

$$\alpha^{\omega} = \sup_{x \in \omega} \alpha^x = \sup_{x \in \omega} \omega^{\alpha_0 \cdot x} = \omega^{\alpha_0 \cdot \omega},$$

so the claim holds when $\gamma = 1$. Suppose it holds for some β; then

$$\alpha^{\omega^{\beta+1}} = (\alpha^{\omega^{\beta}})^{\omega} = (\omega^{\alpha_0 \cdot \omega^{\beta}})^{\omega} = \omega^{\alpha_0 \cdot \omega^{\beta+1}}.$$

Finally, if γ is a limit, and the claim holds when $0 < \beta < \gamma$, then

$$\alpha^{\omega^{\gamma}} = \alpha^{\sup_{x \in \gamma} \omega^x} = \sup_{x \in \gamma} \alpha^{\omega^x} = \sup_{x \in \gamma} \omega^{\alpha_0 \cdot \omega^x} = \omega^{\alpha_0 \cdot \omega^{\gamma}}. \qquad \square$$

Summing up, we have

Theorem 112. *For all infinite ordinals α, limit ordinals β, and positive natural numbers n,*

$$\alpha^{\beta} = \omega^{\alpha_0 \cdot \beta}, \qquad \alpha^{\beta+n} = \omega^{\alpha_0 \cdot \beta} \cdot \alpha^n.$$

For example,

$$
\begin{aligned}
&(\omega^{\omega^\omega\cdot 7+\omega^{31}\cdot 29+13}\cdot 11+\omega^5\cdot 17+19)^{\omega^{37}+\omega\cdot 2+3}\\
&= \omega^{(\omega^\omega\cdot 7+\omega^{31}\cdot 29+13)\cdot(\omega^{37}+\omega\cdot 2)}\cdot(\omega^{\omega^\omega\cdot 7+\omega^{31}\cdot 29+13}\cdot 11+\omega^5\cdot 17+19)^3\\
&= \omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2}\cdot(\,\omega^{(\omega^\omega\cdot 7+\omega^{31}\cdot 29+13)\cdot 3}\cdot 11+\\
&\qquad \omega^{(\omega^\omega\cdot 7+\omega^{31}\cdot 29+13)\cdot 2}\cdot(\omega^5\cdot 17+209)+\\
&\qquad \omega^{\omega^\omega\cdot 7+\omega^{31}\cdot 29+13}\cdot(\omega^5\cdot 17+209)+\\
&\qquad \omega^5\cdot 17+19)\\
&= \omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2}\cdot(\,\omega^{\omega^\omega\cdot 21+\omega^{31}\cdot 29+13}\cdot 11+\\
&\qquad \omega^{\omega^\omega\cdot 14+\omega^{31}\cdot 29+13}\cdot(\omega^5\cdot 17+209)+\\
&\qquad \omega^{\omega^\omega\cdot 7+\omega^{31}\cdot 29+13}\cdot(\omega^5\cdot 17+209)+\\
&\qquad \omega^5\cdot 17+19)\\
&= \omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2+\omega^\omega\cdot 21+\omega^{31}\cdot 29+13}\cdot 11+\\
&\qquad \omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2+\omega^\omega\cdot 14+\omega^{31}\cdot 29+18}\cdot 17+\\
&\qquad \omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2+\omega^\omega\cdot 14+\omega^{31}\cdot 29+13}\cdot 209)+\\
&\qquad \omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2+\omega^\omega\cdot 7+\omega^{31}\cdot 29+18}\cdot 17+\\
&\qquad \omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2+\omega^\omega\cdot 7+\omega^{31}\cdot 29+13}\cdot 209)+\\
&\qquad \omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2+5}\cdot 17+\\
&\qquad \omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2}\cdot 19.
\end{aligned}
$$

6. Cardinality

h:CN

6.1. Cardinality

lity

Definition 35. If a set a is equipollent to some ordinal, then the *least* ordinal to which a is equipollent is called the **cardinality** of a. We may denote the cardinality of a by

$$\operatorname{card}(a).$$

In particular, every ordinal has a cardinality, so every well-ordered set has a cardinality. The converse holds too: every set with a cardinality can be well-ordered. If a is finite, we have

$$\operatorname{card}(a) = |a|.$$

If two sets a and b have cardinalities, then

$$a \approx b \Leftrightarrow \operatorname{card}(a) = \operatorname{card}(b), \qquad a \prec b \Leftrightarrow \operatorname{card}(a) < \operatorname{card}(b).$$

So the following theorem is easy when a and b have cardinalities. However, we do not yet *know* that all sets have cardinalities.

ch-b

Theorem 113 (Schroeder–Bernstein[1]). *For all sets a and b,*

$$a \preccurlyeq b \mathbin{\&} b \preccurlyeq a \Rightarrow a \approx b.$$

Proof. Suppose f embeds a in b, and g embeds b in a. Define $x \mapsto (a_x, b_x)$ recursively on ω by

$$(a_0, b_0) = (a, b), \qquad (a_{n+1}, b_{n+1}) = (g[f[a_n]], f[g[b_n]]).$$

Then, for each n in ω, the relation defined by

$$x \in a_n \smallsetminus a_{n+1} \mathbin{\&} y \in b_n \smallsetminus b_{n+1} \mathbin{\&} (f(x) = y \lor x = g(y))$$

[1] The theorem is also called the Cantor–Bernstein Theorem, as for example by Levy [28, III.2.8, p. 85], who nonetheless observes that Dedekind gave the first proof in 1887.

is a bijection from $a_n \smallsetminus a_{n+1}$ to $b_n \smallsetminus b_{n+1}$. Since f is injective,

$$f[\bigcap\{a_x : x \in \omega\}] = \bigcap\{b_{x+1} : x \in \omega\} = \bigcap\{b_x : x \in \omega\}.$$

Therefore $a \approx b$. □

With a bit more work, the theorem can be shown to hold for arbitrary classes:

Porism. *For all classes $\boldsymbol{C}$ and $\boldsymbol{D}$,*

$$\boldsymbol{C} \preccurlyeq \boldsymbol{D} \mathbin{\&} \boldsymbol{D} \preccurlyeq \boldsymbol{C} \Rightarrow \boldsymbol{C} \approx \boldsymbol{D}.$$

Proof. Suppose $\boldsymbol{F}$ is an embedding of $\boldsymbol{C}$ in $\boldsymbol{D}$, and $\boldsymbol{G}$ is an embedding of $\boldsymbol{D}$ in $\boldsymbol{C}$. We can adjust the proof of the Recursion Theorem (Theorem 35) to show that there is a relation $\boldsymbol{R}$ such that, for all a in $\boldsymbol{C}$ and all n in ω,

$$\begin{gathered} 0 \mathrel{\boldsymbol{R}} a \Leftrightarrow a \in \boldsymbol{C}, \\ n' \mathrel{\boldsymbol{R}} a \Leftrightarrow \exists x\,(n \mathrel{\boldsymbol{R}} x \mathbin{\&} a = \boldsymbol{G}(\boldsymbol{F}(x))). \end{gathered}$$

Likewise, there is a relation $\boldsymbol{S}$ such that, for all b in $\boldsymbol{D}$ and all n in ω,

$$\begin{gathered} 0 \mathrel{\boldsymbol{S}} b \Leftrightarrow b \in \boldsymbol{D}, \\ n' \mathrel{\boldsymbol{S}} b \Leftrightarrow \exists y\,(n \mathrel{\boldsymbol{S}} y \mathbin{\&} b = \boldsymbol{F}(\boldsymbol{G}(y))). \end{gathered}$$

Denote the class $\{x : n \mathrel{\boldsymbol{R}} x\}$ by $\boldsymbol{C}_n$, and $\{y : n \mathrel{\boldsymbol{S}} y\}$ by $\boldsymbol{D}_n$. As before,

$$\boldsymbol{C}_n \smallsetminus \boldsymbol{C}_{n+1} \approx \boldsymbol{D}_n \smallsetminus \boldsymbol{D}_{n+1}, \qquad \bigcap_{n\in\omega} \boldsymbol{C}_n \approx \bigcap_{n\in\omega} \boldsymbol{D}_n$$

(where $\bigcap_{n\in\omega} \boldsymbol{C}_n = \{x : \forall z\,(z \in \omega \Rightarrow z \mathrel{\boldsymbol{R}} x\})$). Therefore $\boldsymbol{C} \approx \boldsymbol{D}$. □

However, the following has no generalization to classes:

thm:pow

Theorem 114 (Cantor)**.** *For all sets a,*

$$a \prec \mathscr{P}(a).$$

Proof. The function $x \mapsto \{x\}$ shows $a \preccurlyeq \mathscr{P}(a)$. Suppose f is an embedding of a in $\mathscr{P}(a)$. Let b be the set $\{x \colon x \in a \mathbin{\&} x \notin f(x)\}$. Then

$$c \in b \Rightarrow c \in b \smallsetminus f(c), \qquad\qquad c \in a \smallsetminus b \Rightarrow c \in f(c) \smallsetminus b$$

Thus, if $c \in a$, then $f(c) \neq b$. So $b \notin \operatorname{rng}(f)$. Therefore, there is no bijection from a to $\mathscr{P}(a)$; so $a \prec \mathscr{P}(a)$. □

The theorem may be *false* when applied to proper classes. Indeed,

$$\mathscr{P}(\mathbf{V}) = \mathbf{V}.$$

As noted on page 95, the following is consistent with GST, and with the rest of our axioms so far, except Infinity:

-set

Axiom 8 (Power Set)**.** *The power class of a set is a set:*

$$\forall x\, \exists y\, y = \mathscr{P}(x).$$

We may now refer to the power class of a set as its **power set.** An infinite set that is not equipollent to ω is called **uncountable;** all other sets are **countable.** So $\mathscr{P}(\omega)$ is uncountable. However, we do not yet know whether it, or any other uncountable set, has a cardinality.

Theorem 115. *For all sets a and b, the class ${}^{b}a$ is a set.*

6.2. Cardinals

Definition 36. An ordinal that is the cardinality of some ordinal is a **cardinal.** The cardinals compose the class denoted by

$$\mathbf{CN};$$

this is a subclass of $\mathbf{ON}$. Cardinals are denoted by minuscule Greek letters like κ, λ, μ, and so on.

The class $\mathbf{CN}$ inherits the ordering $<$ of $\mathbf{ON}$, which is $\in$ and $\subset$; on $\mathbf{CN}$ the ordering is also $\prec$. The finite ordinals are cardinals. Also, ω is a cardinal. But $\omega + 1$ is not a cardinal, since $\omega < \omega + 1$, but $\omega + 1 \approx 1 + \omega = \omega$.

thm:card-lim **Theorem 116.** *Every cardinal is a limit ordinal.*

The converse fails: $\omega \cdot 2$ is a limit ordinal, but $\omega < \omega \cdot 2$, and $\omega \cdot 2 \approx 2 \cdot \omega = \omega$. There are uncountable cardinals:

Lemma 18 (Hartogs). *For every set, there is an ordinal that does not embed in it.*

Proof. Supposing a is a set, let b be the subset of $\mathscr{P}(a) \times \mathscr{P}(a \times a)$ comprising those well-ordered sets $(c, <)$ such that $c \subseteq a$. If $\operatorname{ord}(c, <) = \beta$, and $\gamma < \beta$, then $\operatorname{ord}(d, <) = \gamma$ for some section d of c. This shows that $\{\operatorname{ord}(\mathfrak{c}) \colon \mathfrak{c} \in b\}$ is a transitive subset of **ON**; so it is an ordinal α. If f is an embedding of β in a, then f determines an element of b whose ordinality is β; so $\beta \in \alpha$. Since $\alpha \notin \alpha$, there is no injection of α in a. □

In the proof, α is the class of ordinals that embed in a. If α were simply defined this way, it would not obviously be a set. In any case, we can now make the following:

Definition 37. For every cardinal κ, by Hartogs's Lemma, there is an ordinal α such that $\kappa < \alpha$, but $\kappa \not\approx \alpha$. Therefore $\kappa < \operatorname{card}(\alpha)$. Thus κ has a **cardinal successor,** denoted by

$$\kappa^+;$$

this is the *least* of the cardinals that are greater than κ.

Theorem 117. *The supremum of a set of cardinals is a cardinal.*

Proof. Let a be a set of cardinals. If $\kappa < \sup(a)$, then $\kappa < \lambda$ for some λ in a, and therefore $\kappa \neq \operatorname{card}(\sup(a))$. Therefore $\sup(a)$ must be a cardinal (namely its own cardinality). □

Definition 38. The function

$$x \mapsto \aleph_x$$

from **ON** into **CN** is given recursively by

$$\aleph_0 = \omega, \qquad \aleph_{\alpha'} = (\aleph_\alpha)^+, \qquad \aleph_\beta = \sup_{x \in \beta} \aleph_x,$$

where β is a limit ordinal. Here $\aleph$ is *aleph,* the first letter of the Hebrew alphabet.

thm:aleph **Theorem 118.** *The function $x \mapsto \aleph_x$ is an isomorphism between* **ON** *and the class of infinite cardinals.*

6.3. Cardinal addition and multiplication

rd+.

Definition 39. The **cardinal sum** of two cardinals is the cardinality of their ordinal sum. The **cardinal product** of two cardinals is the cardinality of their ordinal product. The operations of finding cardinal sums and products are **cardinal addition** and **cardinal multiplication,** respectively, and are denoted by $+$ and $\cdot$ (as are ordinal addition and multiplication; context must indicate which operations are meant).

Theorem 119. *For all cardinals κ and λ,*

$$\kappa + \lambda = \operatorname{card}((\kappa \times \{0\}) \cup (\lambda \times \{1\})), \qquad \kappa \cdot \lambda = \operatorname{card}(\kappa \times \lambda).$$

rith

Theorem 120. *For all cardinals κ, λ, and μ,*

$$\begin{gathered}
\kappa + \lambda = \lambda + \kappa, \\
\kappa + 0 = \kappa, \\
(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu), \\
\kappa \cdot \lambda = \lambda \cdot \kappa, \\
\kappa \cdot 1 = \kappa, \\
(\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu), \\
\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu, \\
\kappa \leqslant \lambda \Rightarrow \kappa + \mu \leqslant \lambda + \mu, \\
\kappa \leqslant \lambda \Rightarrow \kappa \cdot \mu \leqslant \lambda \cdot \mu.
\end{gathered}$$

The cardinal operations agree with the ordinal operations on ω.

Lemma 19. *The Cartesian product $a \times b$ is always a set.*

Proof. We have $a \times b = \bigcup\{a \times \{x\} \colon x \in b\}$. □

onon

Lemma 20. *The class* **ON** $\times$ **ON** *is well-ordered by $<$, where*

$$\begin{aligned}
(\alpha, \beta) < (\gamma, \delta) \Leftrightarrow {} & \max(\alpha, \beta) < \max(\gamma, \delta) \vee \\
& \Big(\max(\alpha, \beta) = \max(\gamma, \delta) \mathbin{\&} \big(\alpha < \gamma \vee (\alpha = \gamma \mathbin{\&} \beta < \delta)\big)\Big).
\end{aligned}$$

(See Figure 6.1.) With respect to this ordering, **ON** $\times$ **ON** *is isomorphic to* **ON** *with its usual ordering.*

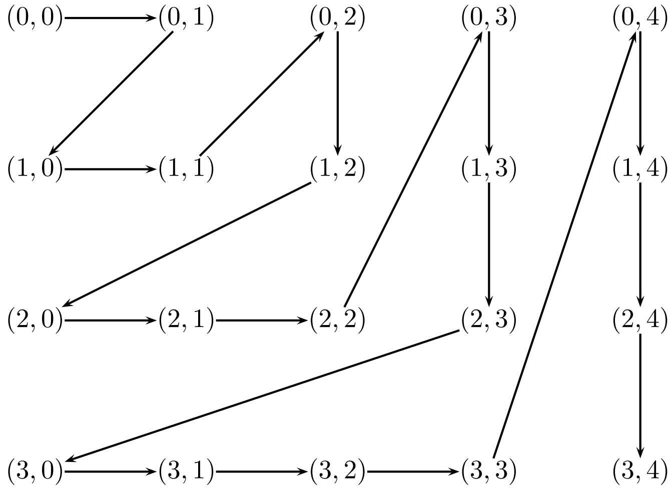


fig:onon

Figure 6.1. $\mathbf{ON} \times \mathbf{ON}$, well-ordered

Proof. It is straightforward to show that the given relation is a linear ordering of $\mathbf{ON} \times \mathbf{ON}$. If a is a nonempty subset of $\mathbf{ON} \times \mathbf{ON}$, we can define

$$\begin{aligned}
\alpha &= \min\{\max(x, y) \colon (x, y) \in a\},\\
\beta &= \min\{x \colon \exists y\, (x, y) \in a \mathbin{\&} \max(x, y) = \alpha)\},\\
\gamma &= \min\{y \colon (\beta, y) \in a\}.
\end{aligned}$$

Then (β, γ) is the least element of a. The linear ordering is left-narrow, since every section is a subset of $\delta \times \delta$ for some δ. So $\mathbf{ON} \times \mathbf{ON}$ is well-ordered. Since it is a proper class, it is isomorphic to $\mathbf{ON}$, by the corollary to Theorem 70. $\square$

Lemma 21. *For all infinite cardinals κ,*

$$\kappa \cdot \kappa = \kappa.$$

Proof. We establish the claim by induction on the infinite cardinals. Suppose λ is an infinite cardinal, and the equation holds whenever $\omega \leqslant \kappa < \lambda$. Let $\boldsymbol{F}$ be the isomorphism from $\mathbf{ON} \times \mathbf{ON}$ onto $\mathbf{ON}$ guaranteed by the last lemma. For every ordinal α, the section $\mathrm{pred}(0, \alpha)$ of $\mathbf{ON} \times \mathbf{ON}$ is just $\alpha \times \alpha$. Then $\boldsymbol{F}[\alpha \times \alpha]$ must be a section of $\mathbf{ON}$: that is, $\boldsymbol{F}[\alpha \times \alpha]$ is an ordinal. Suppose $\boldsymbol{F}[\lambda \times \lambda] = \beta$. Then

$$\lambda = \lambda \cdot 1 \leqslant \lambda \cdot \lambda = \mathrm{card}(\lambda \times \lambda) = \mathrm{card}(\beta) \leqslant \beta.$$

