Dense total orders without endpoints

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These notes repeat and supplement the 2004.10.26 lecture of Math 406 (Introduction to mathematical logic and model-theory).

If $\mathcal{L}$ is a signature (of first-order logic), $\mathfrak{A}$ is an $\mathcal{L}$-structure, and σ is a sentence of $\mathcal{L}$, then we have defined what it means if σ is *true* in $\mathfrak{A}$. In this case, we write

$$\mathfrak{A} \models \sigma.$$

Having defined truth, we can define *logical consequence*. Let $\mathrm{Sn}_{\mathcal{L}}$ be the set of sentences of $\mathcal{L}$. The $\mathcal{L}$-structure $\mathfrak{A}$ is a **model** of a subset Σ of $\mathrm{Sn}_{\mathcal{L}}$ if each sentence in Σ is true in $\mathfrak{A}$; then we can write

$$\mathfrak{A} \models \Sigma.$$

If a sentence σ is true in every model of Σ, then σ is a **(logical) consequence** of Σ, and we can write

$$\Sigma \models \sigma.$$

If $\varnothing \models \sigma$, then we can write just

$$\models \sigma;$$

in this case, σ is a **validity**.

Two sentences are **(logically) equivalent** if each is a logical consequence of the other.

1 Lemma. *Let σ and τ be sentences of $\mathcal{L}$.*

- (∗) *$\{\sigma\} \models \tau$ if and only if $\models (\sigma \to \tau)$, for all σ and τ in $\mathrm{Sn}_{\mathcal{L}}$.*
- (†) *σ and τ are equivalent if and only if $\models (\sigma \to \tau) \land (\tau \to \sigma)$.*
- (‡) *Logical equivalence is an equivalence-relation on $\mathrm{Sn}_{\mathcal{L}}$.*

Proof. **Exercise**. □

Instead of the formula $(\phi \to \chi) \land (\chi \to \phi)$, let us write

$$\phi \leftrightarrow \chi.$$

By the lemma, σ and τ are logically equivalent if and only if $(\sigma \leftrightarrow \tau)$ is a validity. We may blur the distinction between logically equivalent sentences, identifying σ with $\neg\neg\sigma$ for example.

Instead of $\neg\exists v\ \neg\phi$, we may write

$$\forall v\ \phi.$$

Then $\neg\forall v\ \phi$ is (equivalent to) $\exists v\ \neg\phi$.

If $\mathrm{fv}(\phi) = \{u_0, \dots, u_{n-1}\}$, and $\mathfrak{A} \models \forall u_0\ \cdots \forall u_{n-1}\ \phi$, we may write just

$$\mathfrak{A} \models \phi.$$

Here, the sentence $\forall u_0\ \cdots \forall u_{n-1}\ \phi$ is the **(universal) generalization** of ϕ. Now we can define $\Sigma \models \phi$ for arbitrary formulas ϕ (although Σ should still be a set of *sentences*); we can also say that arbitrary formulas ϕ and χ are **(logically) equivalent** if

$$\models (\phi \leftrightarrow \chi).$$

For the formula ϕ with free variables $x_0, \dots, x_{n-1}$, if we have

$$\mathfrak{A} \models \exists u_0\ \cdots \exists u_{n-1}\ \phi,$$

then we can say that ϕ is **satisfied** in $\mathfrak{A}$.

It can happen then that $\mathfrak{A} \not\models \phi$ and $\mathfrak{A} \not\models \neg\phi$. However, if σ is a *sentence*, then either σ or $\neg\sigma$ is true in $\mathfrak{A}$.

2 Example. Each of the following formulas is true in every group:

$$\begin{gathered} x \cdot (y \cdot z) = (x \cdot y) \cdot z, \\ x \cdot 1 = x, \qquad x \cdot x^{-1} = 1, \\ 1 \cdot x = x, \qquad x^{-1} \cdot x = 1. \end{gathered}$$

If $\Sigma \subseteq \mathrm{Sn}_{\mathcal{L}}$, let

$$\mathrm{Con}_{\mathcal{L}}(\Sigma) = \{\sigma \in \mathrm{Sn}_{\mathcal{L}} : \Sigma \models \sigma\}.$$

