Set theory exercises

David Pierce

Spring semester, 2010/11

section 1

The first set of 21 problems was available by March 29, 2011; the first exam was on April 8.

1. Without using the Completeness Theorem, show that it is a logical theorem that sets exist.

2. Ordered pairs are defined by $(a, b) = \{\{a\}, \{b\}\}$, and this ensures

$$(a, b) = (c, d) \Leftrightarrow a = c \;\&\; b = d.$$

If $\boldsymbol{C}$ and $\boldsymbol{D}$ are possibly-proper classes, find a way to define a class $(\boldsymbol{C}, \boldsymbol{D})$ so that

$$(\boldsymbol{C}, \boldsymbol{D}) = (\boldsymbol{E}, \boldsymbol{F}) \Leftrightarrow \boldsymbol{C} = \boldsymbol{E} \;\&\; \boldsymbol{D} = \boldsymbol{F}.$$

3. **Triples** can be defined by

$$\{a, b, c\} = \{a, b\} \cup \{c\}.$$

Given a class $\boldsymbol{C}$, show that there is a bijection between $(\boldsymbol{C} \times \boldsymbol{C}) \times \boldsymbol{C}$ and the class defined by

$$\exists y\, \exists z\, \exists w\, (y \in \boldsymbol{C} \;\&\; z \in \boldsymbol{C} \;\&\; w \in \boldsymbol{C} \;\&\; x = \{\{y\}, \{y, z\}, \{y, z, w\}\}).$$

4. We define $a' = a \cup \{a\}$. Prove that

$$a' = \bigcap\{x \colon a \in x \;\&\; a \subseteq x\}.$$

5. We have defined an **ordinal** as a transitive set that is well-ordered by membership. Show that, if $a = \{a\}$, then a is not an ordinal.

6. The class of all ordinals is $\mathbf{ON}$. Let ω be defined as the class of ordinals that neither are limits nor contain limits. Using this only, show

$$\omega = \bigcup \omega.$$

7. Prove that an ordinal α is 0 or a limit if and only if $\alpha = \bigcup \alpha$.

8. Using that ω satisfies the Peano Axioms (Theorem 29 of the notes), prove the various unproved lemmas and theorems about addition, multiplication, and exponentiation on ω in §3.4, 'Arithmetic', of the notes.

9. We know by the Recursion Theorem that there is a unique homomorphism from $(\omega, 0, ')$ to $(\mathbf{V}, 0, x \mapsto \{x\})$. Let $\boldsymbol{Z}$ be the image of ω under this homomorphism. Find a way to define $\boldsymbol{Z}$, and to prove that $(\boldsymbol{Z}, 0, x \mapsto \{x\})$ satisfies the Peano Axioms, *without* any reference to ω. (Here $\boldsymbol{Z}$ stands for Zermelo, because *his* class of natural numbers was this.)

10. Prove that if ω is a proper class, it is $\mathbf{ON}$.

11. By the Replacement Axiom, every function whose domain is a set is a set, since if $\boldsymbol{F}$ is a function with domain a, then $\boldsymbol{F}$ is the range of the function $x \mapsto (x, \boldsymbol{F}(x))$. Hence there is a class of all functions from a to a given class $\boldsymbol{D}$. We can denote this class of functions by

$${}^{a}\boldsymbol{D}.$$

Given also a good order $(\boldsymbol{C}, <)$, suppose $\boldsymbol{E}$ is the class of all functions whose domains are sections of $\boldsymbol{C}$ and whose ranges are included in $\boldsymbol{D}$. That is,

$$\boldsymbol{E} = \{x \colon \exists y\, (y \in \boldsymbol{C} \mathbin{\&} x \in {}^{\operatorname{pred}(y)}\boldsymbol{D})\}.$$

Say $\boldsymbol{F}$ is a function from $\boldsymbol{E}$ to $\boldsymbol{D}$. Show that there is a unique function $\boldsymbol{G}$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ such that, for all a in $\boldsymbol{C}$,

$$\boldsymbol{G}(a) = \boldsymbol{F}(\boldsymbol{G} \upharpoonright \operatorname{pred}(a)).$$

12. Prove that every initial segment of a well-ordered class is either the class itself or a section of it.

13. Prove that there is at most one embedding of one well-ordered class in another such that the range of the embedding is an initial segment.

14. Prove that the union of a set of transitive sets is transitive, and the union of a set of ordinals is either an ordinal or **ON** itself.

15. Prove that, if b is a set of ordinals, then $\bigcup\{x'\colon x \in b\}$ is the least strict upper bound of b.

16. Prove from the definition that **ON** contains 0 and is closed under $x \mapsto x'$.

17. If $c \subseteq \mathbf{ON}$, prove that $\bigcup\{x'\colon x \in c\}$ is the least of the upper bounds of c that are in *in* c. [This doesn't make sense; 'in *in*' should have been '*not in*'. This would have made the problem a repetition of 15: so the problem should have been deleted.]