So $\lambda \leqslant \beta$. We shall show $\beta \leqslant \lambda$. For this, it is enough to show that, for all infinite cardinals μ,

$$\mu < \beta \Rightarrow \mu < \lambda.$$

Suppose μ is an infinite cardinal, and $\mu < \beta$. Then $\mu = \boldsymbol{F}(\gamma, \delta)$ for some ordinals γ and δ such that $(\gamma, \delta) \in \lambda \times \lambda$. Since λ is a limit ordinal by Theorem 116, the successor ζ of $\max(\beta, \gamma)$ is also less than λ. Hence

$$\mu \in \boldsymbol{F}[\zeta \times \zeta], \qquad \mu \subset \boldsymbol{F}[\zeta \times \zeta],$$

and so

$$\mu \leqslant \mathrm{card}(\zeta \times \zeta) = \mathrm{card}(\zeta) \cdot \mathrm{card}(\zeta) = \mathrm{card}(\zeta) < \lambda$$

by inductive hypothesis. □

easy

Theorem 121. *For all cardinals κ and λ, at least one of which is infinite,*

$$\kappa + \lambda = \max(\kappa, \lambda).$$

If also neither cardinal is 0*,*

$$\kappa \cdot \lambda = \max(\kappa, \lambda).$$

In particular,

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_{\max(\alpha, \beta)}.$$

Proof. We may assume $\kappa \leqslant \lambda$. Then, by the lemma, we need only observe

$$\lambda \leqslant \kappa + \lambda \leqslant \lambda + \lambda = \lambda \cdot 2 \leqslant \lambda \cdot \lambda,$$

and if also $1 \leqslant \kappa$,

$$\lambda = 1 \cdot \lambda \leqslant \kappa \cdot \lambda \leqslant \lambda \cdot \lambda. \qquad \square$$

Suppose we have a set a of cardinality κ, and a function $x \mapsto b_x$ on a such that, for all c in a, the set b_c has a cardinality. Let $\lambda = \sup\{\operatorname{card}(b_x) \colon x \in a\}$. It may appear that $\bigcup_{x\in a} b_x$ has a cardinality, which is bounded above by $\kappa \cdot \lambda$. For, if $c \in a$, let f_c be an embedding of b_c in λ. We may assume that $b_c \cap b_d = \varnothing$ when $c \neq d$. Then we should have an embedding of $\bigcup_{x\in a} b_x$ in $\kappa \times \lambda$, namely

$$\bigcup_{x\in a}\{(y, (x, f_x(y))) \colon y \in b_x\}.$$

The problem with this argument is that it assumes the existence of the function $x \mapsto f_x$ on a, and we do not yet have a way to ensure the existence of this function.

6.4. Cardinalities of ordinal powers

Definition 40. If $n \in \omega$ and $\alpha \in \mathbf{ON}$, an element a of ${}^n\alpha$ can be understood as the function $x \mapsto a_x$ on n. This function can also be written as

$$(a_0, \dots, a_{n-1})$$

and called an n**-tuple.** In case $n = 0$, this n-tuple is the empty set (which is indeed the only function from 0 to α).

Recalling the notation of Definition 33, we have:

Lemma 22. *For all infinite cardinals κ,*

$$\operatorname{card}(\operatorname{fs}({}^\omega\kappa)) = \kappa.$$

Proof. We have

$$\operatorname{fs}({}^\omega\kappa) = \bigcup\{{}^x\kappa \colon x \in \omega\}.$$

Let f be an embedding of $\kappa \times \kappa$ in κ. By finite recursion, we have a function $x \mapsto g_x$ on ω such that

1. g_0 is the embedding $\{(0, 0)\}$ of ${}^0\kappa$ in κ.
2. if g_n is an embedding of ${}^n\kappa$ in κ, then g_{n+1} is the embedding of ${}^{n+1}\kappa$ in κ given by

$$g_{n+1}(a_0, \dots, a_n) = f(g_n(a_0, \dots, a_{n-1}), a_n).$$

Then we have an embedding h of $\operatorname{fs}({}^{\omega}\kappa)$ in $\omega \times \kappa$ given by

$$h(a_0, \dots, a_{n-1}) = (n, f_n(a_0, \dots, a_{n-1})).$$

So we have

$$\kappa \approx {}^{1}\kappa \preccurlyeq \operatorname{fs}({}^{\omega}\kappa) \preccurlyeq \omega \times \kappa \preccurlyeq \kappa.$$

By the Schröder–Bernstein Theorem, we are done. □

Lemma 23. *For all infinite cardinals κ, the set of finite subsets of κ has a cardinality, which is κ.*

Proof. The given set embeds in $\operatorname{fs}({}^{\omega}\kappa)$ under the map that takes every set $\{\alpha_0, \dots, \alpha_{n-1}\}$, where $\alpha_0 < \dots < \alpha_{n-1}$, to $(\alpha_0, \dots, \alpha_{n-1})$. □

Lemma 24. *For all infinite cardinals κ,*

$$\operatorname{card}(\operatorname{fs}({}^{\kappa}\kappa)) = \kappa.$$

Proof. Let f be the function from $\operatorname{fs}({}^{\kappa}\kappa)$ to $\operatorname{fs}({}^{\omega}\kappa)$ such that, if $g \in \operatorname{fs}({}^{\kappa}\kappa)$, and $\operatorname{dom}(g) = \{\alpha_0, \dots, \alpha_{n-1}\}$, where $\alpha_0 < \dots < \alpha_{n-1}$, then $f(g)$ is the function $x \mapsto g(\alpha_x)$ from n to κ. Let b be the set of finite subsets of κ. Then the function $x \mapsto (\operatorname{dom}(x), f(x))$ is an embedding of $\operatorname{fs}({}^{\kappa}\kappa)$ in $b \times \operatorname{fs}({}^{\omega}\kappa)$. Hence

$$\kappa \preccurlyeq \operatorname{fs}({}^{\kappa}\kappa) \preccurlyeq b \times \operatorname{fs}({}^{\omega}\kappa) \approx \kappa \times \kappa \approx \kappa.$$ □

Lemma 25. *For all cardinals κ and λ such that $\kappa > 1$,*

$$\lambda \leqslant \operatorname{card}(\operatorname{fs}({}^{\lambda}\kappa)).$$

Proof. We have an embedding f of λ in $\operatorname{fs}({}^{\lambda}\kappa)$ given by

$$f(\alpha) = \{(\alpha, 1)\} \cup \{(x, 0) \colon x \in \lambda \smallsetminus \{\alpha\}\}.$$ □

Theorem 122. *For all cardinals κ and λ such that $\kappa > 1$, and $\lambda > 0$, and at least one of the two is infinite,*

$$\operatorname{card}(\operatorname{fs}({}^{\lambda}\kappa)) = \max(\kappa, \lambda).$$

Proof. Let $\mu = \max(\kappa, \lambda)$. Then

$$\mu \preccurlyeq \operatorname{fs}({}^{\lambda}\kappa) \preccurlyeq \operatorname{fs}({}^{\mu}\mu) \approx \mu.$$ □

Therefore it would not be of great interest to define a cardinal power as the cardinality of an ordinal power. We should like to define κ^{λ} as the cardinality of ${}^{\lambda}\kappa$. The problem is that this set has no obvious good ordering. We are just going to declare that one exists, in the next section.

6.5. The Axiom of Choice

Theorem 123. *Every natural number embeds in every infinite set.*

Proof. Let a be an infinite set. Trivially, 0 embeds in a. Suppose n embeds in a under a function f. Since a is infinite, we have $f[n] \neq a$. Therefore $a \smallsetminus f[n]$ has an element b, and so $f \cup \{n, b\}$ is an embedding of $n+1$ in a. By finite induction, every natural number embeds in a. □

The theorem is *not* that ω embeds in every infinite set. One might try to adapt the proof of the theorem so as to give a recursive definition of an embedding of ω in a. The problem is that we have no way to select a particular element of $a \smallsetminus f[n]$. A technical term for what we need is given by the following.

Definition 41. A **choice-function** for a set a is a function f on $\mathscr{P}(a) \smallsetminus \{0\}$ such that $f(b) \in b$ for each nonempty subset b of a.

Theorem 124. *The set of natural numbers embeds in every infinite set that has a choice-function.*

Proof. Suppose f is a choice-function for an infinite set a. By transfinite recursion in one part, there is an embedding g of ω in a given by

$$g(n) = f(a \smallsetminus g[\,n\,]).$$

□

thm:ac=wo

Theorem 125. *A set has a choice-function if and only if the set can be well-ordered.*

Proof. Suppose a set a has the choice-function f. We may assume $f(\varnothing)$ is defined, but is not in a. There is a function $\boldsymbol{G}$ on $\mathbf{ON}$ defined recursively by

$$\boldsymbol{G}(\alpha) = f(a \smallsetminus \boldsymbol{G}[\,\alpha\,]).$$

Suppose $\boldsymbol{G}[\alpha] \subseteq a$. If $\gamma < \beta < \alpha$, then $\boldsymbol{G}(\beta) \in a \smallsetminus \boldsymbol{G}[\beta]$, so in particular $\boldsymbol{G}(\beta) \in a \smallsetminus \{\boldsymbol{G}(\gamma)\}$, and therefore $\boldsymbol{G}(\beta) \neq \boldsymbol{G}(\gamma)$. Thus $\boldsymbol{G}$ embeds α in a. The same proof shows that $\boldsymbol{G}$ embeds $\mathbf{ON}$ in a, if $\boldsymbol{G}[\mathbf{ON}] \subseteq a$. Therefore $\boldsymbol{G}(\alpha) \notin a$ for some α. Let β be the least such α. Then $\boldsymbol{G}$ is a bijection from β to a. This induces a good ordering of a.

Now suppose conversely that a is well-ordered. Then there is a choice-function for a that assigns to each non-empty subset of a its least element.

□

-lem **Theorem 126** (Zorn's Lemma). *Assume a has a choice-function. If $(a, <)$ is an order such that every linearly ordered subset of a has an upper bound in a, then a has a maximal element.*

Proof. Let f be the operation on $\mathscr{P}(a)$ taking each element b to the set (possibly empty) of strict upper bounds of b. So f is order-reversing, in the sense that

$$c \subseteq b \Rightarrow f(c) \supseteq f(b).$$

Let g be a choice-function for a, extended so that $g(\varnothing) \notin a$. Then define $\boldsymbol{H}$ on $\mathbf{ON}$ by

$$\boldsymbol{H}(\alpha) = g(f(a \cap \boldsymbol{H}[\,\alpha\,])).$$

As in the proof of the previous theorem, if $\boldsymbol{H}[\alpha] \subseteq a$, then $\boldsymbol{H}$ embeds α in a; in fact it embeds $(\alpha, \in)$ in $(a, <)$, so in particular $\boldsymbol{H}[\alpha]$ is linearly ordered and has an upper bound. Also as in the previous proof, $\boldsymbol{H}(\alpha) \notin a$ for some α. Let β be the least such α. Then $f(\boldsymbol{H}[\beta]) = \varnothing$, so the upper bound of $\boldsymbol{H}[\beta]$ is not strict. This upper bound is therefore a maximal element of a. □

A sort of converse to the last theorem is the following.

zorn **Theorem 127.** *For a set a, let b be the set of functions f such that the domain of f is a subset of $\mathscr{P}(a) \smallsetminus \{0\}$ and, for all x in the domain, $f(x) \in x$. Every maximal element of b with respect to proper inclusion is a choice-function for a.*

Proof. Suppose $f \in b$, but is a not a choice-function for a. Then some nonempty subset c of a is not in the domain of f. But c has an element d, and then $f \cup \{(c, d)\}$ is in b. Thus f is not a maximal element of b. □

In the notation of the theorem, not only is b ordered by proper inclusion, but every linearly ordered subset c of b has the upper bound $\bigcup c$. So the following are equivalent statements about $\mathbf{V}$:

1. Every set has a choice-function.
2. Every set can be well-ordered, so it has a cardinality.
3. Every order has a maximal element, provided every linearly ordered subset of the order has an upper bound.

oice **Axiom 9** (Choice). *Every set has a choice-function.*

The Axiom of Choice, or AC, is a completely new kind of axiom, since it asserts the existence of certain sets (namely, choice-functions) that we do not already have as classes. However, as with the Generalized Continuum Hypothesis (Definition 44 below), so with AC, we shall see in Chapter 8 that we *can* assume it without contradicting our other axioms. The Axiom of Choice is convenient for mathematics in that it allows many theorems to be proved, such as the following, alluded to at the end of § 6.3.

Lemma 26. *For all sets a and functions $x \mapsto c_x$ on a, there is a function f on a such that $f(d) \in c_d$ for all d in a such that c_d is nonempty.*

Proof. Let f be a choice-function for $\bigcup_{x\in a} c_x$. Then we can let $f(x) = g(c_x)$. □

Theorem 128. *For all sets a and functions $x \mapsto b_x$ on a,*

$$\operatorname{card}(\bigcup_{x\in a} b_x) \leqslant \operatorname{card}(a) \cdot \sup_{x\in a} \operatorname{card}(b_x).$$

In particular, the union of a countable set of countable sets is countable.

Proof. We can use the argument at the end of §6.3, since the function $x \mapsto f_x$ there does exist, by the lemma. □

6.6. Exponentiation

ect:card-exp

Cardinal exponentiation is quite different from ordinal exponentiation.

Definition 42. If κ and λ are cardinals, then

$$\kappa^\lambda = \operatorname{card}({}^\lambda\kappa);$$

this is the λ-th **cardinal power** of κ. The operation $(x, y) \mapsto x^y$ on **CN** is **cardinal exponentiation.**

-exp **Theorem 129.** *For all cardinals κ, λ, μ and ν,*

$$\kappa^0 = 1,$$
$$0 < \lambda \Rightarrow 0^\lambda = 0,$$
$$1^\lambda = 1,$$
$$\kappa^1 = \kappa,$$
$$\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu,$$
$$\kappa^{\lambda\cdot\mu} = (\kappa^\lambda)^\mu, \tag{6.1}$$
eqn:exp-exp
$$\kappa \leqslant \mu \;\&\; \lambda \leqslant \nu \Rightarrow \kappa^\lambda \leqslant \mu^\nu. \tag{6.2}$$
eqn:exp-inec

<2^k **Theorem 130.** *For all sets a,*

$$\mathscr{P}(a) \approx 2^a.$$

Proof. There is a bijection between ${}^a2$ and $\mathscr{P}(a)$ that takes the function f to the set $\{x \in a \colon f(x) = 1\}$. □

Corollary. *For all cardinals κ,*

$$\kappa < 2^\kappa.$$

Theorem 131. *Suppose κ and λ are cardinals such that*

$$2 \leqslant \kappa, \qquad 1 \leqslant \lambda, \qquad \aleph \leqslant \max(\kappa, \lambda).$$

Then

$$\max(\kappa, 2^\lambda) \leqslant \kappa^\lambda \leqslant 2^{\max(\kappa,\lambda)}. \tag{6.3}$$
eqn:card-exp

In particular, if also $\kappa \leqslant 2^\lambda$, then

$$\kappa^\lambda = 2^\lambda,$$

while if $2^\lambda < \kappa$, then

$$\kappa \leqslant \kappa^\lambda \leqslant 2^\kappa. \tag{6.4}$$
eqn:exp«

Proof. The first inequality in (6.3) follows from (6.2) in Theorem 130. For the second inequality, we have by the corollary of the last theorem, by (6.1) in Theorem 129, and by Theorem 121,

$$\kappa^\lambda \leqslant (2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^{\max(\kappa,\lambda)}. \qquad \square$$

For example, (6.4) gives us

$$2^{2^{\aleph_0}} \leqslant (2^{2^{\aleph_0}})^{\aleph_0} \leqslant 2^{2^{2^{\aleph_0}}};$$

but we could already compute directly that the first weak inequality is equality:

$$(2^{2^{\aleph_0}})^{\aleph_0} = 2^{2^{\aleph_0} \cdot \aleph_0} = 2^{2^{\aleph_0}}.$$

However, if we just let $\kappa \geqslant (2^{\aleph_0})^+$, then (6.4) gives us

$$\kappa \leqslant \kappa^{\aleph_0} \leqslant 2^\kappa,$$

and it is not clear how we can be more precise. For a last example, (6.4) gives us

$$\aleph_\omega \leqslant {\aleph_\omega}^{\aleph_0} \leqslant 2^{\aleph_\omega};$$

but now we can make the first inequality strict as follows. Suppose f is a function from $\aleph_\omega$ into ${}^\omega\aleph_\omega$. Define an element g of ${}^\omega\aleph_\omega$ by

$$g(n) = \min(\aleph_\omega \smallsetminus \{f(\alpha)(n)\colon \alpha < \aleph_n\}).$$

Since $\operatorname{card}(\{g(\alpha)(n)\colon \alpha < \aleph_n\}) \leqslant \aleph_n < \aleph_\omega$, the function g is well defined. Also, if $\alpha \in \aleph_\omega$, then $\alpha \in \aleph_n$ for some n in ω, and then

$$g(n) \neq f(\alpha)(n),$$

so $g \neq f(\alpha)$. Thus f is not surjective onto ${}^\omega\aleph_\omega$. Therefore

$$\aleph_\omega < {\aleph_\omega}^{\aleph_0}.$$

6.7. The Continuum

ct:continuum

It may be convenient to have names for certain powers of 2.

Definition 43. The function

$$x \mapsto \beth_x$$

from **ON** into **CN** is given recursively by

$$\beth_0 = \omega, \qquad \beth_{\alpha'} = 2^{\beth_\alpha}, \qquad \beth_\beta = \sup_{x \in \beta} \beth_x,$$

where β is a limit. Here $\beth$ is *beth,* the second letter of the Hebrew alphabet.