3 Lemma. $\mathrm{Con}_{\mathcal{L}}(\mathrm{Con}_{\mathcal{L}}(\Sigma)) = \mathrm{Con}_{\mathcal{L}}(\Sigma)$.

Proof. Since $\Sigma \subseteq \mathrm{Con}_{\mathcal{L}}(\Sigma)$, we have $\mathrm{Con}_{\mathcal{L}}(\Sigma) \subseteq \mathrm{Con}_{\mathcal{L}}(\mathrm{Con}_{\mathcal{L}}(\Sigma))$. Suppose $\sigma \in \mathrm{Con}_{\mathcal{L}}(\mathrm{Con}_{\mathcal{L}}(\Sigma))$. Then $\mathrm{Con}_{\mathcal{L}}(\Sigma) \models \sigma$. But if $\mathfrak{A} \models \Sigma$, then $\mathfrak{A} \models \mathrm{Con}_{\mathcal{L}}(\Sigma)$, so in this case $\mathfrak{A} \models \sigma$. Thus $\sigma \in \mathrm{Con}_{\mathcal{L}}(\Sigma)$. □

A subset T of $\mathrm{Sn}_{\mathcal{L}}$ is a **theory** of $\mathcal{L}$ if $\mathrm{Con}_{\mathcal{L}}(T) = T$. A subset Σ of a theory T is a set of **axioms** for T if

$$T = \mathrm{Con}_{\mathcal{L}}(\Sigma);$$

we may also say then that Σ **axiomatizes** T.

4 Example. The theory of groups is axiomatized by

$$\begin{gathered} \forall x\ \forall y\ \forall z\ x \cdot (y \cdot z) = (x \cdot y) \cdot z, \\ \forall x\ x \cdot 1 = x, \qquad \forall x\ x \cdot x^{-1} = 1, \\ \forall x\ 1 \cdot x = x, \qquad \forall x\ x^{-1} \cdot x = 1. \end{gathered}$$

If $\mathfrak{A}$ is an $\mathcal{L}$-structure, let

$$\mathrm{Th}(\mathfrak{A}) = \{\sigma \in \mathrm{Sn}_{\mathcal{L}} : \mathfrak{A} \models \sigma\}.$$

5 Lemma. $\mathrm{Th}(\mathfrak{A})$ *is a theory.*

Proof. Say $\mathrm{Th}(\mathfrak{A}) \models \sigma$. Since $\mathfrak{A} \models \mathrm{Th}(\mathfrak{A})$, we have $\mathfrak{A} \models \sigma$, so $\sigma \in \mathrm{Th}(\mathfrak{A})$. □

We can now call $\mathrm{Th}(\mathfrak{A})$ the **theory of** $\mathfrak{A}$. Note that, if T is $\mathrm{Th}(\mathfrak{A})$, then

$$T \models \sigma \iff T \not\models \neg\sigma$$

for all sentences σ. An arbitrary theory T need not have this property; if it does, then T is **complete**. So, the theory *of* a structure is always complete. The set $\mathrm{Sn}_{\mathcal{L}}$ is a theory, but it is not complete by this definition. Complete theories are 'maximal' in the following sense:

6 Lemma. *Let T be a theory of $\mathcal{L}$.*

(∗) *If T has no model, then T is $\mathrm{Sn}_{\mathcal{L}}$ itself.*

(†) *If T has a model, namely $\mathfrak{A}$, then T is included in a complete theory, namely $\mathrm{Th}(\mathfrak{A})$.*

(‡) *If T has a model, then*

$$T \models \sigma \implies T \not\models \neg\sigma$$

for all σ in $\mathrm{Sn}_{\mathcal{L}}$.

(§) *Hence, to prove that T is complete, it is enough to show that T has models and*

$$T \not\models \sigma \implies T \models \neg\sigma$$

for all σ in $\mathrm{Sn}_{\mathcal{L}}$.

Proof. If T is a theory with no models, and σ is a sentence, then σ is true in every model of T, so $T \models \sigma$, whence $\sigma \in T$. The second statement is obvious. The third statement follows since $\{\sigma, \neg\sigma\}$ has no models. The last statement is now obvious. □

We can also speak of the theory of a *class* of $\mathcal{L}$-structures. If K is such a class, then $\mathrm{Th}(K)$ is the set of sentences of $\mathcal{L}$ that are true in *every* structure in K.

In particular, if $\Sigma \subseteq \mathrm{Sn}_{\mathcal{L}}$, then we can define

$$\mathrm{Mod}(\Sigma)$$

to be the class of all models of Σ. Then

$$\mathrm{Th}(\mathrm{Mod}(\Sigma)) = \mathrm{Con}_{\mathcal{L}}(\Sigma).$$

7 Example. By definition, a group is just a model of the theory of groups, as axiomatized in **4**. Hence this theory is $\mathrm{Th}(K)$, where K is the class of all groups.

