MATH 303, FINAL EXAMINATION *SOLUTIONS*

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Problem 1. *In the 8th century* B.C.E., *the colony of Cumae (Κύμη) was founded, near what is now Naples, by settlers from Euboea (Eğriboz), and also from Cyme (Κύμη) in western Anatolia near what is now Aliağa.*[1] *From the Greek alphabet as used in Cumae, the Latin alphabet was ultimately derived; this came to have 23 letters:*

A B C D E F G H I K L M N O P Q R S T V X Y Z.

In the year 863 C.E., *a monk from Salonica named Cyril invented the so-called Glagolitic alphabet in order to translate holy scripture from Greek into Old Bulgarian. Soon after that, the simpler Cyrillic alphabet was invented.*[2] *After some changes (such as the abolition of a few letters by the Soviet government in 1918), the Cyrillic alphabet became the 33-letter Russian alphabet of today:*

А Б В Г Д Е Ё Ж З И Й К Л М Н О П Р С Т У Ф Х Ц Ч Ш Щ Ъ Ы Ь Э Ю Я.

This alphabet retains 19 of the 24 letters of the Greek alphabet, in their original order, though not always in the original form. What are *the 24 letters of the Greek alphabet?*

Solution. *Α Β Γ Δ Ε Ζ Η Θ Ι Κ Λ Μ Ν Ξ Ο Π Ρ Σ Τ Υ Φ Χ Ψ Ω*, or

α β γ δ ϵ ζ η θ ι κ λ μ ν ξ ο π ρ σ τ υ ϕ χ ψ ω.

Remark. Most people seem to have learned the alphabet for this exam. If this had been so on the first exam, I may not have asked for the alphabet on *this* exam.

Problem 2. *Does a square have a ratio to its side? Explain.*

Solution. No, since no multiple of the side can exceed the square.

Remark. This problem alludes to Definition 4 of Book V of the *Elements:*

> Magnitudes are said to *have a ratio* to one another which are capable, when multiplied, of exceeding one another.

Euclid does not seem to *refer* to this definition later; but (as we discussed in class) he *uses* the definition implicitly, in Proposition V.16 for example, where there is an unstated assumption that A and C have a ratio, and (therefore) B and D have a ratio. In his 'quadrature of the parabola,' discussed on the last day of class, Archimedes assumes that, if two areas are unequal, then their difference *has a ratio* (in the sense of Euclid) to either of the areas.

Date: Tuesday, January 12, 2010.

[1]Paul Harvey, *The Oxford Companion to Classical Literature* (1980); Bilge Umar, *Türkiye'deki Tarihsel Adlar* (İstanbul: İnkilâp, 1993).

[2]S. H. Gould, *Russian for the Mathematician* (Springer-Verlag, Berlin–Heidelberg–New York, 1972). Many alphabets can be seen in Carl Faulmann, *Yazı Kitabı* (Türkiye İş Bankası Kültür Yayınları, 2001).

Problem 3. *Suppose a magnitude A has a ratio to a magnitude B, and a magnitude C has a ratio to a magnitude D. What does it mean to say that A has the* same *ratio to B that C has to D (according to Definition 5 of Book V of Euclid's* Elements*)?*

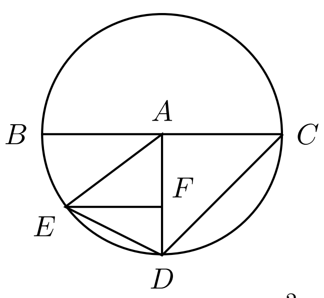
Solution. If equimultiples mA and mC of A and C be taken, and other equimultiples nB and nD of B and D be taken, then

$$mA > nB \text{ if and only if } mC > nD,$$
$$mA = nB \text{ if and only if } mC = nD,$$
$$mA < nB \text{ if and only if } mC < nD.$$

Remark. The definition of ratio is perhaps the most important sentence in Euclid. Euclid of course does not use special notation for a multiple of a magnitude.

Problem 4. *Suppose a straight line AB is bisected at C, and another point, D, is chosen on AB. What is the relation between the squares on AC and CD and the rectangle contained by AD and DB?*

Solution. $AC^2 = CD^2 + AD.DB$ [by Euclid's II.5].



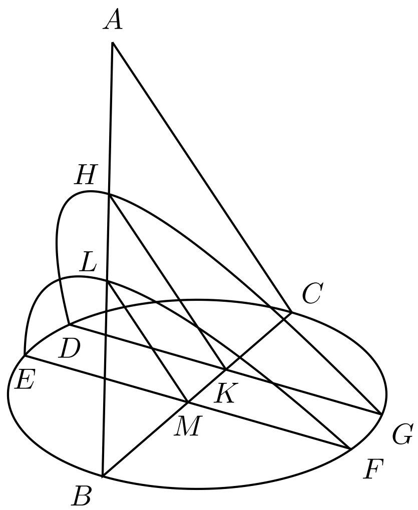
Problem 5. *In the diagram, BAC is the diameter of a circle, A is the center, and AD is at right angles to BC. Straight line DC is drawn. From a point E on the circumference between B and D, the straight line EF is drawn at right angles to AD, and EA and ED are drawn.*

*Show that the square on DE has the same ratio to the square on DC that the straight line DF has to DA. (*Suggestion: *express DE^2 and DC^2 in terms of DF, FA, and DA.)*

Solution. Just compute: $DC^2 = 2DA^2$, while

$$DE^2 = DF^2 + FE^2 = DF^2 + EA^2 - FA^2 = DF^2 + DA^2 - FA^2$$
$$= 2DF^2 + 2DF.FA = 2DF.DA, \quad (*)$$

so $DE^2 : DC^2 :: 2DF.DA : 2DA.DA :: DF : DA$.