Theorem 132. *The function* $x \mapsto \beth_x$ *is an embedding of* **ON** *in* **CN**, *and*

$$\aleph_\alpha \leqslant \beth_\alpha. \tag{6.5}$$

eqn:aleph-be

Proof. Theorem 130 and induction. □

:GCH

Definition 44. The **Continuum Hypothesis,** or CH, is

$$\aleph_1 = \beth_1;$$

the **Generalized Continuum Hypothesis,** or GCH, is

$$\aleph_\alpha = \beth_\alpha$$

for all ordinals α.

We shall see in Chapter 8 that we *can* make these hypotheses without contradicting our other axioms about sets. We shall not see what is also the case, that these hypotheses are not *implied* by our axioms [8].

The Continuum Hypothesis is so called because it is that $\aleph_1$ is the cardinality of the **continuum,** namely the set of *real numbers.* Indeed, full details are a lengthy exercise in ordered algebra; but one approach to the real numbers can be sketched out as follows.

Suppose a relation $\sim$ on a set s is an **equivalence relation,** that is,

$$a \sim a, \qquad a \sim b \Rightarrow b \sim a, \qquad a \sim b \;\&\; b \sim c \Rightarrow a \sim c,$$

for all a, b, and c in s. Then we let

$$[a] = [a]_\sim = \{x \colon x \in s \;\&\; x \sim a\},$$

and we let

$$s/{\sim} = \{[x] \colon x \in s\}.$$

Let $\mathbb{Z}^+$ denote $\omega \smallsetminus \{0\}$. There is an equivalence-relation $\sim$ on $\mathbb{Z}^+ \times \mathbb{Z}^+$ given by

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc.$$

Then $(\mathbb{Z}^+ \times \mathbb{Z}^+)/{\sim}$ is denoted by

$$\mathbb{Q}^+,$$

and its element $[(a,b)]$ is denoted by

$$\frac{a}{b}$$

or a/b. The set $\mathbb{Q}^+$ has binary operations $+$ and $\cdot$ such that

$$\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}, \qquad\qquad \frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d},$$

and a linear ordering $<$ so that

$$\frac{a}{b}<\frac{c}{d}\Leftrightarrow a\cdot d<b\cdot c,$$

and the function $x\mapsto x/1$ embeds $\mathbb{Z}^+$ in $\mathbb{Q}^+$. In the present context we identify a in $\mathbb{Z}^+$ with $a/1$ in $\mathbb{Q}^+$.

A **cut** of $\mathbb{Q}^+$ is a proper nonempty subset c of $\mathbb{Q}^+$ with no greatest element that contains all predecessors of its elements. We denote the set of cuts of $\mathbb{Q}^+$ by

$$\mathbb{R}^+;$$

this is the set of **positive** real numbers. The set is linearly ordered by proper inclusion, and $\mathbb{Q}^+$ embeds in it under $x\mapsto \operatorname{pred}(x)$. The binary operations $+$ and $\cdot$ are defined on $\mathbb{R}^+$ by

$$a+b=\{x+y\colon x\in a\ \&\ y\in b\}, \qquad a\cdot b=\{x\cdot y\colon x\in a\ \&\ y\in b\}.$$

A nonempty subset of $\mathbb{R}^+$ with an upper bound has a supremum, namely the union of the subset. We identify an element a of $\mathbb{Q}^+$ with its image $\operatorname{pred}(a)$ in $\mathbb{R}^+$.

On $\mathbb{R}^+\times\mathbb{R}^+$ there is an equivalence relation $\sim$ given by

$$(a,b)\sim(c,d)\Leftrightarrow a+d=b+c.$$

Then $(\mathbb{R}^+\times\mathbb{R}^+)/{\sim}$ is denoted by

$$\mathbb{R},$$

and $\mathbb{R}^+$ embeds in this under $x\mapsto[(x+1,1)]$. The element $[(1,1)]$ of $\mathbb{R}$ is also denoted by

$$0.$$

Then

$$\mathbb{R} = \{[(1, 1 + x)] \colon x \in \mathbb{R}^+\} \cup \{0\} \cup \{[(x + 1, 1)] \colon x \in \mathbb{R}^+\},$$

and the three sets of the union are disjoint. We identify a in $\mathbb{R}^+$ with $[(a + 1, 1)]$.

We have $\mathbb{R} \approx \mathbb{R}^+$, and $\mathbb{R}^+ \subseteq \mathscr{P}(\mathbb{Q}^+)$, and $\operatorname{card}(\mathbb{Q}^+) = \aleph_0$, so

$$\operatorname{card}(\mathbb{R}) \leqslant 2^{\aleph_0}.$$

The reverse inequality follows as well, because there is an embedding f of ${}^{\omega}2$ into $\mathbb{R}$ defined by

$$f(\sigma) = \sup\Big\{\sum_{k=0}^{x} \frac{2 \cdot \sigma(k)}{3^{k+1}} : x \in \omega\Big\}.$$

The function f is indeed well-defined, since, by induction,

$$\sum_{k=0}^{n} \frac{2 \cdot \sigma(k)}{3^{k+1}} \leqslant 1 - \frac{1}{3^{n+1}} < 1.$$

Also, f is injective, since, if $\sigma \restriction n = \tau \restriction n$, but $\sigma(n) = 0 < 1 = \tau(n)$, then

$$f(\sigma) \leqslant \sum_{k=0}^{n-1} \frac{2 \cdot \sigma(k)}{3^{k+1}} + \frac{1}{3^n} < \sum_{k=0}^{n-1} \frac{2 \cdot \sigma(k)}{3^{k+1}} + \frac{2}{3^n} \leqslant f(\tau).$$

So $2^{\aleph_0} \leqslant \operatorname{card}(\mathbb{R})$; by the Schröder–Bernstein Theorem,

$$\operatorname{card}(\mathbb{R}) = 2^{\aleph_0}.$$

Here $f[{}^{\omega}2]$ is called the **Cantor set;** it is the intersection of the sets depicted in Figure 6.2.

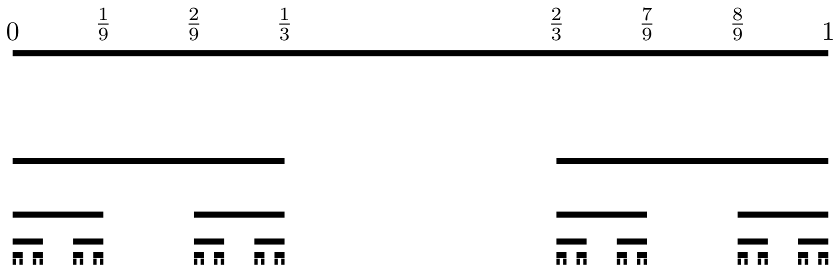


fig:cantor

Figure 6.2. Towards the Cantor set

7. Incompleteness

lete

7.1. Formulas as sets

sets

In this book, our *propositions* (labelled as lemmas, theorems, porisms, corollaries, or just axioms) are of three kinds. They may be:

1. Statements *about* the sentences and other formulas of our logic.
2. English versions of sentences *in* this logic—that is, $\in$-sentences—that are axioms or are derivable from these axioms. (This includes the logical theorems.)
3. Statements concerning how well our logic captures our informal notions of mathematics.

In Chapter 2, the propositions are of the first kind, until the Russell Paradox (Theorem 17), which is of the second kind. In Chapter 4, propositions such as Theorem 28 are of the third kind. In Chapters 5 and 6, the propositions are all of the second kind.

Now we return to proving some propositions of the first kind, *about* formulas. These propositions will be a bridge between propositions of the first and second kinds. Now that we have developed some understanding of sets, we can use it to understand formulas *as* sets; our first propositions below will develop this understanding. This will allow some propositions about formulas (propositions of the first kind) to be written out as $\in$-sentences and so be understood as propositions of the second kind. Then a *fourth* kind of proposition will come to light:

4. English versions of $\in$-sentences that are true, but are not derivable from our axioms.

(We could consider an axiom as being of this kind if it is not derivable from any *other* of our axioms.)

We convert formulas to sets as follows. We first assign to each symbol s of our logic a set, in fact a number, to be denoted by

$$\ulcorner s \urcorner.$$

This will be a **code** for s. To define these codes, we must first be clear about what our symbols *are*. We may now suppose that our variables are v_k, and our constants are c_k, where $k \in \omega$. Then we can make an assignment thus:

s	$\in$	$\neg$	$($	$\Rightarrow$	$)$	$\exists$	v_k	c_k
$\ulcorner s \urcorner$	0	1	2	3	4	5	$6+2k$	$7+2k$

The details of the assignment are not important, as long as the function $s \mapsto \ulcorner s \urcorner$ is injective. In actually writing particular formulas, we can continue our usual practice of letting a, b, and c be constants; and x, y, and z, variables.

Each formula φ of our logic is a string $s_0 \cdots s_n$ for some n in ω. Given this string, we can now form the $(n+1)$-tuple

$$(\ulcorner s_0 \urcorner, \dots, \ulcorner s_n \urcorner),$$

which we can denote by

$$\ulcorner \varphi \urcorner.$$

This tuple is a **code** for φ; it is an element of ${}^{n+1}\omega$. In order to work with such codes, we may use the following notation: If $\vec{a}$ is the m-tuple $(a_0, \dots, a_{m-1})$, and $\vec{b}$ is the n-tuple $(b_0, \dots, b_{n-1})$, then we let

$$\vec{a} \frown \vec{b} = (a_0, \dots, a_{m-1}, b_0, \dots, b_{n-1}),$$

which is an $m+n$-tuple. Thus $(x, y) \mapsto x \frown y$ is a binary operation on $\bigcup_{n \in \omega} {}^n\omega$, namely the operation of **concatenation.**

em:codes-set

Lemma 27. *The codes of the formulas compose a set.*

Proof. Since these codes are all elements of the set $\bigcup_{n \in \omega} {}^n\omega$, it is enough to show that they compose a class. The proof is just a translation of the recursive definition of formula (Definition 1 in §2.4) into the language of sets. The desired class consists of all tuples $\vec{b}$ such that, for some n in ω, there is an $(n+1)$-tuple $(\vec{a}_0, \dots, \vec{a}_n)$ of tuples such that $\vec{a}_n = \vec{b}$ and, for all k in $n+1$, one of the following holds:

1. For some t and u in $\omega \setminus 6$ (that is, for some t and u that are codes of terms),
$$\vec{a}_k = (t, 0, u).$$
2. For some j in k,
$$\vec{a}_k = (1) \frown \vec{a}_j.$$

3. For some i and j in k,

$$\vec{a}_k = (2) \frown \vec{a}_i \frown (3) \frown \vec{a}_j \frown (4).$$

4. For some j in k, for some m in $\omega \smallsetminus 3$,

$$\vec{a}_k = (5) \frown (2m) \frown \vec{a}_j. \qquad \square$$

With a bit more work, we have:

Lemma 28. *The codes of the sentences compose a set.*

Proof. Each sentence is such because there is a tuple

$$((\ulcorner\varphi_0\urcorner, a_0), \dots, (\ulcorner\varphi_n\urcorner, a_n))$$

of ordered pairs, where φ_n is the sentence, and for each k in $n+1$, the set a_k consists of those j in ω such that v_j is a free variable of φ_k. (In particular then, $a_n = 0$.) Now proceed as in the proof of the last lemma. $\square$

By the same methods, we now have:

Lemma 29. *The codes of the logical theorems compose a set.*

7.2. Incompleteness

lete

Let us now say that a collection of $\in$-formulas is **set-like** if the codes of those sentences compose a set.[1] By Lemma 27, it is sufficient that these codes compose a class. Therefore, a collection Γ of formulas is set-like if and only if there is a singulary formula φ such that, for all formulas ψ,

$$\psi \text{ belongs to } \Gamma \text{ if and only if } \varphi(\ulcorner\psi\urcorner) \text{ is true.}$$

We now have more lemmas of the first kind like those of the previous section.

-rec **Lemma 30.** *For every set-like collection Γ of sentences, the collection of sentences that are derivable from Γ is also set-like.*

[1] I do not know whether this definition has been made elsewhere.

Recall that binary relations are particular kinds of classes, as developed in §4.2.

lem:aR **Lemma 31.** *For every set-like collection Δ of sentences, there is a binary relation $\boldsymbol{R}$ such that, for all sets a and singulary formulas ψ,*

$$a\ \boldsymbol{R}\ \ulcorner\psi\urcorner \text{ if and only if } \Delta \vdash \psi(a).$$

Gödel [19] establishes something like the foregoing lemmas of this chapter, not in the logic of sets that we are using, but in a logic for ω. Gödel then proves the original (and more difficult) version of the following.[2]

n:incomplete **Theorem 133** (Incompleteness). *For every set-like collection Δ of true sentences, there is a true sentence that is not derivable from Δ.*

Proof. Assuming $\boldsymbol{R}$ is as in the last lemma, let $\varphi(x)$ be the formula $\neg(x\ \boldsymbol{R}\ x)$, and let σ be the sentence $\varphi(\ulcorner\varphi\urcorner)$. Then

$$\neg\sigma \Rightarrow \sigma.$$

Indeed, $\neg\sigma$ is $\neg\varphi(\ulcorner\varphi\urcorner)$, that is, $\ulcorner\varphi\urcorner\ \boldsymbol{R}\ \ulcorner\varphi\urcorner$, which means $\Delta \vdash \varphi(\ulcorner\varphi\urcorner)$, that is, $\Delta \vdash \sigma$. So we can write $\neg\sigma \Rightarrow \sigma$ as

$$\text{If } \Delta \vdash \sigma, \text{ then } \sigma \text{ is true.}$$

This is true by Theorem 6 of §2.7. Therefore σ itself is true, by the tautology $(\neg\sigma \Rightarrow \sigma) \Rightarrow \sigma$. That is, $\neg(\ulcorner\varphi\urcorner\ \boldsymbol{R}\ \ulcorner\varphi\urcorner)$ is true, which means $\varphi(\ulcorner\varphi\urcorner)$ or rather σ is *not* derivable from Δ. □

[2] If we say that our logic has the *signature* $\{\in\}$, then the logic covered by Gödel's original theorem has the signature $\{+, \cdot\}$. It is possible to say precisely what it means for a sentence in this signature to be true in ω; we shall not be concerned with the precise definition, but it is a generalization of Definition 45 in the next section. Relying on the Fundamental Theorem of Arithmetic, Gödel defines the code of a formula $s_0 \cdots s_n$ as the natural number

$$p_0{}^{\ulcorner s_0\urcorner} \cdots p_n{}^{\ulcorner s_n\urcorner},$$

where $k \mapsto p_k$ is the order-preserving bijection from ω onto the set $\{2, 3, 5, \dots\}$ of prime numbers. Gödel's concern is not just whether a collection of such codes is a set, but whether it is of the special kind called a *recursive set.* Informally, a recursive set is a computable set, a set for which there is a computer program that can tell whether a given number belongs to the set. The precise definition of recursive sets of natural numbers will not concern us; but by Gödel's theorem, for every collection of sentences in the signature $\{+, \cdot\}$ that are true in ω and whose codes compose a recursive set, there is a sentence that is true in ω but cannot be derived from the collection.

The theorem is called the Incompleteness Theorem for the following reason. In §2.7, we referred to the collection of all true sentences as *set theory.* To be more precise, it is **complete** set theory, the *complete* theory of sets. In general, a **theory** is a collection of sentences (in some logic) that contains all sentences (in that logic) that are derivable from it. In §4.3, we named such a theory: GST, or General Set Theory. By the Incompleteness Theorem, GST is incomplete.

More generally, if Δ is a collection of sentences, then the collection of sentences that are derivable from Δ is a theory, namely the theory **axiomatized by** Δ. If Δ is a set-like collection of $\in$-sentences, then, by Lemma 30, so is the theory that it axiomatizes. Then the Incompleteness Theorem is just that no set-like theory of sets is complete.[3]

Our system of codes converts formulas into sets. But a singulary formula defines a *class,* and some of these are already sets. This observation allows us to give a proof of the Russell Paradox (Theorem 17 in §2.7) that is analogous to the foregoing proof of Theorem 133. Let $\varphi(x)$ be the formula $x \notin x$. Suppose if possible that the class $\{x\colon \varphi(x)\}$ is a set a, and let σ be the sentence $\varphi(a)$. Then

$$\neg\sigma \Rightarrow \sigma.$$

Indeed, $\neg\sigma$ is $\neg\varphi(a)$, that is, $a \in a$, which means $\varphi(a)$, that is, σ (since generally $b \in a$ means $\varphi(b)$). Therefore σ itself is true. That is, $\varphi(a)$ is true, which means $a \notin a$, so $\neg\varphi(a)$, that is, $\neg\sigma$. In short,

$$\neg\sigma \Leftrightarrow \sigma,$$

which is a contradiction. Hence there is no such set a.

The Incompleteness Theorem, as it is stated above, is of the first kind. There is no class $\boldsymbol{C}$ such that, for all set-like collections Δ of sentences, the set of codes of elements of Δ belongs to $\boldsymbol{C}$ if and only if each sentence

[3] If Δ is a collection of sentences whose codes in *Gödel's* sense compose a recursive set, then the set of codes of the sentences derivable from Δ is just what is called *recursively enumerable;* it might not be recursive. By Gödel's Incompleteness Theorem, no recursively axiomatizable theory of natural numbers in the signature $\{+, \cdot\}$ is complete. However, there *are* complete, recursively axiomatizable theories. Indeed, Tarski's student Presburger showed that the complete theory of ω in the signature $\{+\}$ is recursively axiomatizable. Tarski himself showed that the theories of $\mathbb{R}$ and $\mathbb{C}$ in $\{+, \cdot\}$ are recursively axiomatizable.

in Δ is true. This is a consequence of the following, which is attributed to Tarski.[4]

thm:U-of-T

Theorem 134 (Undefinability of Truth). *The collection of all true sentences is not set-like.*

We can understand this theorem as a corollary of the Incompleteness Theorem. However, there is a direct proof, more nearly analogous to that of the Russell Paradox. Suppose the collection of true sentences is set-like. Then there is a formula φ defining the class of codes of singulary formulas ψ such that $\neg\psi(\ulcorner\psi\urcorner)$ is true. Let σ be the sentence $\varphi(\ulcorner\varphi\urcorner)$. Then

$$\neg\sigma \Leftrightarrow \sigma.$$

This is a contradiction. So the collection of true sentences is not set-like.