In general, if we have some sentences, how might we show that the theory that they axiomatize is complete? If the theory is *not* complete, this is easy to show:

8 Example. The theory of groups is not complete, since the sentence

$$\forall x\ \forall y\ xy = yx$$

is true (by definition) only in abelian groups, but there are non-abelian groups (such as the group of permutations of three objects). The theory of abelian groups is not complete either, since (in the signature $\{+, -, 0\}$) the sentence

$$\forall x\ (x + x = 0 \to x = 0)$$

is true in $(\mathbb{Z}, +, -, 0)$, but false in $(\mathbb{Z}/2\mathbb{Z}, +, -, 0)$.

Let TO be the theory of *strict* total orders; this is axiomatized by the universal generalizations of:

$$\neg(x < x),$$
$$x < y \to \neg(y < x),$$
$$x < y \land y < z \to x < z,$$
$$x < y \lor y < x \lor x = y.$$

This theory is not complete, since $(\omega, <)$ and $(\mathbb{Z}, <)$ are models of TO with different complete theories (**exercise**).

Let TO^* be the theory of **dense total orders without endpoints**, namely, TO^* has the axioms of TO, along with the universal generalizations of:

$$\exists z\ (x < z \land z < y),$$
$$\exists y\ y < x,$$
$$\exists y\ x < y.$$

The theory TO^* has a model, namely $(\mathbb{Q}, <)$. We shall show that TO^* is complete. In order to do this, we shall first show that the theory admits *(full) elimination of quantifiers.*

An arbitrary theory T admits **(full) elimination of quantifiers** if, for every formula ϕ of $\mathcal{L}$, there is an *open* formula χ of $\mathcal{L}$ such that

$$T \models (\phi \leftrightarrow \chi)$$

—in words, ϕ **is equivalent to** χ ***modulo*** T.

9 Lemma. *An $\mathcal{L}$-theory T admits quantifier-elimination, provided that, if ϕ is an open formula, and v is a variable, then $\exists v\ \phi$ is equivalent* modulo T *to an open formula.*

Proof. Use induction on formulas. Specifically:

Every atomic formula is equivalent *modulo* T to an open formula, namely itself.

Suppose ϕ is equivalent *modulo* T to an open formula α. Then $T \models (\neg\phi \leftrightarrow \neg\alpha)$; but $\neg\alpha$ is open.

Suppose also χ is equivalent *modulo* T to an open formula β. Then

$$T \models ((\phi \to \chi) \leftrightarrow (\alpha \to \beta));$$

but $(\alpha \to \beta)$ is open.

Finally, $T \models (\exists v\ \phi \leftrightarrow \exists v\ \alpha)$ (**exercise**); but by assumption, $\exists v\ \alpha$ is equivalent to an open formula γ; so $T \models (\exists v\ \phi \leftrightarrow \gamma)$ (**exercise**). This completes the induction. □

The lemma can be improved slightly. Every open formula is logically equivalent to a formula in *disjunctive normal form*:

$$\bigvee_{i<m} \bigwedge_{j<n} \alpha_i^{(j)},$$

where each $\alpha_i^{(j)}$ is either an atomic or a negated atomic formula. (See § 2.6 of this year's notes for Math 111.) This formula in disjunctive normal form can also be written

$$\bigvee_{i<m} \bigwedge \Sigma_i$$

where $\Sigma_i = \{a_i^{(j)} : j < n\}$. Note that

$$\models (\exists v \bigvee_{i<m} \bigwedge \Sigma_i \leftrightarrow \bigvee_{i<m} \exists v \ \bigwedge \Sigma_i) \tag{1}$$

(**exercise**). The formulas $\exists v\ \bigwedge \Sigma_i$ are said to be *primitive*. In general, a **primitive** formula is a formula

$$\exists u_0 \ \cdots \exists u_{n-1} \ \bigwedge \Sigma,$$

where Σ is a *finite* non-empty set of atomic and negated atomic formulas. (Remember that $\bigwedge \Sigma$ is just an abbreviation for $\phi_0 \wedge \ldots \wedge \phi_{n-1}$, where the formulas ϕ_i compose Σ; so Σ must be finite since formulas must have finite length. Also, formulas have *positive* length, so Σ must be non-empty. However, the notation $\bigwedge \varnothing$ could be understood to stand for a validity.)