18. Prove that $a \times b$ is a set.

19. Prove that the lexicographic ordering is indeed a linear ordering.

20. Use transfinite induction to prove that, if $\alpha \leqslant \beta$, then $\alpha + x = \beta$ has a solution.

21. Find an ordinal α such that $\omega + \alpha = \alpha$.

section 2

The next exam, on May 6, concerned the following two problems:

22. Find the Cantor normal forms of sums, products, and powers of ordinals given in Cantor normal form.

23. Using transfinite induction, prove all of our theorems about ordinal arithmetic from the *recursive* definitions of addition, multiplication, and exponentiation of ordinals.

section 3

The third exam, May 25, concerns cardinality and some basics of the 'well-founded universe' **WF** as discussed in class. Such things are treated by the following problems. First, the Axiom of Choice is not assumed:

24. Show that Zorn's Lemma (every ordered set whose every linearly ordered subset has an upper bound has a maximal element) implies the Axiom of Choice (every set has a choice-function). *Suggestion:* Given a set a, find an appropriate ordered set $(b, \subset)$ of functions such that a maximal element of b will be a choice-function for a. *Note:* The set of choice-functions for subsets of a is *not* an appropriate ordered set b. (Earlier editions of the course notes suggested that it was; this was a mistake.)

Now the Axiom of Choice is assumed, so that cardinal exponentiation is defined.

25. Show that, for all cardinals κ, λ, μ, and ν,

$$\begin{gathered}\kappa^0 = 1, \qquad 0^\lambda = \begin{cases} 1, & \text{if } \lambda = 0, \\ 0, & \text{if } \lambda > 0, \end{cases} \\ \kappa^1 = \kappa, \qquad 1^\lambda = 1, \\ \kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu, \quad \kappa^{\lambda\cdot\mu} = (\kappa^\lambda)^\mu, \\ \kappa \leqslant \mu \;\&\; \lambda \leqslant \nu \Rightarrow \kappa^\lambda \leqslant \mu^\nu. \end{gathered}$$

26. We have various operations on sets, such as $\smallsetminus$, $\cap$, $\cup$, $\bigcap$, $\bigcup$, $\mathscr{P}$, $\times$, and $(x, y) \mapsto {}^x y$. If various compositions of these operations are applied to sets of known cardinality, what are the possible cardinalities of the results? (These cardinalities may have to be given in the form $\beth_\alpha$ or $2^{\aleph_\alpha}$.)

27. We can define a function $\boldsymbol{F}$ from $\mathbf{ON} \times \mathbf{CN}$ to $\mathbf{CN}$ by

$$\boldsymbol{F}(0, \kappa) = 1, \qquad \boldsymbol{F}(\alpha', \kappa) = \kappa^{\boldsymbol{F}(\alpha,\kappa)}, \qquad \boldsymbol{F}(\beta, \kappa) = \sup_{\alpha<\beta} \boldsymbol{F}(\alpha, \kappa),$$

where β is a limit. Now define $\boldsymbol{G}$ on the class of infinite cardinals by

$$\boldsymbol{G}(\kappa) = \boldsymbol{F}(\alpha, \kappa),$$

where $\kappa = \aleph_\alpha$. Show that

$$\boldsymbol{G}(\aleph_\alpha) = \begin{cases} 1, & \text{if } \alpha = 0, \\ \beth_{\alpha-1}(\aleph_\alpha), & \text{if } 0 < \alpha < \omega, \\ \beth_\alpha(\aleph_\alpha), & \text{if } \omega \leqslant \alpha, \end{cases}$$

for all α, where

$$\beth_0(\kappa) = \kappa, \qquad \beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}, \qquad \beth_\beta(\kappa) = \sup_{\alpha<\beta} \beth_\alpha(\kappa)$$

where β is a limit.

In the last problem, note that $\beth_\alpha(\aleph_0) = \beth_\alpha$. Recall that the function $\mathbf{R}$ is defined on $\mathbf{ON}$ by

$$\mathbf{R}(0) = 0, \qquad \mathbf{R}(\alpha') = \mathscr{P}(\mathbf{R}(\alpha)), \qquad \mathbf{R}(\beta) = \bigcup_{\alpha<\beta} \mathbf{R}(\alpha).$$

28. Show that

$$\alpha < \beta \Rightarrow \mathbf{R}(\alpha) \subseteq \mathbf{R}(\beta), \qquad \alpha < \beta \Rightarrow \mathbf{R}(\alpha) \in \mathbf{R}(\beta).$$

29. Show that

$$\operatorname{card}(\mathbf{R}(\omega + \alpha)) = \beth_\alpha.$$

30. If $\alpha \leqslant \beta$, show

$${\aleph_\alpha}^{\aleph_\beta} = 2^{\aleph_\beta}, \qquad {\beth_\alpha}^{\beth_\beta} = \beth_{\beta+1}.$$

31. If $\alpha > \beta$, show

$$\aleph_\alpha \leqslant {\aleph_\alpha}^{\aleph_\beta} \leqslant 2^{\aleph_\alpha}, \qquad \beth_\alpha \leqslant {\beth_\alpha}^{\beth_\beta} \leqslant \beth_{\alpha+1}.$$

32. Supposing

$$2 \leqslant \kappa, \qquad 1 \leqslant \lambda, \qquad \aleph_0 \leqslant \kappa + \lambda,$$

show that

$$\max(\kappa, 2^\lambda) \leqslant \kappa^\lambda \leqslant \max(2^\kappa, 2^\lambda).$$

33. If $k \in \omega$, show that

$${\beth_{\alpha+k}}^{\beth_\alpha} = \beth_{\alpha+\max(1,k)}.$$

34. Compute

$${\aleph_{\omega^2+\omega}}^{\aleph_{\omega^3}}.$$

35. Find

$$\sup\{\beth_1, {\beth_1}^{\beth_1}, {\beth_1}^{{\beth_1}^{\beth_1}}, \dots\}.$$