Note then that the collection of *codes* of all true sentences is not a class. Thus there is a collection of sets that is not a class.

7.3. Models

:structures

We can formalize the notion of an *interpretation* of atomic sentences (see §2.7) by means of the following:

fn:structure

Definition 45. A **structure** (more precisely, an **$\in$-structure**) is a class $\boldsymbol{M}$ together with a binary relation $\boldsymbol{R}$. We may denote the structure by

$$(\boldsymbol{M},\boldsymbol{R}),$$

though we may omit mention of $\boldsymbol{R}$ if it is just $\in$. Two structures $(\boldsymbol{M},\boldsymbol{R})$ and $(\boldsymbol{M},\boldsymbol{S})$ can be considered as the same if

$$\boldsymbol{R}\cap(\boldsymbol{M}\times\boldsymbol{M})=\boldsymbol{S}\cap(\boldsymbol{M}\times\boldsymbol{M}).$$

Hence an $\in$-structure can be considered as a set if $\boldsymbol{M}$ is a set. In any case, we may abbreviate $(\boldsymbol{M},\boldsymbol{R})$ by

$$\mathfrak{M}$$

[4] The theorem appears as Theorem I of §5 of Tarski's article 'The Concept of Truth in Formalized Languages' [37, p. 247], which is based mostly on work of 1929; but according to Tarski's footnote, Theorem I was not one of those results, and Tarski's proof is based on Gödel's work.

(see Appendix B). A sentence σ without constants is **true in $\mathfrak{M}$** if σ is true under the assumptions:

1. Every set belongs to $\boldsymbol{M}$.
2. Membership is the relation $\boldsymbol{R}$.

In other words, in $\mathfrak{M}$, variables range over $\boldsymbol{M}$, and membership is **interpreted as $\boldsymbol{R}$**. If σ is true in $\mathfrak{M}$, we may express this by writing

$$\models_{\mathfrak{M}} \sigma.$$

Note then that being *true,* simply, means being true in the structure $\mathbf{V}$, that is, $(\mathbf{V}, \in)$.

If we allow formulas to have constants, then we must consider $(\in, \omega)$-structures $(\boldsymbol{M}, \boldsymbol{R}, f)$, where f is a function from ω into $\boldsymbol{M}$. Then σ is true in such a structure if it is true when the above conditions are met, along with:

3. Each constant c_k denotes the set $f(k)$.

Here f is an **interpretation** of the constants. If each sentence in a collection Γ is true in $\mathfrak{M}$, then $\mathfrak{M}$ is a **model** of Γ, and we may write

$$\models_{\mathfrak{M}} \Gamma.$$

If also $\boldsymbol{M}$ is a countable set, then $\mathfrak{M}$ is a **countable model** of Γ.

A structure itself can now be called an **interpretation;** this is the promised refinement of the definition given in §2.7. We could think of a collection of sets that is not a class, such as the collection of codes of true sentences, as an interpretation too; but again, by the Completeness Theorem in the next section, this will not be necessary.

To be more precise about what truth in a structure means, we make the following:

Definition 46. Each formula φ has a **relativization** to a structure $(\boldsymbol{M}, \boldsymbol{R}, f)$ or $\mathfrak{M}$. This relativization is denoted by

$$\varphi_{\mathfrak{M}}.$$

The definition is recursive:

1. $\varphi_{\mathfrak{M}}$ is φ if this is atomic.
2. $(\neg\varphi)_{\mathfrak{M}}$ is $\neg(\varphi_{\mathfrak{M}})$.
3. $(\varphi \Rightarrow \psi)_{\mathfrak{M}}$ is $\varphi_{\mathfrak{M}} \Rightarrow \psi_{\mathfrak{M}}$.

4. $(\exists x\, \varphi)_{\mathfrak{M}}$ is
$$\exists x\,(x \in \boldsymbol{M} \mathbin{\&} \varphi_{\mathfrak{M}}).$$
Note then that $(\forall x\, \varphi)_{\mathfrak{M}}$ can be understood as
$$\forall x\,(x \in \boldsymbol{M} \Rightarrow \varphi_{\mathfrak{M}}).$$
A sentence σ is true in $\mathfrak{M}$ if and only if $\sigma_{\mathfrak{M}}$ is true in $(\mathbf{V}, \in, f)$.

capitals

It is important to distinguish between structures that are sets and structures that are not. By the undefinability of truth (Theorem 134), the following is false for arbitrary structures (since it is false for $\mathbf{V}$). It is true for structures that are *uncountable* sets; but there is no need to consider these. We now begin to allow capital plainface letters like M and R to denote sets.

-model-truth

Lemma 32. *Suppose $\mathfrak{M}$ is a structure (M, R, f), where M is a countable set. The collection of sentences that are true in $\mathfrak{M}$ is set-like.*

Proof. First note that, by re-indexing constants and enlarging f as necessary, we may assume that $M = f[\omega]$, that is, every element of M is the interpretation in $\mathfrak{M}$ of a constant. Now let A be the set of all ordered pairs (t, f), where both t and f consist of codes of sentences, and the following conditions hold. The conditions ensure that, if $\ulcorner\tau\urcorner \in t$, then τ is true in $\mathfrak{M}$, but if $\ulcorner\tau\urcorner \in f$, then τ is false in $\mathfrak{M}$:

1. a) If $\ulcorner \mathsf{c}_i \in \mathsf{c}_j \urcorner \in t$, then $f(i)\ R\ f(j)$.
   b) If $\ulcorner \mathsf{c}_i \in \mathsf{c}_j \urcorner \in f$, then $\neg(f(i)\ R\ f(j))$.
2. a) If $\ulcorner\neg\tau\urcorner \in t$, then $\ulcorner\tau\urcorner \in f$.
   b) If $\ulcorner\neg\tau\urcorner \in f$, then $\ulcorner\tau\urcorner \in t$.
3. a) If $\ulcorner(\tau \Rightarrow \rho)\urcorner \in t$, then $\ulcorner\rho\urcorner \in t$ or $\ulcorner\tau\urcorner \in f$.
   b) If $\ulcorner(\tau \Rightarrow \rho)\urcorner \in f$, then $\ulcorner\rho\urcorner \in f$ and $\ulcorner\tau\urcorner \in t$.
4. a) If $\ulcorner\exists x\, \varphi\urcorner \in t$, then for some i in ω, the code $\ulcorner\varphi(\mathsf{c}_i)\urcorner$ is in t.
   b) If $\ulcorner\exists x\, \varphi\urcorner \in f$, then for all i in ω, the code $\ulcorner\varphi(\mathsf{c}_i)\urcorner$ is in f.

The collection of codes of sentences that are true in $\mathfrak{M}$ is a set, namely the union of the set of all sets t such that, for some set f, the ordered pair (t, f) belongs to A. $\square$

The notion of consistency was introduced in §2.7:

Definition 47. We use the symbol
$$\bot$$

to denote a contradiction. (Note that every sentence, and in particular every contradiction, is derivable from a constradiction.) A collection Γ of sentences is **inconsistent** if

$$\Gamma \vdash \bot;$$

otherwise Γ is **consistent.** If Γ has a single member σ, we say σ is consistent or inconsistent when Γ is.

cons

Lemma 33. *For all set-like collections Γ of sentences, there is a sentence σ such that*

$$\Gamma \textit{ is inconsistent if and only if } \sigma \textit{ is true.}$$

Proof. Let φ define the class of sentences that are derivable from Γ. Then we can let σ be $\neg\varphi(\ulcorner\bot\urcorner)$. $\square$

The following is of the first kind.

Theorem 135. *Every collection of sentences with a model is consistent.*

Proof. Suppose $\models_{\mathfrak{M}} \Delta$ and $\Delta \vdash \sigma$. Then $\Delta \models \sigma$ by Theorem 6 of §2.7, so σ is true in $\mathfrak{M}$. Therefore σ cannot be a contradiction. $\square$

However, the following special case is of the second kind; we shall establish its converse in the next section.

Corollary. *Every set-like collection of sentences with a countable model is consistent.*

7.4. Completeness

lete

We show that the syntactical notion of derivation is the same *in extension* (see §2.1) as the semantic notion of logical entailment: the syntactic turnstile $\vdash$ is interchangeable with the semantic turnstile $\models$. Recall the convention (established in Definition 7 in § 2.7) whereby $\sigma \Rightarrow \tau$ means $(\sigma \Rightarrow \tau)$, and then $\rho \Rightarrow \sigma \Rightarrow \tau$ means $\rho \Rightarrow (\sigma \Rightarrow \tau)$.

From an inconsistent collection of sentences, *every* sentence is derivable. Indeed, suppose by

$$\bot$$

we denote some contradiction. Then $\bot \Rightarrow \sigma$ is a tautology for all sentences σ, so if $\Gamma \vdash \bot$, then $\Gamma \vdash \sigma$.

lem:nvc **Lemma 34.** *If $\neg\sigma$ is not a logical theorem, then σ is consistent.*

Proof. Suppose σ is not consistent. Then $\{\sigma\} \vdash \bot$, so $\vdash \sigma \Rightarrow \bot$ by the Deduction Lemma. But $(\sigma \Rightarrow \bot) \Rightarrow \neg\sigma$ is a tautology, so $\vdash \neg\sigma$. □

Lemma 35. *If every* finite *subset of Γ is consistent, then Γ must be consistent.*

Proof. If Γ is inconsistent, there is a formal proof of this, and this proof uses only finitely many formulas from Γ. □

In a word, the lemma is that consistency is **finitary.**

lem:or-not **Lemma 36.** *If Γ is a consistent, then $\Gamma \cup \{\sigma\}$ or $\Gamma \cup \{\neg\sigma\}$ is consistent.*

Proof. Suppose both $\Gamma \cup \{\sigma\}$ and $\Gamma \cup \{\neg\sigma\}$ are inconsistent. Since consistency is finitary, there is a finite subset $\{\tau_0, \dots, \tau_{n-1}\}$ of Γ such that both $\{\tau_0, \dots, \tau_{n-1}, \sigma\}$ and $\{\tau_0, \dots, \tau_{n-1}, \neg\sigma\}$ are inconsistent. By Deduction, we have

$$\vdash \sigma \Rightarrow \tau_0 \Rightarrow \dots \Rightarrow \tau_{n-1} \Rightarrow \bot,$$
$$\vdash \neg\sigma \Rightarrow \tau_0 \Rightarrow \dots \Rightarrow \tau_{n-1} \Rightarrow \bot.$$

By the tautology $(\sigma \Rightarrow \rho) \Rightarrow (\neg\sigma \Rightarrow \rho) \Rightarrow \rho$, we conclude

$$\vdash \tau_0 \Rightarrow \dots \Rightarrow \tau_{n-1} \Rightarrow \bot,$$

so $\{\tau_0, \dots, \tau_{n-1}\} \vdash \bot$, and therefore Γ is inconsistent. □

lem:const **Lemma 37.** *If $\Gamma \cup \{\exists x\, \varphi(x)\}$ is consistent, and a does not occur in any of its formulas, then $\Gamma \cup \{\exists x\, \varphi(x), \varphi(a)\}$ is consistent.*

Proof. Suppose $\Gamma \cup \{\exists x\, \varphi(x), \varphi(a)\}$ is inconsistent. Then, as in the proof of the last lemma, for some subset $\{\tau_0, \dots, \tau_{n-1}\}$ of $\Gamma \cup \{\exists x\, \varphi(x)\}$ we have

$$\vdash \varphi(a) \Rightarrow \tau_0 \dots \Rightarrow \tau_{n-1} \Rightarrow \bot,$$
$$\vdash \exists x\, \varphi(x) \Rightarrow \tau_0 \dots \Rightarrow \tau_{n-1} \Rightarrow \bot,$$

and therefore $\Gamma \cup \{\exists x\, \varphi(x)\}$ is inconsistent. □

lem:not **Lemma 38.** *$\{\sigma, \neg\sigma\}$ is inconsistent.*

:imp **Lemma 39.** *Assume Γ is consistent. Then $\Gamma \cup \{\sigma \Rightarrow \tau\}$ is consistent if and only if one of $\Gamma \cup \{\neg\sigma\}$ or $\Gamma \cup \{\tau\}$ is consistent.*

A form of the following theorem was proved by Gödel [18]; but our proof is due to Henkin [21]. The proposition is of the first kind.

lete **Theorem 136** (Completeness). *Every logically true sentence is a logical theorem:*

$$\textit{If} \models \sigma, \textit{ then} \vdash \sigma.$$

Proof. Suppose σ is not a logical theorem. Then $\neg\sigma$ is consistent, by Lemma 34. We shall find a model of $\{\neg\sigma\}$. This model will be $(\omega, r, \mathrm{id}_\omega)$ for some r. There is a surjective function $n \mapsto \tau_n$ from ω onto the collection of sentences. We recursively define a function $n \mapsto \Gamma_n$ on ω, where each Γ_n is a finite collection of sentences:

1. $\Gamma_0 = \{\neg\sigma\}$.
2. Γ_{n+1} includes Γ_n. Also, if $\Gamma_n \cup \{\tau_n\}$ is consistent, then Γ_{n+1} contains τ_n; otherwise, Γ_{n+1} contains $\neg\tau_n$. If Γ_{n+1} contains τ_n, and this sentence is $\exists x\, \varphi(x)$ for some variable x and formula φ, then Γ_n contains $\varphi(\mathsf{c}_k)$, where k is the *least* number ℓ such that the constant c_ℓ does not occur in τ_n or any formula in Γ_n.

By Lemmas 36 and 37, each Γ_n is consistent. Let

$$\Delta = \bigcup_{n\in\omega} \Gamma_n.$$

Since $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$, every finite subcollection of Δ is included in some Γ_n, so by finitariness Δ is consistent. Also, for each τ_k, either it or its negation is in Δ, but not both, by Lemma 38. In short, $\Delta \cup \{\tau_k\}$ is consistent if and only if τ_k is in Δ.

Now let r be the set (and it is a set) of those pairs (j, k) such that the atomic sentence $\mathsf{c}_j \in \mathsf{c}_k$ is in Δ. By induction, every sentence τ_k is true in the structure $(\omega, r, \mathrm{id}_\omega)$ if and only if it belongs to Δ:

1. The claim is true by definition when τ_k is atomic.
2. If the claim is true for some τ_k, then it is true for $\neg\tau_k$, since τ_k belongs to Δ if and only if $\neg\tau_k$ does not.
3. If the claim is true when τ_k is ρ and when τ_k is π, then it is true when τ_k is $\rho \Rightarrow \pi$, since $\rho \Rightarrow \pi$ belongs to Δ if and only if either ρ does not or π does, by Lemma 39.

4. If, for some singulary formula $\varphi(x)$, the claim is true whenever τ_k is $\varphi(\mathsf{c}_j)$ for some j, then the claim is true when τ_k is $\exists x\, \varphi(x)$, by Lemma 37.

In particular, since $\neg\sigma$ is in Δ, it is true in $(\omega, r, \mathrm{id}_\omega)$. Thus σ is false in some interpretation of atomic sentences, so σ is not logically true. □

The following proposition is however of the second kind.

Porism. *Every consistent set-like collection of sentences has a countable model.*

Proof. In the proof of the Completeness Theorem, we show that, if $\{\neg\sigma\}$ is consistent, then it has a countable model. We can adapt the proof to show that, if Γ is a consistent set-like collection of sentences, then it has a model. We let $\Gamma_0 = \Gamma$, then continue as before. To obtain Γ_{n+1} from Γ_n though, we should assume (as we may) that every constant used in Γ is c_{2k} for some k. This ensures that there is always some constant that is not used in Γ_n. □

This gives us the converse of Theorem 6.

Corollary. *Every logical consequence of a set-like collection of sentences is derivable from that collection:*

$$\textit{If } \Gamma \models \sigma, \textit{ then } \Gamma \vdash \sigma.$$

Proof. Suppose σ is not derivable from Γ. Then $\Gamma \cup \{\neg\sigma\}$ is consistent, so it has a model. Then Γ cannot entail σ. □

7.5. Set theories

We defined *well-* and *ill-founded* sets in Definition 11 in §4.3. In that section, the possible existence of ill-founded sets caused a difficulty in the formulation of the class of natural numbers. We might have made things easier for ourselves by assuming the **Axiom of Foundation** (Axiom 10 in §8.3 below), namely that all sets are well-founded; equivalently, every nonempty set has an element that is disjoint from it.

However, it is good that we have not yet assumed the Foundation Axiom. The axioms that we have now are:

1) Equality,	4) Separation,	7) Infinity,
2) Null Set,	5) Replacement,	8) Power Set,
3) Adjunction,	6) Union,	9) Choice.

(The list is in Appendix C, with page references.) If these axioms are indeed consistent, then they will still be consistent when the Foundation Axiom is added: this is Theorem 167 in §8.2 below. The theory GST is axiomatized by the first four axioms above.

As axioms of set theory, Zermelo [40] proposed

1) Extension,	4) Power Set,	7) Infinity.
2) Pairing,	5) Union,	
3) Separation,	6) Choice,	

However, Zermelo did not have a formal logic. If he had had a formal logic, it might have had the equals sign $=$ as one of its official symbols, as described at the end of §3.3. Extension is the axiom mentioned there. The existence of sets is a logical truth, so our Null Set Axiom is a consequence of Zermelo's (and now our) Separation Axiom. Zermelo's **Pairing Axiom** corresponds to our Theorem 22; this and Union entail our Adjunction Axiom.

Fraenkel (see [16, p. 50, n. 3]), and independently Skolem [35], proposed the Replacement Axiom (which makes Separation redundant). Skolem and more definitely von Neumann [38] proposed Foundation. The collection of all of these axioms, besides Choice, axiomatizes the theory called

$$\mathrm{ZF}$$

for Zermelo and Fraenkel. When Choice is added, the resulting theory is called

$$\mathrm{ZFC}\,.$$

If Foundation is removed as an axiom of either of these theories, the resulting weaker theory is respectively

$$\mathrm{ZF}^-$$

or

$$\mathrm{ZFC}^-\,.$$

Our own axioms so far entail exactly the same sentences as the axioms of ZFC^-. In particular, the Completeness Theorem and its porism and corollary are in ZFC^-.