Using (1), we can adjust the induction above to show that T *admits quantifier-elimination, provided that every primitive formula with one (existential) quantifier is equivalent* modulo T *to an open formula.*

Henceforth suppose $\mathcal{L}$ is $\{<\}$, and $\mathrm{TO} \subseteq T$; so T is *a* theory of total orders. Then we can improve **9** even more. Indeed, the atomic formulas of $\mathcal{L}$ now are $x = y$ and $x < y$, where x and y are variables. Moreover,

$$\mathrm{TO} \models (\neg(x < y) \leftrightarrow (x = y \vee y < x)),$$
$$\mathrm{TO} \models (\neg(x = y) \leftrightarrow (x < y \vee y < x)).$$

Hence, in $\mathcal{L}$, any formula is equivalent, *modulo* TO, to the result of replacing each negated atomic sub-formula with the appropriate disjunction of atomic formulas. If this replacement is done to a formula in disjunctive normal form, then the new formula will have a disjunctive normal form that involves no negations. So T admits quantifier-elimination, provided that every formula

$$\exists v \ \bigwedge \Sigma$$

is equivalent, *modulo* T, to an open formula, where now Σ is a set of atomic formulas.

Using this criterion, we shall show that TO^* admits quantifier-elimination:

10 Theorem. TO* *admits full elimination of quantifiers.*

Proof. Let Σ be a finite, non-empty set of atomic formulas (in the signature $\{<\}$). Let X be the set of variables appearing in formulas in Σ; that is,

$$X = \bigcup_{\alpha\in\Sigma} \operatorname{fv}(\alpha).$$

Then X is a finite non-empty set; say

$$X = \{x_0, \dots, x_n\}.$$

Suppose $\mathfrak{A}$ is an $\mathcal{L}$-structure, and $\vec{a} \in A^{n+1}$. If α is an atomic formula of $\mathcal{L}$ with variables from X, we can let $\alpha(\vec{a}\,)$ be the result of replacing each x_i in α with a_i. Then we can let

$$\Sigma(\vec{a}\,) = \{\alpha(\vec{a}\,) : \alpha \in \Sigma\}.$$

Suppose in fact

$$\mathfrak{A} \models \mathrm{TO} \cup \{\bigwedge \Sigma(\vec{a}\,)\}.$$

Let us define $\Sigma_{(\mathfrak{A},\vec{a}\,)}$ as the set of atomic formulas α such that $\operatorname{fv}(\alpha) \subseteq X$ and $\mathfrak{A} \models \alpha(\vec{a}\,)$. Then

$$\Sigma \subseteq \Sigma_{(\mathfrak{A},\vec{a}\,)}.$$

Moreover, once Σ has been chosen, *there are only finitely many possibilities for the set* $\Sigma_{(\mathfrak{A},\vec{a}\,)}$. Let us list these possibilities as

$$\Sigma_0, \dots, \Sigma_{m-1}.$$

Now, possibly $m = 0$ here. In this case,

$$\mathrm{TO} \models (\exists v \, \bigwedge \Sigma \leftrightarrow v \neq v),$$

so we are done. Henceforth we may assume $m > 0$. If $\mathfrak{B} \models \mathrm{TO} \cup \{\bigwedge \Sigma(\vec{b}\,)\}$, then

$$\mathfrak{B} \models \bigwedge \Sigma_i(\vec{b}\,)$$

for some i in m. Therefore

$$\mathrm{TO} \models (\bigwedge \Sigma \leftrightarrow \bigvee_{i<m} \bigwedge \Sigma_i),$$

and hence

$$\mathrm{TO} \models (\exists v \, \bigwedge \Sigma \leftrightarrow \bigvee_{i<m} \exists v \, \bigwedge \Sigma_i).$$

Therefore, for our proof of quantifier-elimination, we may assume that Σ *is* one of the sets $\Sigma_{(\mathfrak{A},\vec{a}\,)}$ (so that, in particular, $m = 1$).

Now partition Σ as $\Gamma \cup \Delta$, where no formula in Γ, but every formula in Δ, contains v. There are two extreme possibilities:

($*$) Suppose $\Gamma = \varnothing$. Then $X = \{v\}$ (since if $x \in X \smallsetminus \{v\}$, then $(x = x) \in \Gamma$). Also, $\Sigma = \Delta = \{v = v\}$, so

$$\models (\exists v \bigwedge \Sigma \leftrightarrow v = v),$$

and we are done in this case.