Lemma 40. *The collections* ZF, ZFC, ZF^-, *and* ZFC^- *are set-like.*

By applying the porism of Completeness to ZFC^- itself, we obtain the following proposition of the second kind:

Theorem 137 (Skolem Paradox)**.** *If* ZFC^- *is consistent, then it has a countable model.*

The paradox is that, according to ZFC^-, uncountable sets exist; but then some countable model says this is true. The paradox is resolved by observing that not every sentence that is true in a given model is true simply.

Since ZFC^- is consistent, we have the following proposition, but it is of the *third* kind:

Theorem 138. ZFC^- *has a countable model.*

This last theorem is *not* a logical consequence of ZFC^-, by the Second Incompleteness Theorem, Theorem 141 below. However, let the **Model Axiom** be the statement that ZFC^- has a countable model. Then the following is a proposition of the second kind:

Theorem 139. *If* ZFC^- *is consistent, then the Model Axiom is consistent with it.*

In short, there is no logical barrier to assuming the Model Axiom. But the model guaranteed by it might be strange. Indeed, suppose $\mathfrak{M}$ is a countable model (M, R) of ZFC^-. If a and b are in M, we need *not* have $a \in b \Leftrightarrow a\ R\ b$; that is, we need not have $a \in b \Leftrightarrow (a \in b)_{\mathfrak{M}}$. We *do* have

$$a = b \Rightarrow (a = b)_{\mathfrak{M}},$$

but perhaps not the converse. The sets a and b are equal *in* $\mathfrak{M}$ if and only if $c\ R\ a \Leftrightarrow c\ R\ b$ for all c in M; this is implied by, but does not imply, $a = b$.

7.6. Compactness

We can obtain the following as a corollary of the Completeness Theorem and its porism:

Theorem 140 (Compactness). *If every finite subcollection of a set-like collection* Γ *of sentences has a model, then* Γ *has a model, in fact a countable model.*

Proof. If every finite subcollection of Γ has a model, then every finite subset is consistent, and therefore the whole collection is consistent, so it has a model. $\square$

We can now prove what was stated in §4.3: that we cannot prove $\mathbb{N} = \bigcup \omega$. Note that we *have* proved $\bigcup \omega = \omega$. For each n in ω, there is a singulary formula φ_n that defines the class $\{n\}$, and there is a singulary formula ψ that defines ω. Let Δ be the collection

$$\mathrm{ZFC}^- \cup \{\neg\varphi_n(\mathsf{c}_0) \colon n \in \omega\} \cup \{\psi(\mathsf{c}_0)\}.$$

Then Δ is set-like. Also, every finite subcollection of Δ is a subcollection of $\mathrm{ZFC}^- \cup \{\neg\varphi_k(\mathsf{c}_0) \colon k < n\} \cup \{\psi(\mathsf{c}_0)\}$ for some n in ω. This last collection has the model $(\mathbf{V}, \in, f)$, provided $f(0) = n$. by the Compactness Theorem, Δ itself has a model $\mathfrak{M}$. In this, c_0 is interpreted as an *infinite* element of ω: it is greater than 0, 1, 2, and the rest of 'our' natural numbers. This is not a contradiction; it is a proof that the set of 'our' natural numbers is not an element of $\boldsymbol{M}$.

A similar use of the Compactness Theorem gives us infinite and also *infinitesimal* elements of $\mathbb{R}$. Here arises the field of *non-standard analysis,* created by Abraham Robinson [32] as a way to justify Leibniz's approach to calculus and more generally to simplify various proofs in mathematics and to encourage the discovery of new theorems.

7.7. Second incompleteness

omp2

Many propositions in this chapter have required no axioms besides those of GST. In particular, we have the following refinements (of the second kind) of Lemmas 33 and 31.

Lemma 41. *For all set-like collections Δ of sentences, there is a sentence σ such that the following are equivalent:*

$$\Delta \textit{ is inconsistent}, \qquad \mathrm{GST} \vdash \sigma, \qquad \sigma \textit{ is true.}$$

The sentence σ of the lemma can be denoted by

$$\mathrm{Incons}_{\Delta}.$$

If $\mathrm{GST} \vdash \neg\mathrm{Incons}_{\Delta}$, then, since GST is consistent, the sentence Incons_{Δ} must not be derivable from GST, so Δ must be consistent; but this is a proposition of the third kind.

Lemma 42. *For every set-like collection Δ of sentences, there is a binary relation $\boldsymbol{R}$ such that, for all sets a and singulary formulas ψ, the following are equivalent:*

$$\Delta \vdash \psi(a), \qquad \mathrm{GST} \vdash a \mathrel{\boldsymbol{R}} \ulcorner\psi\urcorner, \qquad a \mathrel{\boldsymbol{R}} \ulcorner\psi\urcorner \textit{ is true.}$$

thm:2incomp

Theorem 141 (Second Incompleteness). *Suppose Δ is a set-like collection of sentences from which every axiom of* GST *is derivable. Then*

$$\Delta \vdash \neg\mathrm{Incons}_{\Delta} \textit{ if and only if } \Delta \textit{ is inconsistent.}$$

Proof. Assuming $\boldsymbol{R}$ is as in the last lemma, let $\varphi(x)$ be the formula $\neg(x \mathrel{\boldsymbol{R}} x)$, and let σ be the sentence $\varphi(\ulcorner\varphi\urcorner)$. Then

$$\neg\mathrm{Incons}_{\Delta} \Rightarrow \sigma.$$

Indeed, suppose σ is false. Then $\ulcorner\varphi\urcorner \mathrel{\boldsymbol{R}} \ulcorner\varphi\urcorner$ is true, so by the last lemma, $\Delta \vdash \varphi(\ulcorner\varphi\urcorner)$. We can rewrite this as $\Delta \vdash \sigma$. But we also have by the lemma $\mathrm{GST} \vdash \ulcorner\varphi\urcorner \mathrel{\boldsymbol{R}} \ulcorner\varphi\urcorner$, so $\mathrm{GST} \vdash \neg\sigma$, hence $\Delta \vdash \neg\sigma$. Thus Δ is inconsistent, so $\neg\mathrm{Incons}_{\Delta}$ is false.

By the Completeness Theorem now, if $\Delta \vdash \neg\mathrm{Incons}_{\Delta}$, then $\Delta \vdash \sigma$, but then also $\Delta \vdash \neg\sigma$ as before, so Δ is inconsistent. □

Thus, among theories that entail GST, only inconsistent theories prove their own consistency.

In Morse–Kelley set theory (see Appendix D), the proof of Lemma 32 can be adapted to show that the collection of true $\in$-sentences is set-like. Also, this collection includes ZFC^{-}, but not $\perp$. Thus it is a theorem of Morse–Kelley set theory that ZFC^{-} is consistent.

8. Models

dels

8.1. The well-founded universe

t:wf

Using our current axioms, essentially ZFC$^-$, we shall find a class **WF**, the *well-founded universe,* that is a model of ZFC. Then we shall have proved that, if ZFC$^-$ is consistent, then so is ZFC.

Note that we cannot just let **WF** be the class of well-founded sets. For, consider again some examples in §4.3:

1. Suppose there are sets a and b such that $a = \{b\}$ and $b = \{a\}$. If $a \neq b$, then a and b are well-founded, but $\{a, b\}$ is not. Thus, Zermelo's Pairing Axiom is false in the class of well-founded sets. Note that a and b are not transitive. However, $\{a, b\} = \{b\} \cup b = a \cup \bigcup a$, and this is transitive, but ill-founded.
2. Similarly, if a, b, and c are distinct sets such that $a \in b$, $b \in c$, and $c \in a$, then:
   a) a is well-founded, but not transitive;
   b) $\{a, b\}$, which is $a \cup \bigcup a$, is well-founded, but not transitive;
   c) $\{a, b, c\}$, which is $a \cup \bigcup a \cup \bigcup \bigcup a$, is transitive, but ill-founded.

Again, such sets were a difficulty that had to be met in the formulation of the class of natural numbers. Now that we do have this class, we can make the following:

Definition 48. Given a set a, we define the function $x \mapsto \bigcup^x a$ on ω recursively by

$$\bigcup^0 a = a, \qquad \bigcup^{n+1} a = \bigcup \bigcup^n a.$$

Then we let

$$\operatorname{tc}(a) = \bigcup \Bigl\{ \bigcup^x a \colon x \in \omega \Bigr\}.$$

We call this set the **transitive closure** of a, because of the following.

m:tc **Theorem 142.** *For all sets a, the set* $\operatorname{tc}(a)$

1) *includes* a,
2) *is transitive,*
3) *is included in every class that includes* a *and is transitive.*

By definition, a class is well-founded if every nonempty sub*set* has an element that is disjoint from it. By using transitive closures, we can show that the same is true for arbitrary sub*classes* of well-founded classes:

Theorem 143. *Every non-empty well-founded class* $\boldsymbol{C}$ *has an element* a *such that*

$$\boldsymbol{C} \cap a = 0.$$

Proof. Suppose $\boldsymbol{C}$ is well-founded, and $b \in \boldsymbol{C}$, but $\boldsymbol{C} \cap b \neq 0$. Since $\boldsymbol{C}$ is well-founded, its nonempty subset $\boldsymbol{C} \cap b$ has an element a such that $\boldsymbol{C} \cap b \cap a = 0$. We would be done if we could show $\boldsymbol{C} \cap a = 0$. We could conclude this if b were transitive, so that $a \subseteq b$ and therefore $\boldsymbol{C} \cap a = \boldsymbol{C} \cap b \cap a$. So we start over, replacing b with $\operatorname{tc}(b)$:

Since $b \subseteq \operatorname{tc}(b)$, the subclass $\boldsymbol{C} \cap \operatorname{tc}(b)$ of $\boldsymbol{C}$ is nonempty, so it has an element a such that $\boldsymbol{C} \cap \operatorname{tc}(b) \cap a = 0$. But $a \subseteq \operatorname{tc}(b)$, so $\boldsymbol{C} \cap a = 0$, as desired. □

Above we described examples of a well-founded set a such that $a \cup \bigcup a$ or $a \cup \bigcup a \cup \bigcup \bigcup a$ was ill-founded. The latter set is $\operatorname{tc}(a)$ in each case. This suggests that the following will be useful:

Definition 49. We denote by

$$\mathbf{WF}$$

the class of all sets whose transitive closures are well-founded.[1]

thm:v=wf

Theorem 144. *Every set is well-founded if and only if every set belongs to* $\mathbf{WF}$*; so the Foundation Axiom can be expressed as* $\mathbf{V} = \mathbf{WF}$.

Proof. If every set is well-founded, then in particular every transitive closure of a set is well-founded, so every set belongs to $\mathbf{WF}$. Conversely, if $a \in \mathbf{WF}$, then $\operatorname{tc}(a)$ is well-founded; but $a \subseteq \operatorname{tc}(a)$, so a is well-founded. □

[1] It is the elements of this class that are called *well-founded* in [16].

We shall establish another characterization of **WF**, which will be more useful and suggestive. We start with the following.

rank **Definition 50.** The function $\mathbf{R}$ on $\mathbf{ON}$ is defined recursively by

$$\mathbf{R}(0) = 0, \qquad \mathbf{R}(\alpha + 1) = \mathscr{P}(\mathbf{R}(\alpha)), \qquad \mathbf{R}(\beta) = \bigcup \mathbf{R}[\beta],$$

where β is a limit. If $c \in \bigcup \mathbf{R}[\mathbf{ON}]$, then the least ordinal α such that $c \in \mathbf{R}(\alpha)$ must be a successor, $\beta + 1$. In this case, β is called the **rank** of c and is denoted by $\operatorname{rank}(c)$. That is,

$$\begin{aligned} \operatorname{rank}(c) &= \min\{x \colon x \in \mathbf{ON} \ \&\ c \in \mathbf{R}(x+1)\} \\ &= \min\{x \colon x \in \mathbf{ON} \ \&\ c \subseteq \mathbf{R}(x)\}. \end{aligned}$$

We may refer to $\bigcup \mathbf{R}[\mathbf{ON}]$ as the class of **ranked sets.**

We shall show in Theorem 150 that the class of ranked sets is precisely **WF**.

Theorem 145. *For all ordinals α,*

$$\operatorname{card}(\mathbf{R}(\omega + \alpha)) = \beth_\alpha.$$

rans **Lemma 43.**
1. *The empty set is transitive.*
2. *The power set of a transitive set is transitive.*
3. *The union of a set of transitive sets is transitive.*

rans **Theorem 146.** *Each set $\mathbf{R}(\alpha)$ is transitive, and so is the whole class of ranked sets.*

Corollary. *Every element of a ranked set is ranked and has a lower rank than that set.*

wf-< **Theorem 147.** *For all ordinals α and β,*

$$\alpha < \beta \Leftrightarrow \mathbf{R}(\alpha) \subset \mathbf{R}(\beta).$$

Proof. We show $\alpha < \beta \Rightarrow \mathbf{R}(\alpha) \subset \mathbf{R}(\beta)$ by induction on β. (The converse follows from the linearity of the ordering of **ON**.)

1. The claim holds vacuously when $\beta = 0$.

2. Suppose the claim holds when $\beta = \gamma$. If $\alpha < \gamma + 1$, then $\alpha \leqslant \gamma$, so $\mathbf{R}(\alpha) \subseteq \mathbf{R}(\gamma)$ and therefore $\mathbf{R}(\alpha) \in \mathscr{P}(\mathbf{R}(\gamma))$, which is $\mathbf{R}(\gamma+1)$. By the last theorem then, $\mathbf{R}(\alpha) \subseteq \mathbf{R}(\gamma+1)$. Since a set is never equal to its power set (by Cantor's Theorem, Theorem 114 in §6.1), $\mathbf{R}(\alpha) \subset \mathbf{R}(\gamma+1)$.

3. Suppose γ is a limit, and the claim is true when $\beta < \gamma$. If $\alpha < \gamma$, then $\alpha + 1 < \gamma$, so $\mathbf{R}(\alpha) \subset \mathbf{R}(\alpha+1) \subseteq \mathbf{R}(\gamma)$. □

A partial converse of Theorem 146 is the following.

m:sub-is-el

Theorem 148. *Every subset of an element of $\mathbf{R}(\alpha)$ is an element of $\mathbf{R}(\alpha)$. Every set of ranked sets is itself a ranked set.*

Proof. Suppose $b \in \mathbf{R}(\alpha)$. Then $b \subseteq \mathbf{R}(\beta)$ for some β in α. If $a \subseteq b$, then $a \in \mathbf{R}(\alpha+1)$, so $a \in \mathbf{R}(\beta)$.

Suppose c is a set of ranked sets. Let $\beta = \sup\{\mathrm{rank}(x) \colon x \in c\}$; then $c \subseteq \mathbf{R}(\beta+1)$, so $c \in \mathbf{R}(\beta+2)$. □

thm:wf-wf

Theorem 149. *Every ranked set is well-founded.*

Proof. Suppose a is ranked, and b is a nonempty subset of a. The elements of b are ranked. Let c be an element of b of minimal rank. Since any element of $b \cap c$ would have lower rank than c, this intersection must be empty. □

thm:wfron

Theorem 150. *The class of ranked sets is just* $\mathbf{WF}$.

Proof. Suppose a is ranked. Then a is *included* in the class of ranked sets, since this class is transitive. But then this class must then include $\mathrm{tc}(a)$, by Theorem 142. Then $\mathrm{tc}(a)$ itself is ranked by Theorem 148, so $\mathrm{tc}(a)$ is well-founded by Theorem 149.

Now suppose a is not ranked. Then a has unranked elements, so $\mathrm{tc}(a)$ has unranked elements. Let b be the set of unranked elements of $\mathrm{tc}(a)$, and let c be an arbitrary element of b. Then $c \subseteq \mathrm{tc}(a)$, and since c is unranked, it has unranked elements. But then these are also elements of b. Thus $b \cap c \neq 0$. Therefore $\mathrm{tc}(a)$ is not well-founded. □

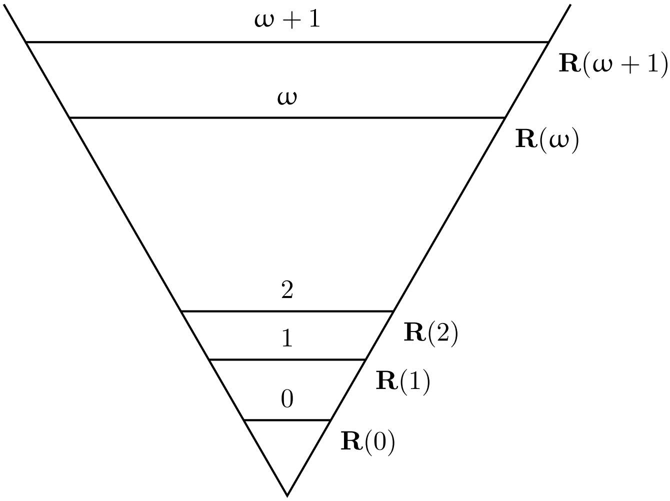
Since each ordinal is transitive and well-founded, we have $\mathbf{ON} \subseteq \mathbf{WF}$. Moreover:

Theorem 151. *For all ordinals α,*

$$\mathrm{rank}(\alpha) = \alpha.$$

Proof. By induction, $\alpha \subseteq \mathbf{R}(\alpha)$, so $\operatorname{rank}(\alpha) \leqslant \alpha$. If $\operatorname{rank}(\alpha) = \beta < \alpha$, then $\operatorname{rank}(\beta) < \operatorname{rank}(\alpha) = \beta$ by the Corollary to Theorem 146. Thus the class of α such that $\operatorname{rank}(\alpha) < \alpha$ has no least element; so it must be empty. □

One might picture **WF** as in Figure 8.1.[2]



g:wf

Figure 8.1. The well-founded universe

8.2. Absoluteness

sist

We shall establish that **WF** is a model of ZFC, along with similar results. How can we do this? We can establish one part of this result directly, along the lines of Theorem 149:

Theorem 152. *The Foundation Axiom is true in every subclass of* **WF**.

[2]Like much of the mathematics in this section, the picture is adapted from Kunen [26, Ch. 3, §4, p. 101].

Proof. Suppose $\boldsymbol{M} \subseteq \mathbf{WF}$, and $\boldsymbol{M}$ has elements a and b such that a is a nonempty subset of b in $\boldsymbol{M}$, that is,

$$\exists x\,(x \in \boldsymbol{M} \mathbin{\&} x \in a) \mathbin{\&} \forall x\,(x \in \boldsymbol{M} \Rightarrow x \in a \Rightarrow x \in b).$$

Let c be an element of $a \cap \boldsymbol{M}$ of minimal rank. Then $a \cap \boldsymbol{M} \cap c$ must be empty, since any element would have lesser rank than c. In particular,

$$\forall y\,(y \in \boldsymbol{M} \Rightarrow y \notin a \cap c).$$

So the sentence $\exists x\,(x \in a \mathbin{\&} a \cap x = 0)$ is true in $\boldsymbol{M}$. □

Consider next the Equality Axiom,

$$a = b \Rightarrow \forall x\,(a \in x \Leftrightarrow b \in x).$$

Here, by definition of equality,

$$a = b \Leftrightarrow \forall x\,(x \in a \Leftrightarrow x \in b).$$

To decide whether the Equality Axiom is true in a particular class $\boldsymbol{M}$, we should first check whether equality *in* $\boldsymbol{M}$ is the same as 'real' equality, or equality in $\mathbf{V}$. If $\boldsymbol{M}$ is not transitive, then it may have a nonempty element a, none of whose elements is in $\boldsymbol{M}$. Then $a \neq 0$, but

$$\models_{\boldsymbol{M}} a = 0$$

($a = 0$ in $\boldsymbol{M}$). In particular, equality may fail to be *absolute* for $\boldsymbol{M}$. The precise definition of this notion is as follows.

defn:def-in

Definition 51. If $\boldsymbol{M}$ is a class, then every formula whose constants are from $\boldsymbol{M}$ can be said to be **over** $\boldsymbol{M}$. Suppose φ is such a formula.

1. We first consider the case when φ is singulary. We define

$$\varphi^{\boldsymbol{M}} = \{x \colon x \in \boldsymbol{M} \mathbin{\&} \varphi_{\boldsymbol{M}}(x)\}.$$

That is, $\varphi^{\boldsymbol{M}}$ is the class of all a in $\boldsymbol{M}$ such that $\varphi(a)$ is true in $\boldsymbol{M}$. In particular, $\varphi^{\mathbf{V}}$ is just $\{x \colon \varphi(x)\}$, the class defined by φ. We may say then that $\varphi^{\boldsymbol{M}}$ is the class **defined in** $\boldsymbol{M}$ by φ. The formula φ is **absolute** for $\boldsymbol{M}$ if

$$\varphi^{\boldsymbol{M}} = \boldsymbol{M} \cap \varphi^{\mathbf{V}}.$$

2. Now suppose φ is not necessarily singulary. We may assume that each variable in our formulas is x_k for some k in ω. Then there is a finite subset p of ω comprising those k such that x_k is a free variable of φ. We may then refer to φ as a **p-ary formula,** and we may refer to p as the **arity** of φ.[3] An element of ${}^p\boldsymbol{M}$ can be called a **p-tuple;** such an element can be denoted by $\vec{a}$ or $x \mapsto a_x$ or $\{(x, a_x) \colon x \in p\}$. Then we can denote by

$$\varphi(\vec{a})$$

the result of replacing each free occurrence of x_k in φ with a_k, for each k in p. Correspondingly, we may denote φ itself by

$$\varphi(\vec{x}).$$

We denote by

$$\varphi^{\boldsymbol{M}}$$

the class of all $\vec{a}$ in ${}^p\boldsymbol{M}$ such that $\varphi(\vec{a})$ is true in $\boldsymbol{M}$, that is, $\varphi_{\boldsymbol{M}}(\vec{a})$ is true. This is the class **defined in $\boldsymbol{M}$** by φ. The formula φ is **absolute for $\boldsymbol{M}$** if

$$\varphi^{\boldsymbol{M}} = {}^p\boldsymbol{M} \cap \varphi^{\mathbf{V}}.$$

Every subset of ${}^p\boldsymbol{M}$ can be referred to as a **p-ary relation on $\boldsymbol{M}$.** If $\boldsymbol{A}$ is such a relation, and $\boldsymbol{K} \subseteq \boldsymbol{M}$, then $\boldsymbol{A}$ is **definable over $\boldsymbol{K}$** if there is a formula ψ over $\boldsymbol{K}$ such that $\boldsymbol{A} = \psi^{\boldsymbol{M}}$. Then $\boldsymbol{A}$ is **definable,** simply, if it is definable over $\boldsymbol{M}$.

3. Note the following special case. For a sentence σ with constants from $\boldsymbol{M}$, we have

$$\sigma^{\boldsymbol{M}} = \{0 \colon \sigma_{\boldsymbol{M}}\},$$

that is, $\sigma^{\boldsymbol{M}}$ is 1 if σ is true in $\boldsymbol{M}$, and otherwise $\sigma^{\boldsymbol{M}} = 0$. So σ is absolute for $\boldsymbol{M}$ if and only if σ is true in $\boldsymbol{M}$.

Theorem 153.

1. *The formula $x \in y$ and all other quantifier-free formulas are absolute for all classes.*

[3] We could restrict ourselves to n-ary formulas, where $n \in \omega$. That is, we could require the free variables of a formula to be indexed by an initial segment of ω. However, this restriction causes its own complications. I have decided here not to consider the variables as being ordered, but to consider any set of n variables to be as good as any other.

2. *If φ is absolute for $\boldsymbol{M}$, then so is $\neg\varphi$.*
3. *If φ and ψ are absolute for $\boldsymbol{M}$, then so is $(\varphi \Rightarrow \psi)$.*

Definition 52. If p and q are disjoint finite subsets of ω, we may denote an element of ${}^{p\cup q}\boldsymbol{M}$ by

$$(\vec{a}, \vec{b}),$$

where $\vec{a} \in {}^{p}\boldsymbol{M}$ and $\vec{b} \in {}^{q}\boldsymbol{M}$; correspondingly, we may denote a $p \cup q$-ary formula by

$$\varphi(\vec{x}, \vec{y}).$$

Then $\varphi(\vec{a}, \vec{y})$ and $\varphi(\vec{x}, \vec{b})$ are the obvious formulas.

Theorem 154. *If $\varphi(\vec{x}, \vec{y})$ is absolute for $\boldsymbol{M}$, and $\vec{a}$ is from $\boldsymbol{M}$, then $\varphi(\vec{a}, \vec{y})$ is absolute for $\boldsymbol{M}$.*

Definition 53. The Δ_0 formulas are defined by:
1. Atomic formulas are Δ_0.
2. If φ is Δ_0, then so is $\neg\varphi$.
3. If φ and ψ are Δ_0, then so is $(\varphi \Rightarrow \psi)$.
4. If φ is Δ_0, then so is $\exists x\,(x \in y \mathbin{\&} \varphi)$.

Theorem 155. *All Δ_0 formulas are absolute for all transitive classes.*

We may have to analyze the abbreviations that we use in formulas in order to see that the underlying formula is Δ_0:

Theorem 156. *If φ is Δ_0, then so is*

$$\forall x\,(x \in y \Rightarrow \varphi).$$

Proof. The given formula is $\neg\exists x\,(x \in y \mathbin{\&} \neg\varphi)$. □

thm:easy-abs **Theorem 157.** *The following formulas are Δ_0:*
1. *$x = y$.*
2. *The Equality Axiom.*
3. *$x = 0$.*
4. *$x = y \cup \{z\}$.*
5. *$x = \bigcup y$.*

Proof. The formula $x = y$ is $\forall z\,(z \in x \Leftrightarrow z \in y)$, which can be understood as

$$\forall z\,(z \in x \Rightarrow z \in y)\ \&\ \forall z\,(z \in y \Rightarrow z \in x).$$

Similarly for the rest. □

Theorem 158.

1. *The Null Set Axiom is true in every class that contains* 0.
2. *The Adjunction Axiom is true in every transitive class that is closed under the operation* $(x, y) \mapsto x \cup \{y\}$.
3. *The Union Axiom is true in every transitive class that is closed under the operation* $x \mapsto \bigcup x$.

Theorem 159. *The Null Set, Adjunction, and Union Axioms are true in every class* $\mathbf{R}(\alpha)$ *such that* α *is a limit, and in* $\mathbf{WF}$.

Proof. Since $0 \subseteq \mathbf{R}(0)$, we have $0 \in \mathbf{R}(1)$, so 0 is in $\mathbf{WF}$ and in all $\mathbf{R}(\alpha)$ such that $\alpha > 0$.

If a and b are in $\mathbf{R}(\alpha)$, then, by transitivity of this class, the sets $a \cup \{b\}$ and $\bigcup a$ are subsets of $\mathbf{R}(\alpha)$, so they are elements of $\mathbf{R}(\alpha + 1)$. □

Theorem 160. *The Power Set Axiom is true in every transitive class* $\boldsymbol{M}$ *that contains* $\boldsymbol{M} \cap \mathscr{P}(a)$ *for every element* a *of* $\boldsymbol{M}$.

Proof. The formula $x \in \mathscr{P}(y)$ is Δ_0. Therefore the formula $x = \mathscr{P}(a)$, relativized to $\boldsymbol{M}$, is

$$\forall y\,(y \in x \Rightarrow y \in \mathscr{P}(a))\ \&\ \forall y\,(y \in \boldsymbol{M} \cap \mathscr{P}(a) \Rightarrow y \in x).$$

Hence the sentence $b = \mathscr{P}(a)$ is true in $\boldsymbol{M}$ if $b = \boldsymbol{M} \cap \mathscr{P}(a)$. □

Theorem 161. *The Power Set Axiom is true in* $\mathbf{WF}$ *and in* $\mathbf{R}(\alpha)$ *when* α *is a limit.*

Proof. Each of these classes both contains and includes the power set of each of its elements. □

Theorem 162. *The Separation Axiom is true in every set* $\mathbf{R}(\alpha)$ *and in the class* $\mathbf{WF}$.

Proof. The Separation Axiom consists of a sentence $\exists x\, x = a \cap \varphi^{\mathbf{V}}$ for every set a and every singulary formula φ. This sentence can be understood as $\exists x\, \psi$, where ψ is

$$\forall y\, (y \in x \Leftrightarrow y \in a \mathbin{\&} \varphi(y)).$$

This is Δ_0, if φ is. In any case, the relativization of ψ to a transitive class $\boldsymbol{M}$ that contains a can be understood as

$$\forall y\, (y \in x \Leftrightarrow y \in a \mathbin{\&} \varphi_{\boldsymbol{M}}(y)),$$

that is, $x = a \cap \varphi^{\boldsymbol{M}}$. But $a \cap \varphi^{\boldsymbol{M}}$ is just a subset of a. Therefore, if $\boldsymbol{M}$ contains all subsets of all of its elements, then $\exists x\, \psi$ is true in $\boldsymbol{M}$. We are done by Theorem 148. □

Theorem 163. *The following formulas are absolute for all transitive subclasses of* $\mathbf{WF}$*:*

1. *x is an ordinal: $x \in \mathbf{ON}$.*
2. *x is a successor ordinal.*
3. *x is a limit ordinal.*
4. *$x = \omega$.*

In particular, the Axiom of Infinity is true in all transitive subclasses of $\mathbf{WF}$ *that contain* ω*.*

Proof. In a subclass $\boldsymbol{M}$ of $\mathbf{WF}$, since the Foundation Axiom is true, $\mathbf{ON}$ is the class of transitive sets that are linearly ordered by membership. Then all of the needed formulas are Δ_0. □

Lemma 44. *The following are absolute for all transitive classes:*

1. *x is a singleton: $\exists y\, x = \{y\}$.*
2. *x is a pair: $\exists y\, \exists z\, x = \{y, z\}$.*
3. *x is an ordered pair: $\exists y\, \exists z\, x = (y, z)$.*
4. *x is a binary relation: $\forall y\, (y \in x \Rightarrow \exists z\, \exists w\, y = (z, w))$.*
5. *x is a function: x is a relation, and*

$$\forall y\, \forall z\, \forall w\, ((y, z) \in x \mathbin{\&} (y, w) \in x \Rightarrow z = w).$$

Theorem 164. *The Axiom of Choice is true in* $\mathbf{WF}$ *and in* $\mathbf{R}(\alpha)$ *for each limit ordinal* α*.*

Proof. Suppose α is a limit ordinal, and $a \in \mathbf{R}(\alpha)$. Then $a \in \mathbf{R}(\beta)$ for some β in α. Then $\mathscr{P}(a) \subseteq \mathbf{R}(\beta)$, so $\mathscr{P}(a) \cup a \subseteq \mathbf{R}(\beta)$, and both $\mathscr{P}(a)$ and $\mathscr{P}(a) \cup a$ are elements of $\mathbf{R}(\beta + 1)$.

Let f be a choice-function for a. Then $f \subseteq \mathscr{P}(a) \times a$. Each element (b, c) of $\mathscr{P}(a) \times a$ is $\{\{b\}, \{b, c\}\}$, where b and c are in $\mathscr{P}(a) \cup a$ and hence are in $\mathbf{R}(\beta)$; so $(b, c) \in \mathbf{R}(\beta+2)$. Thus $f \in \mathbf{R}(\beta+3)$, so $f \in \mathbf{R}(\alpha)$. By the last lemma and transitivity of $\mathbf{R}(\alpha)$, the sentence 'f is a choice-function for a' is true in $\mathbf{R}(\alpha)$. □

(oo) **Theorem 165.** *In* $\mathbf{R}(\omega)$, *the axioms of* ZFC *besides Infinity are true, but Infinity is false.*

Proof. Every element of $\mathbf{R}(\omega)$ is finite; in particular, $\omega \notin \mathbf{R}(\omega)$. Therefore, because of all of the foregoing theorems, we need only show that Replacement is true in $\mathbf{R}(\omega)$. Every subset of $\mathbf{R}(\omega)$ is an element of some $\mathbf{R}(n)$, where $n \in \omega$. Therefore, if a formula defines in $\mathbf{R}(\omega)$ a function, then the image of a finite set under this function is again an element of $\mathbf{R}(\omega)$. □

oo2) **Theorem 166.** *In* $\mathbf{R}(\omega \cdot 2)$, *the axioms of* ZFC, *besides Replacement, are true, but Replacement is false.*

Proof. Since GST is true in $\mathbf{R}(\omega \cdot 2)$, we can define in this set, by finite induction, the function $x \mapsto \omega + x$. But $\mathbf{R}(\omega \cdot 2)$ contains ω. The image of ω under $x \mapsto \omega + x$ is $\{\omega + x \colon x \in \omega\}$, whose union is $\omega \cdot 2$, which is not in $\mathbf{R}(\omega \cdot 2)$. Therefore the image itself is not in $\mathbf{R}(\omega \cdot 2)$. □

m:wf **Theorem 167.** *In* **WF**, *the axioms of* ZFC *are true.*

Proof. The Replacement Axiom is true in **WF** by Theorem 148. □

8.3. Collections of equivalence classes

:ceq

Let us finally complete our list of axioms with the following.

tion **Axiom 10** (Foundation). *All sets are well-founded:*

$$\forall x\, \exists y\, (x \neq 0 \Rightarrow (y \in x \;\&\; y \cap x = 0).$$

By Theorem 144, this axiom is expressed by the equation

$$\mathbf{V} = \mathbf{WF}.$$

For the purposes of this chapter, the rest of this section is merely a curiosity.

Definition 54. For every nonempty class $\boldsymbol{C}$, if α is the least rank of an element of $\boldsymbol{C}$, we let

$$\tau(\boldsymbol{C}) = \mathbf{R}(\alpha + 1) \cap \boldsymbol{C}.$$

We also let

$$\tau(0) = 0.$$

So $\tau(\boldsymbol{C})$ is always a set, and

$$\tau(\boldsymbol{C}) \subseteq \boldsymbol{C}, \qquad \boldsymbol{C} \neq 0 \Rightarrow \tau(\boldsymbol{C}) \neq 0.$$

The following is now immediate:

thm:e-wf **Theorem 168.** *If $\boldsymbol{E}$ is an equivalence-relation on $\boldsymbol{C}$, and for all a in $\boldsymbol{C}$ we write*

$$[a] = \{x \colon x \in \boldsymbol{C} \;\&\; x \mathrel{\boldsymbol{E}} a\},$$

then $\tau([a])$ is the set $\{x \colon x \in \boldsymbol{C} \;\&\; x \mathrel{\boldsymbol{E}} a \;\&\; \operatorname{rank}(x) = \operatorname{rank}(a)\}$, and the function $x \mapsto \tau([x])$ is a function $\boldsymbol{F}$ on $\boldsymbol{C}$ such that, for all a and b in $\boldsymbol{C}$,

$$\boldsymbol{F}(a) = \boldsymbol{F}(b) \Leftrightarrow [a] = [b].$$

In the notation of the theorem then, the collection of equivalence classes $[a]$ can be identified with the class $\boldsymbol{F}[\boldsymbol{C}]$.