($\dagger$) Suppose $\Delta = \varnothing$. Then $v \notin X$, and

$$\models (\exists v \bigwedge \Sigma \leftrightarrow \bigwedge \Sigma),$$

so we are done in *this* case.

Henceforth, suppose neither Γ nor Δ is empty. Then

$$\models (\exists v \bigwedge \Sigma \leftrightarrow \bigwedge \Gamma \wedge \exists v \bigwedge \Delta).$$

We shall show that

$$\mathrm{TO}^* \models (\exists v \bigwedge \Sigma \leftrightarrow \bigwedge \Gamma), \tag{2}$$

which will complete the proof. To show (2), it is enough to show

$$\mathrm{TO}^* \models (\bigwedge \Gamma \to \exists v \bigwedge \Delta).$$

But this follows from the definition of TO^*:

Indeed, remember that Σ is $\Sigma_{(\mathfrak{A},\vec{a}\,)}$. Hence, for all i and j in $n+1$, we have

$$\begin{aligned} a_i < a_j &\iff (x_i < x_j) \in \Sigma; \\ a_i = a_j &\iff (x_i = x_j) \in \Sigma. \end{aligned}$$

We have $v \in X$. We can relabel the elements of X as necessary so that v is x_n and

$$a_0 \leqslant \ldots \leqslant a_{n-1}.$$

(Here, $a_i \leqslant a_{i+1}$ means $a_i < a_{i+1}$ or $a_i = a_{i+1}$ as usual.) Suppose $\mathfrak{B} \models \mathrm{TO}^*$, and B^n contains $\vec{b}$ such that $\mathfrak{B} \models \bigwedge \Gamma(\vec{b}\,)$. We have to show that there is c in B such that $\mathfrak{B} \models \bigwedge \Delta(\vec{b}\,, c)$. Now, for all i and j in n, we have

$$\begin{aligned} b_i < b_j &\iff a_i < a_j; \\ b_i = b_j &\iff a_i = a_j. \end{aligned}$$

Because $\mathfrak{B}$ is a model of TO^* (and not just TO), we can find c as needed according to the relation of a_n with the other a_i:

($*$) If $a_n = a_i$ for some i in n, then let $c = b_i$.

($\dagger$) If $a_{n-1} < a_n$, then let c be greater than b_{n-1}.

($\ddagger$) If $a_n < a_0$, then let c be less than b_0.

($\S$) If $a_k < a_n < a_{k+1}$, then we can let c be such that $b_k < c < b_{k+1}$.

This completes the proof that TO^* admits quantifier-elimination. □

We have proved more than quantifier-elimination: we have shown that, *modulo* TO^*, the formula $\exists v\ \bigwedge\Sigma$ is equivalent to $v \neq v$ or $v = v$ or an open formula *with the same free variables as* $\exists v\ \bigwedge\Sigma$. In the proof, we introduced $v \neq v$ simply as a formula ϕ such that $\mathfrak{A} \not\models \phi$ for every structure $\mathfrak{A}$. Such a formula corresponds to a nullary Boolean connective, namely an **absurdity** (the negation of a validity). We used 0 as such a connective; but let us now use $\bot$.

Likewise, instead of $v = v$, we can use, as a validity, the nullary Boolean connective $\top$. From the last proof, therefore, we have:

11 Porism. *In the signature* $\{<\}$*, with the nullary connectives* $\bot$ *and* $\top$ *allowed, every formula is equivalent* modulo TO^* *to an open formula with the same free variables.*

In a signature of first-order logic without constants, an open *sentence* consists entirely of Boolean connectives, with no propositional variables; so it is either an absurdity or a validity. As a consequence, we have:

12 Theorem. TO^* *is a complete theory.*

Proof. By the porism, every *sentence* is equivalent to an open *sentence*; as just noted, such a sentence is an absurdity or a validity. Suppose $\mathrm{TO}^* \models (\sigma \leftrightarrow \bot)$. But $\models (\sigma \leftrightarrow \bot) \leftrightarrow \neg\sigma$; so $\mathrm{TO}^* \models \neg\sigma$. Similarly, if $\mathrm{TO}^* \models (\sigma \leftrightarrow \top)$, then $\mathrm{TO}^* \models \sigma$. Hence, for all sentences σ, if $\mathrm{TO}^* \not\models \sigma$, then $\mathrm{TO}^* \models \neg\sigma$. Therefore TO^* is complete by **6**. □