8.4. Constructible sets

In showing that the Axiom of Choice was true in $\mathbf{WF}$, we assumed the Axiom of Choice was true simply. Now, using ZF alone, we define a subclass of $\mathbf{WF}$ in which all of ZFC is true.

thm:const **Theorem 169.** *For all finite subsets p of ω, the collection of p-ary definable relations on a set is a set.*

Proof. We first work with an arbitrary class $\boldsymbol{M}$. Let $\boldsymbol{A}$ be a p-ary definable relation on $\boldsymbol{M}$. Then $\boldsymbol{A} = \varphi^{\boldsymbol{M}}$ for some p-ary formula φ whose constants are from $\boldsymbol{M}$. We can put the subformulas of φ in a string,

$$\varphi_0 \cdots \varphi_v,$$

where φ_v is φ, and for each u in $v+1$, for some i and j in ω, for some a and b in $\boldsymbol{M}$, and for some s and t in u, the formula φ_u is one of:
1) $x_i \in x_j$,
2) $x_i \in a$,
3) $a \in x_i$,
4) $a \in b$,
5) $\neg\varphi_s$,
6) $(\varphi_s \Rightarrow \varphi_t)$,
7) $\exists x_i\, \varphi_s$.

Letting p_s be the arity of φ_s, and writing $\boldsymbol{A}_s$ for $\varphi_s{}^{\boldsymbol{M}}$, we have correspondingly that $\boldsymbol{A}_u$ is:
1) $\{\{(i,x),(j,y)\} \colon x \in \boldsymbol{M} \;\&\; y \in \boldsymbol{M} \;\&\; x \in y\}$,
2) $\{\{(i,x)\} \colon x \in \boldsymbol{M} \;\&\; x \in a\}$,
3) $\{\{(i,x)\} \colon x \in \boldsymbol{M} \;\&\; a \in x\}$,
4) $\{0 \colon a \in b\}$ (which is 1 if $a \in b$, and otherwise 0),
5) ${}^{p_u}\boldsymbol{M} \smallsetminus \boldsymbol{A}_s$,
6) $\{\vec{x} \colon \vec{x} \in {}^{p_u}\boldsymbol{M} \;\&\; (\vec{x} \restriction p_s \in \boldsymbol{A}_s \Rightarrow \vec{x} \restriction p_t \in \boldsymbol{A}_t)\}$,
7) $\pi[\boldsymbol{A}_s]$, where π is $\vec{x} \mapsto \vec{x} \restriction p_u$ on ${}^{p_u}\boldsymbol{M}$.

Conversely, $\boldsymbol{A}$ is definable if there exists such a string $\boldsymbol{A}_0 \cdots \boldsymbol{A}_v$ of relations on $\boldsymbol{M}$. When $\boldsymbol{M}$ is a set, then so are the relations $\boldsymbol{A}_s$, and then the collection of definable p-ary relations on $\boldsymbol{M}$ is the subclass—in fact, subset—of $\mathscr{P}({}^{p}\boldsymbol{M})$ comprising those subsets $\boldsymbol{A}$ of ${}^{p}\boldsymbol{M}$ for which there is, for some v in ω, a v-tuple $(\boldsymbol{A}_0, \dots, \boldsymbol{A}_v)$ of sets meeting the appropriate conditions. □

Definition 55. For all finite subsets p of ω, the set of p-ary definable relations on a set a is denoted by

$$\mathscr{D}_p(a).$$

Then $\mathscr{D}_p(M) \subseteq \mathscr{P}({}^{p}M)$. The function $x \mapsto \mathbf{L}(x)$ on $\mathbf{ON}$ is defined recursively by

$$\mathbf{L}(0) = 0, \qquad \mathbf{L}(\alpha+1) = \mathscr{D}_1(\mathbf{L}(\alpha)), \qquad \mathbf{L}(\beta) = \bigcup \mathbf{L}[\beta]$$

where β is a limit. We now denote $\bigcup\{\mathbf{L}(x) \colon x \in \mathbf{ON}\}$ by

$$\mathbf{L}.$$

(So this letter, by itself, will not denote the function $x \mapsto \mathbf{L}(x)$.) The elements of $\mathbf{L}$ are called the **constructible sets.**

The following supplements Lemma 43 in §8.1.

Lemma 45. *If a is a transitive set, then so is $\mathscr{D}_1(a)$.*

Proof. If $b \in \mathscr{D}_1(a)$, then $b \subseteq a$. If also $c \in b$, then $c \in a$, and also c is the subset of a defined by $x \in c$, since a is transitive; so $c \in \mathscr{D}_1(a)$. Thus $b \subseteq \mathscr{D}_1(a)$. □

Then the following are analogous to Theorems 146 and 147.

Theorem 170. *Each set $\mathbf{L}(\alpha)$ is transitive, and so is the whole class $\mathbf{L}$.*

Theorem 171. *For all ordinals α and β,*

$$\alpha < \beta \Leftrightarrow \mathbf{L}(\alpha) \subset \mathbf{L}(\beta).$$

We now generalize Definition 51.

Definition 56. If $\boldsymbol{M} \subseteq \boldsymbol{N}$, and the p-ary formula φ takes its constants from $\boldsymbol{M}$, then φ is **absolute for** $(\boldsymbol{M}, \boldsymbol{N})$ if

$$\varphi^{\boldsymbol{M}} = {}^{p}\boldsymbol{M} \cap \varphi^{\boldsymbol{N}}.$$

So absoluteness for $\boldsymbol{M}$ is just absoluteness for $(\boldsymbol{M}, \mathbf{V})$.

Theorem 157 is still true with the more general sense of absoluteness. Hence we obtain the following test for absoluteness.

Theorem 172 (Tarski–Vaught Test). *If $\boldsymbol{M} \subseteq \boldsymbol{N}$, and the formula φ takes its constants from $\boldsymbol{M}$, then φ is absolute for $(\boldsymbol{M}, \boldsymbol{N})$, provided that, for every subformula of φ of the form $\exists x\, \psi(x, \vec{y})$, for all $\vec{b}$ from $\boldsymbol{M}$, if $\exists x\, \psi(x, \vec{b})$ is true in $\boldsymbol{N}$, then $\psi(a, \vec{b})$ is true in $\boldsymbol{N}$ for some a in $\boldsymbol{M}$.*

Proof. We argue by induction on a string of subformulas of φ such as was considered in the proof of Theorem 169. It is enough to suppose that ψ is absolute for $(\boldsymbol{M}, \boldsymbol{N})$ and prove the same for $\exists x\, \psi$, assuming that, for all $\vec{b}$ from $\boldsymbol{M}$, if $\exists x\, \psi(x, \vec{b})$ is true in $\boldsymbol{N}$, then $\psi(a, \vec{b})$ is true in $\boldsymbol{N}$ for some a in $\boldsymbol{M}$. But in this case the following are equivalent:

1. $\exists x\, \psi(x, \vec{b})$ is true in $\boldsymbol{N}$;
2. $\psi(a, \vec{b})$ is true in $\boldsymbol{N}$ for some a in $\boldsymbol{M}$;
3. $\psi(a, \vec{b})$ is true in $\boldsymbol{M}$ for some a in $\boldsymbol{M}$;
4. $\exists x\, \psi(x, \vec{b})$ is true in $\boldsymbol{M}$. □

In the notation of the theorem, even if φ is *not* necessarily absolute for $(\boldsymbol{M}, \boldsymbol{N})$, the theorem suggests a method of finding a class $\boldsymbol{M}^\dagger$ such that

$$\boldsymbol{M} \subseteq \boldsymbol{M}^\dagger \subseteq \boldsymbol{N}$$

and φ is absolute for $(\boldsymbol{M}^\dagger, \boldsymbol{N})$. The method is given given by the following corollary in case $\boldsymbol{M}$ is a set. We shall make a couple of applications of the method.

Corollary. *Suppose φ is a formula over $\boldsymbol{N}$, and there is a function $x \mapsto x^*$ such that, if $m \subseteq \boldsymbol{N}$, and φ is over m, then*

$$m \subseteq m^* \subseteq \boldsymbol{N},$$

and for every subformula of φ of the form $\exists x\, \psi(x, \vec{y})$, for all $\vec{b}$ from m, if $\exists x\, \psi(x, \vec{b})$ is true in $\boldsymbol{N}$, then $\psi(a, \vec{b})$ is true in $\boldsymbol{N}$ for some a in m^*. Then there is a set $m^\dagger$ such that

$$m \subseteq m^\dagger \subseteq \boldsymbol{N}$$

and φ is absolute for $(m^\dagger, \boldsymbol{N})$. Namely,

$$m^\dagger = \bigcup\{m_k \colon k \in \omega\},$$

where $x \mapsto m_x$ is defined by

$$m_0 = m, \qquad\qquad m_{k+1} = {m_k}^*.$$

beta **Theorem 173.** *For every formula with constants from $\mathbf{L}$, there is β such that the formula has its constants in $\mathbf{L}(\beta)$ and is absolute for $(\mathbf{L}(\beta), \mathbf{L})$.*

Proof. In the notation of the corollary of the Tarski–Vaught Test, if $m = \mathbf{L}(\alpha)$, we let m^* be $\mathbf{L}(\alpha^*)$, where α^* is the least ordinal γ such that $\alpha \leqslant \gamma$ and, for all subformulas of φ of the form $\exists x\, \psi(x, \vec{y})$, for all tuples $\vec{b}$ from $\mathbf{L}(\alpha)$, if $\exists x\, \psi(x, \vec{b})$ is true in $\mathbf{L}$, then $\psi(a, \vec{b})$ is true in $\mathbf{L}$ for some a in $\mathbf{L}(\gamma)$. We choose α_0 so that φ is over $\mathbf{L}(\alpha_0)$; and then $m^\dagger$ is $\mathbf{L}(\beta)$, where $\beta = \sup\{\alpha_k \colon k \in \omega\}$, where $\alpha_{k+1} = {\alpha_k}^*$. □

By analogy with Definition 50:

Definition 57. If $a \in \mathbf{L}$, we define

$$\operatorname{rank}_{\mathbf{L}}(a) = \min\{x \colon a \in \mathbf{L}(x+1)\}.$$

Note however that possibly $a \in \mathbf{L}$, and $a \subseteq \mathbf{L}(\beta)$, but a is not a *definable* subset of $\mathbf{L}(\beta)$, so $a \notin \mathbf{L}(\beta+1)$, and so $\beta < \operatorname{rank}_{\mathbf{L}}(a)$.

Theorem 174. $\mathbf{ON} \subseteq \mathbf{L}$*, and for all ordinals* α*,*

$$\operatorname{rank}_{\mathbf{L}}(\alpha) = \alpha.$$

thm:zf-l **Theorem 175.** ZF *is true in* $\mathbf{L}$*.*

Proof. By the theorems in §8.2 and the last theorem, equality is absolute for $\mathbf{L}$, and the Equality, Null Set, Adjunction, Union, Foundation, and Infinity Axioms are true in $\mathbf{L}$. Indeed, if a and b are in $\mathbf{L}(\alpha)$, then, since this set is transitive, $a \cup \{b\}$ and $\bigcup a$ are definable subsets of $\mathbf{L}(\alpha)$, so they are in $\mathbf{L}(\alpha+1)$.

Suppose $a \in \mathbf{L}$. Let $\beta = \sup\{\operatorname{rank}_{\mathbf{L}}(x) \colon x \in \mathbf{L} \mathbin{\&} x \subseteq a\}$. Then $\mathbf{L} \cap \mathscr{P}(a) \in \mathbf{L}(\beta+2)$. Thus Power Set is true in $\mathbf{L}$.

Suppose $\varphi(x,y)$ defines in $\mathbf{L}$ a function $\boldsymbol{F}$, and $a \in \mathbf{L}$. Some $\mathbf{L}(\alpha)$ contains all constants in φ and elements of $\boldsymbol{F}[\,a\,]$. Then $\varphi(a,y)^{\mathbf{L}} \subseteq \mathbf{L}(\alpha)$. By Theorem 173, there is β such that $\varphi(a,y)^{\mathbf{L}} = \varphi(a,y)^{\mathbf{L}(\beta)}$. Thus $\boldsymbol{F}[\,a\,] \in \mathbf{L}(\beta+1)$. Therefore Replacement is true in $\mathbf{L}$. □

thm:ac **Theorem 176.** *The Axiom of Choice is true in* $\mathbf{L}$*; indeed,* $\mathbf{L}$ *itself is well-ordered. In particular,* ZFC *is consistent (assuming* ZF *is consistent).*

Proof. There is a binary formula φ such that $\mathbf{L}$ is well-ordered by $\varphi_{\mathbf{L}}$. Indeed, because of the recursive construction of the sets $\mathscr{D}_1(a)$, there is a ternary formula ψ such that, if $\mathbf{L}(\alpha)$ is well-ordered by r, then $\mathbf{L}(\alpha+1)$ is well-ordered by $\psi(r,x,y)^{\mathbf{L}}$, and this ordering agrees with r on $\mathbf{L}(\alpha)$. By transfinite recursion, there is a function $x \mapsto r_x$ on $\mathbf{ON}$ such that $\mathbf{L}(\alpha)$ is well-ordered by r_α for each ordinal α, and r_α agrees with r_β on $\mathbf{L}(\alpha)$ if $\alpha < \beta$. Then $\mathbf{L}$ is well-ordered by $\bigcup\{r_x \colon x \in \mathbf{ON}\}$. This argument does not use the Axiom of Choice. □

8.5. The Generalized Continuum Hypothesis

Our second application of the corollary of the Tarski–Vaught Test is the following.

Theorem 177 (Löwenheim–Skolem). *For every set m_0 and every formula φ over m_0, there is a set $m^\dagger$ such that $m_0 \subseteq m^\dagger$, and*

$$\operatorname{card}(m^\dagger) \leqslant \operatorname{card}(m_0) + \aleph_0,$$

and φ is absolute for $m^\dagger$.

Proof. Given m, we let a be the set of all *nonempty* sets $\psi(x, \vec{b})^{\mathbf{V}}$, where $\exists x\, \psi(x, \vec{y})$ is a subformula of φ, and $\vec{b}$ is from m. Then we let

$$\alpha = \sup\{\min\{\operatorname{rank}(z) \colon z \in x\} \colon x \in a\}.$$

This ensures that $\mathbf{R}(\alpha+1) \cap c \neq 0$ for all c in a. The set $\mathbf{R}(\alpha+1)$ is well-ordered by some binary relation r. We now define

$$m^*_r = m \cup \{\min(\mathbf{R}(\alpha+1) \cap x) \colon x \in a\}.$$

Note that $\operatorname{card}(m^*_r) \leqslant \operatorname{card}(m) + \aleph_0$.

Since m^*_r depends on r as well as m, we do not yet have the desired function $x \mapsto x^*$. We can get it by considering the set of appropriate sequences $((m_x, r_x) \colon x \in \alpha)$, where $\alpha \leqslant \omega$, and $m_{k+1} = (m_k)^*_{r_k}$. By Zorn's Lemma, there is such a sequence where $\alpha = \omega$. Then we can define ${m_k}^* = m_{k+1}$. □

A class is well-founded if and only if the relation of membership on the class is *well-founded* in the following sense.

nded **Definition 58.** A binary relation $\boldsymbol{R}$ is **well-founded** on a class $\boldsymbol{C}$ if

1) for all a in $\boldsymbol{C}$, the class $\{x \colon x \in \boldsymbol{C} \;\&\; x \mathrel{\boldsymbol{R}} a\}$ is a set;
2) every nonempty subset b of $\boldsymbol{C}$ has an element c such that $\{x \colon x \in b \;\&\; x \mathrel{\boldsymbol{R}} c\}$ is empty.

In particular, a linear ordering that meets the first part of this definition is just a left-narrow linear ordering, and a well-founded linear ordering is just a good ordering. (See Definition 21 in §5.1.) The relation $\boldsymbol{R}$ is **extensional** on $\boldsymbol{C}$ if the Extension Axiom is true in the structure $(\boldsymbol{C}, \boldsymbol{R})$ in the sense that

$$\{x \colon x \in \boldsymbol{C} \;\&\; x \mathrel{\boldsymbol{R}} a\} = \{x \colon x \in \boldsymbol{C} \;\&\; x \mathrel{\boldsymbol{R}} b\} \Rightarrow a = b.$$

Theorem 178 (Mostowski Collapsing)**.** *Let $\boldsymbol{R}$ be a well-founded relation on $\boldsymbol{C}$. There is a unique function $\boldsymbol{F}$ on $\boldsymbol{C}$ given by*

$$\boldsymbol{F}(a) = \boldsymbol{F}[\,\{x\colon x \in \boldsymbol{C} \mathbin{\&} x \mathrel{\boldsymbol{R}} a\}\,].$$

Then $\boldsymbol{F}[\,\boldsymbol{C}\,]$ is transitive. If $\boldsymbol{R}$ is extensional, then $\boldsymbol{F}$ is an isomorphism from $(\boldsymbol{C}, \boldsymbol{R})$ to $(\boldsymbol{F}[\,\boldsymbol{C}\,], \in)$.

Proof. We follow the proof of Theorem 67, though since $\boldsymbol{R}$ need not be transitive, we shall need the following. If $a \in \boldsymbol{C}$, then by recursion we define

$$\mathrm{cl}_0(a) = \{a\}, \qquad \mathrm{cl}_{n+1}(a) = \{x\colon \exists y\,(y \in \mathrm{cl}_n(a) \mathbin{\&} x \mathrel{\boldsymbol{R}} y)\},$$

Now we let

$$\mathrm{cl}(a) = \bigcup\{\mathrm{cl}_n(a)\colon n \in \omega\}.$$

We shall also use the notation

$$\boldsymbol{R}a = \{x\colon x \in \boldsymbol{C} \mathbin{\&} x \mathrel{\boldsymbol{R}} a\}.$$

We first show that the structure $(\boldsymbol{C}, \boldsymbol{R})$ admits induction in the sense that, if $\boldsymbol{C}_0 \subseteq \boldsymbol{C}$ and, for all a in $\boldsymbol{C}$, we have $a \in \boldsymbol{C}_0$ whenever $\boldsymbol{R}a \subseteq \boldsymbol{C}_0$, then $\boldsymbol{C}_0 = \boldsymbol{C}$. Indeed, suppose $\boldsymbol{C}_0 \subset \boldsymbol{C}$, and $a \in \boldsymbol{C} \smallsetminus \boldsymbol{C}_0$. By definition, the set $(\boldsymbol{C} \smallsetminus \boldsymbol{C}_0) \cap \mathrm{cl}(a)$ has an element b such that

$$(\boldsymbol{C} \smallsetminus \boldsymbol{C}_0) \cap \mathrm{cl}(a) \cap \boldsymbol{R}b = 0.$$

But $\boldsymbol{R}b \subseteq \mathrm{cl}(a)$ (since $b \in \mathrm{cl}_n(a)$ for some n, and then $\boldsymbol{R}b \subseteq \mathrm{cl}_{n+1}(a)$). Hence $(\boldsymbol{C} \smallsetminus \boldsymbol{C}_0) \cap \boldsymbol{R}b = 0$, so $\boldsymbol{R}b \subseteq \boldsymbol{C}_0$.

Now we can show by induction that, for all a in $\boldsymbol{C}$, there is a unique function f_a on $\mathrm{cl}(a)$ such that

$$f_a(c) = f_a[\,\boldsymbol{R}c\,].$$

Indeed, suppose the claim holds when $a \mathrel{\boldsymbol{R}} b$. If a and d are in $\boldsymbol{R}b$, then f_a and f_d must agree on $\mathrm{cl}(a) \cap \mathrm{cl}(d)$ since, if f_a and g disagree on $\mathrm{cl}(a) \cap \mathrm{cl}(d)$, then by well-foundedness this set has an element e such that f_a and g agree on $\boldsymbol{R}e$, but

$$g(e) \neq f_a(e) = f_a[\,\boldsymbol{R}e\,] = g[\,\boldsymbol{R}e\,].$$

Now we can define f_b on $\mathrm{cl}(b)$ so that, if $c \in \mathrm{cl}(b) \smallsetminus \{b\}$, then $f_b(c) = f_a(c)$, where a is such that $a \boldsymbol{R} b$ and $c \in \mathrm{cl}(a)$; and $f_b(b) = f_b[\boldsymbol{R}b]$.

The desired function $\boldsymbol{F}$ is now $\bigcup\{f_a : a \in \boldsymbol{C}\}$. Indeed, this is a function, since any two functions f_a agree as before on the intersection of their domains. Likewise, $\boldsymbol{F}$ itself is unique. Since $\boldsymbol{F}(a) = \boldsymbol{F}[\boldsymbol{R}a] \subseteq \boldsymbol{F}[\boldsymbol{C}]$, it follows that $\boldsymbol{F}[\boldsymbol{C}]$ is transitive.

We have $a \boldsymbol{R} b \Rightarrow \boldsymbol{F}(a) \in \boldsymbol{F}(b)$. If $\boldsymbol{F}$ is injective, then $\boldsymbol{F}(a) \in \boldsymbol{F}(b) \Rightarrow a \boldsymbol{R} b$, so $\boldsymbol{F}$ is an isomorphism. Suppose $\boldsymbol{F}$ is not injective. Let $\boldsymbol{C}_0$ comprise those a in $\boldsymbol{C}$ for which there is no distinct b such that $\boldsymbol{F}(a) = \boldsymbol{F}(b)$. As in the proof that $(\boldsymbol{C}, \boldsymbol{R})$ admits induction, there is an element a of $\boldsymbol{C} \smallsetminus \boldsymbol{C}_0$ such that $\boldsymbol{R}a \subseteq \boldsymbol{C}_0$. Then $\boldsymbol{F}(a) = \boldsymbol{F}(b)$ for some distinct b. This means

$$\{\boldsymbol{F}(x) \colon x \in \boldsymbol{R}a\} = \{\boldsymbol{F}(y) \colon y \in \boldsymbol{R}b\}.$$

Since $\boldsymbol{R}a \subseteq \boldsymbol{C}_0$, we conclude $\boldsymbol{R}b = \boldsymbol{R}a$. Thus $\boldsymbol{R}$ is not extensional. □

Lemma 46. *For all ordinals α, in* $\mathbf{V}$ *and in* $\mathbf{L}$.

$$\mathrm{card}(\mathbf{L}(\beta)) = \mathrm{card}(\beta),$$

:gch

Theorem 179. *The Generalized Continuum Hypothesis is true in* $\mathbf{L}$. *Thus* GCH *is consistent with* ZFC *(assuming* ZF *is consistent).*

Proof. Suppose $a \in \mathscr{P}(\mathbf{L}(\alpha)) \cap \mathbf{L}$, where α is infinite. We shall show

$$a \in \mathbf{L}(\mathrm{card}(\alpha)^+).$$

By the last lemma, it will follow that, in $\mathbf{L}$,

$$\mathrm{card}(\mathscr{P}(\kappa)) = \kappa^+.$$

Apply the Löwenheim–Skolem Theorem to the set $\mathbf{L}(\alpha) \cup \{a\}$ and the formula

$$x = y \mathbin{\&} \exists z\, a \in \mathbf{L}(z).$$

We get a set $m^\dagger$ such that $\mathbf{L}(\alpha) \subseteq m^\dagger$, $a \in m^\dagger$, $\mathrm{card}(m^\dagger) = \mathrm{card}(\alpha)$, $(m^\dagger, \in)$ is extensional, and

$$\models_{m^\dagger} \exists x\, a \in \mathbf{L}(x).$$

By the Mostowski Collapsing Theorem, we may assume further that $m^\dagger$ is *transitive.* In particular, an element β of $m^\dagger$ such that $\models_{m^\dagger} a \in \mathbf{L}(\beta)$ really is an ordinal. Then $a \in \mathbf{L}(\beta)$, but $\mathrm{card}(\beta) = \mathrm{card}(\alpha)$, so $a \in \mathbf{L}(\mathrm{card}(\alpha)^+)$. $\square$

9. Independence

9.1. Models

(Under construction)

Theorem 180. *If* Γ *is a collection of sentences that is consistent with* ZFC^-*, then* Γ *has a set model.*

About a quarter century after Gödel proved that AC and GCH are consistent with ZF, Cohen (see [8]) proved the same of their negations.

A. The Greek alphabet

app:Greek

capital	minuscule	transliteration	name
Α	α	a	alpha
Β	β	b	beta
Γ	γ	g	gamma
Δ	δ	d	delta
Ε	ε	e	epsilon
Ζ	ζ	z	zeta
Η	η	ê	eta
Θ	ϑ	th	theta
Ι	ι	i	iota
Κ	ϰ	k	kappa
Λ	λ	l	lambda
Μ	μ	m	mu
Ν	ν	n	nu
Ξ	ξ	x	xi
Ο	ο	o	omicron
Π	π	p	pi
Ρ	ρ	r	rho
Σ	σ, ς	s	sigma
Τ	τ	t	tau
ϒ	υ	y, u	upsilon
Φ	φ	ph	phi
Χ	χ	ch	chi
Ψ	ψ	ps	psi
Ω	ω	ô	omega

The following remarks pertain to *ancient* Greek. The vowels are α, ε, η, ι, ο, υ, ω, where η is a long ε, and ω is a long ο; the other vowels (α, ι, υ) can be long or short. Some vowels may be given tonal accents (ά, ᾶ, ὰ). An initial vowel takes either a rough-breathing mark (as in ἁ) or a smooth-breathing mark (ἀ): the former mark is transliterated by a preceding h, and the latter can be ignored, as in ὑπερβολή hyperbolê *hyperbola,* ὀρθογώνιον orthogônion *rectangle.* Likewise, ῥ is transliterated as rh, as in ῥόμβος rhombos *rhombus.* A long vowel may have an iota subscript (ᾳ, ῃ, ῳ), especially in case-endings of nouns. Of the two forms of minuscule sigma, the ς appears at the ends of words; elsewhere, σ appears, as in βάσις basis *base.*

B. The German script

app:German

Writing in 1993, Wilfrid Hodges [23, Ch. 1, p. 21] observes

> Until about a dozen years ago, most model theorists named structures in horrible Fraktur lettering. Recent writers sometimes adopt a notation according to which all structures are named M, M', M^*, $\bar{M}$, M_0, M_i or occasionally N.

For Hodges, structures are A, B, C, and so forth; he refers to their universes as **domains** and denotes these by $\operatorname{dom}(A)$ and so forth. This practice is convenient if one is using a typewriter (as in the preparation of another of Hodges's books [24], from 1985). In 2002, David Marker [30] uses 'calligraphic' letters for structures, so that M is the universe of $\mathcal{M}$. I still prefer the Fraktur letters:

$$\begin{array}{ccccccccc}
\mathfrak{A} & \mathfrak{B} & \mathfrak{C} & \mathfrak{D} & \mathfrak{E} & \mathfrak{F} & \mathfrak{G} & \mathfrak{H} & \mathfrak{I} \\
\mathfrak{J} & \mathfrak{K} & \mathfrak{L} & \mathfrak{M} & \mathfrak{N} & \mathfrak{O} & \mathfrak{P} & \mathfrak{Q} & \mathfrak{R} \\
\mathfrak{S} & \mathfrak{T} & \mathfrak{U} & \mathfrak{V} & \mathfrak{W} & \mathfrak{X} & \mathfrak{Y} & \mathfrak{Z} &
\end{array}$$

$$\begin{array}{ccccccccc}
\mathfrak{a} & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} & \mathfrak{e} & \mathfrak{f} & \mathfrak{g} & \mathfrak{h} & \mathfrak{i} \\
\mathfrak{j} & \mathfrak{k} & \mathfrak{l} & \mathfrak{m} & \mathfrak{n} & \mathfrak{o} & \mathfrak{p} & \mathfrak{q} & \mathfrak{r} \\
\mathfrak{s} & \mathfrak{t} & \mathfrak{u} & \mathfrak{v} & \mathfrak{w} & \mathfrak{x} & \mathfrak{y} & \mathfrak{z} &
\end{array}$$

A way to write these by hand is shown in Figure B.1, which is taken from a 1931 textbook of German for English-speakers [20].

fig:German

Figure B.1. The German alphabet by hand

C. The Axioms

app:axioms

We work in a logic whose only predicate is $\in$. We introduce constants for the purpose of defining truth of sentences (in Definitions 3 and 5 on pp. 29 and 34). We expand our logic so that, by definition (p. 48),

$$\forall x\,\forall y\,(x = y \Leftrightarrow \forall z\,(z \in x \Leftrightarrow z \in y)).$$

We define certain classes (by means of formulas). Then our axioms are:

1. Equality (p. 49):

$$\forall x\,\forall y\,(x = y \Leftrightarrow \forall z\,(x \in z \Leftrightarrow y \in z).$$

2. Null set (p. 54): The empty class is a set.
3. Adjunction (p. 54): The adjunction of one set to another is a set.
4. Separation (p. 63): Every subclass of a set is a set.
5. Replacement (p. 84): The image of a set under a function is a set.
6. Union (p. 89): The union of every set is a set.
7. Infinity (p. 96): The class ω of natural numbers is a set.
8. Power Set (p. 113): The power class of a set is a set.
9. Choice (p. 121): Every set has a choice-function.
10. Foundation (p. 155): All sets are well-founded.

The Separation and Replacement Axioms are schemes of sentences, one for each class (its parameters universally quantified; functions are kinds of classes).

D. Other set theories and approaches

ries

As I note in the Preface, I aim in this book to introduce axioms are only when further progress is otherwise hindered. This approach is taken also by Lemmon, whose *Introduction to axiomatic set theory* [27] can be analyzed as in Table D.1. However, Lemmon explicitly (on his p. 4) follows

chapter	title	page	axiom		page
I	The General Theory of Classes	14	A1	[Extension]	14
			A2	[Classification]	15
II	Sets, Relations, and Functions	47	A3	[Null Set]	51
			A4	[Pairs]	51
			A5	[Union]	51
			A6	[Power Set]	51
			A7	[Separation]	51
			A8	[Replacement]	89
III	Numbers	92	A9	[Infinity]	105
			A10	[Foundation]	105
			A11	[Choice]	118
Appendix on some variant set theories		121			

ab:L

Table D.1. Lemmon's *Introduction to axiomatic set theory*

von Neumann, Bernays, Gödel, and Morse in giving formal existence to classes. He uses only one kind of variable, which ranges over the classes. So really Lemmon's theory is a theory of classes. His 'classification axiom scheme', A2, is that, for every formula in one free variable, there is a class of all *sets* that satisfy the formula. He notes in the appendix that he thus gives up the finite axiomatizability of Gödel's theory (the theory called below NBG, after von **N**eumann, **B**ernays, and **G**ödel).

Meanwhile, Lemmon's A2 allows the next six axioms, A3–A8, to be that certain classes are sets. But then Lemmon states A9, the Axiom of Infinity, in the traditional way: there exists a set, called ω, with certain closure properties. He writes:

> A9, the axiom of infinity, is an outright declaration that a certain set exists; in this respect it is like A3 [the Null Set Axiom]. In fact A3 is a consequence of A9... [27, p. 105]

Since he has already defined ω informally on page 99, it is not clear why he does not take the extra step of observing that ω does exist as a class, regardless of the Axiom of Infinity. I prefer to take this step, then see what can be said *before* going on to declare that the class ω is a set.

I prefer in addition to remain agnostic about infinity as long as possible: somewhat longer than Lemmon. Kunen [26, IV.3.13, p. 123] shows that $\mathbf{R}(\omega)$ is a model of ZFC with the Axiom of Infinity replaced with its negation; but he assumes ZF without Foundation; in particular, he assumes Infinity. This is not necessary. All one needs is GST (see p. 63).

Lemmon's classification axiom scheme is non-minimalistic in a strong sense. His theory is a version of so-called Morse–Kelley set theory; in particular, it assumes more than our ZFC. I quote here some criticism of this theory by Joseph Shoenfield.[1] Shoenfield apparently uses the word *collection* as I do in the text, as the most general collective noun (see p. 15):

> In response to some rather unfavorable remarks I made about MK (Morse–Kelly set-class theory), Friedman has defended MK as natural and important. Let me try to describe briefly what (in my opinion) is the origin and purpose of NBG and MK.
>
> For the moment, let us take a class to be a collection of sets definable from set parameters in the language of ZFC. Classes in this sense play an important role, even if one is working in ZFC. For example, the Separation Scheme is most easily stated as: every class which is included in a set is a set. In the language of ZFC, we can only state this as a scheme. If we introduce variables for classes, we can state it as a single axiom. Of course, we then need axioms to insure us that all classes are values of the class variables. This can be done by a simple scheme. As Friedman and others have pointed out, this scheme can be

[1] This Shoenfield is apparently the author of the text *Mathematical Logic* [34], who was born in 1927, and who died on November 15, 2000. The quotation is from an email, dated February 14, 2000, sent to the FOM (Foundations of Mathematics) email list and archived at `http://www.cs.nyu.edu/pipermail/fom/2000-February/003740.html` (accessed March 5, 2011). A reference to this email is given at `http://en.wikipedia.org/wiki/Morse-Kelley_set_theory` (accessed same date). I have imposed the special mathematical typography on the original plain-text email, and I have corrected one or two typographical errors.

derived from a finite number of its instances; but the proof of this is a bit tedious, and is quite useless if one wishes to develop set theory in NBG.

The nice thing about NBG it that every model M of ZFC has a least extension to a model of NBG; the classes in the extension are just the classes in M. From this it follows that NBG is a conservative extension of ZFC. Thus whether we do set theory in ZFC or NBG is a matter of taste.

Now all of this naturally suggests an extended notion of class, in which a class is an arbitrary collection of sets. We then extend our class existence scheme to make every collection of sets definable in our extended language a class. Of course not every class (in the extended sense) is so definable; but these are the only ones we can assert are classes in our extended language.

Unfortunately, it is no longer true that any model of ZFC can be extended to a model of MK. We can prove Con(ZFC)[2] in MK by proving that the class $\mathbf{V}$ is a model of ZFC; and Con(ZFC) is a statement in the language of ZFC not provable in ZFC. If the model of ZFC has strong enough closure properties, we can extend it. For example, if the model is closed under forming subsets, it is clear that the Separation Scheme will hold independent of the choice of the classes in the model. In this way we can show (as Friedman observes) that a model V_κ where κ is an inacessible cardinal can be extended to a model of MK. The trouble with such models is that they have strong absoluteness properties; most interesting set theoretic statements are true in V_κ iff they are true in $\mathbf{V}$. This makes the models useless for most independence proofs.

Friedmann has given a sketch of an independence proof in MK by forcing; but many of the details are unclear to me. He takes a model M of MK, lets M' be the included model of ZFC and N' a generic extension of M'. He then says N' canonically generates a model N of MK. I do not understand how one selects the classes of N, nor how one can prove the axioms of MK hold in N. I would be surprised if the details wouls lead me to agree with Friedman that the question he was considering is 'not very much easier to solve for NBG than it is for MK'. In any case, there seems to be little reason to solve it for MK.

Friedman concludes with some predictions about the future of MK and similar systems, He says:

[2]That is, consistency of ZFC, which we have denoted by $\neg\mathrm{Incons}_{\mathrm{ZFC}}$.

> We have only the bare beginnings of where the axioms of large cardinals come from or why they are canonical or why they should be accepted or why they are consistent.

I agree whole-heartedly with this, and with the implied statement that these are important questions. He then says:

> I have no doubt that further substantial progess on these crucial issues will at least partly depend on deep philosophical introspection, and I have no doubt concepts of both class and set and their 'interaction' will play a crucial role in the future.

Here I strongly disagree. I think that if there is one thing we can learn from the development of mathematical logic in the last century, it is that all the major accomplishments of this subject consist of mathematical theorems, which, in the most interesing cases, have evident foundational consequences. I do not know of any major result in the field which was largely achieved by means of philosophical introspection, as I understand the term. I do not see the the study of the interaction of sets and classes has led to any very interesting results.

If the problems about large cardinals cannot be solved by philo- sophical introspection, how can they be solved? Fortunately, I have available an example of how to proceed, furnished by the recent communication of John Steel. I think it says more about the problems of large cardinals then all the previous FOM communications combined. The idea is to examine all the results which have been proved about large cardinals and related concepts, and see if they give some hint of which large cardinals we should accept and what further results we might prove to further justify these axioms. We are still a long way from accomplishing the goal, but, as Steel shows, we have advanced a great deal since large cardinals first appeared on the scene forty years ago.

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