Remark. The equation $(*)$ or rather $DA^2 + DF^2 = 2DF.DA + FA^2$, happens to be the symbolic expression of Euclid's Proposition II.7. I obtained this problem from Isaac Newton, who writes in the *Principia,* in the scholium after the Laws of Motion:

> It is a proposition very well known to geometers that the velocity of a pendulum at the lowest point is as the chord of the arc which it describes in falling.

Problem 6. *In the diagram, ABC is an axial triangle of a cone whose base is the circle $CDEBFG$, and DKG and EMF are at right angles to BC. Planes through DKG and EMF cut the cone, making sections DHG and ELF, with diameters HK and LM, respectively; and these diameters are parallel to AC. The* **parameters** *(the 'upright sides' or* latera recta*) of the sections are not shown; but let them be HN and LP. What is the ratio of HN to LP (in terms of straight lines that* are *shown in the diagram)?*

Solution. Since $HN : HA :: BC^2 : BA.AC$ and $LP : LA :: BC^2 : BA.AC$ [By I.11 of Apollonius], we have $HN : HA :: LP : LA$, and alternately

$$HN : LP :: HA : LA.$$

Remark. One may alternatively observe that $DK^2 = HN.HK$, but also $DK^2 = BK.KC$, and similarly for EM. Hence

$$DK^2 : EM^2 :: HN : LP \,\&\, HK : LM, \qquad (\dagger)$$

but also

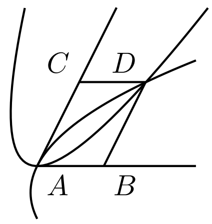
$$\begin{aligned} DK^2 : EM^2 &:: BK : BM \,\&\, KC : MC \\ &:: HK : LM \,\&\, HA : LA, \end{aligned}$$

and therefore $HN : LP :: HA : LA$. Now, from (†), one might write

$$\begin{aligned} HN : LP &:: DK^2 : EM^2 \,\&\, LM : HK \\ &:: DK^2.LM : EM^2.HK; \end{aligned}$$

but this isn't the best answer. A better answer is $HN : LP :: CK : CM$, but this still refers to the particular choice of base for the cone, when the parabolas themselves do not depend on this choice.

Problem 7. *We know that an ellipse or an hyperbola has two 'conjugate' diameters, each diameter being situated ordinatewise with respect to the other. A parabola cannot have conjugate diameters in this sense. Nonetheless, suppose, in the diagram, AB is the diameter of a parabola, and AC is drawn ordinatewise, and AC is also the diameter of another parabola, and AB is situated ordinatewise with respect to AC. Suppose the two parabolas meet at D (as well as at A). Let the respective ordinates DB and DC be dropped. Finally, suppose the parabola with diameter AB has parameter E (not shown), and the parabola with diameter AC has parameter F.*

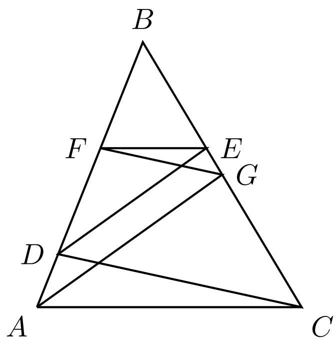


Show that

$$E : AC :: AC : AB, \qquad AC : AB :: AB : F.$$

*(*Remark. *It follows then that E is to F as the* cube *on AC is to the cube on AB. In particular, if E is twice F, then the cube on AC is double the cube on AB. According to Eutocius in his* Commentary on Archimedes's Sphere and Cylinder, *Menaechmus discovered this method of 'duplicating' the cube, along with another method involving a parabola and a hyperbola. This work is the earliest known use of conic sections. For Menaechmus however, the angle BAC would have been right.)*

Solution. Since $AB.E = BD^2 = AC^2$, we have $E : AC :: AC : AB$; the other proportion is similar.



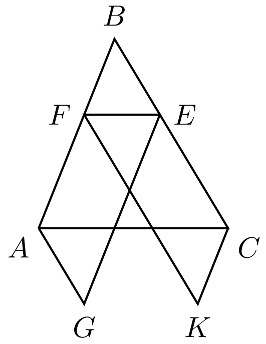
Problem 8. *In the triangle ABC shown, FG is parallel to DC, and DE is parallel to AG. Show that AC is parallel to FE. (You may use the theory of proportion developed in Books V and VI of the* Elements. *In that case, you will probably want to use* alternation: *if $A : B :: C : D$, then $A : C :: B : D$. You may use also that if $A : B :: E : F$ and $B : C :: D : E$, then $A : C :: D : F$. Alternatively, it is possible to avoid the theory of proportion by showing, as a lemma, that, in the diagram,*

FE is parallel to AC if and only if the parallelogram bounded by BF and BC, in the angle B, is equal to the parallelogram bounded by BE and BA. Or maybe you can find another method. In modern terms, this problem can be set in a two-dimensional vector-space; but if the scalar field of that space is non-commutative, then the claim is false.)

Solution. Because of the parallels, we have

$$BF : BD :: BG : BC, \qquad BD : BA :: BE : BG;$$

therefore [by the suggested result, which is V.23 of Euclid] $BF : BA :: BE : BC$, which yields the parallelism of FE and AC.

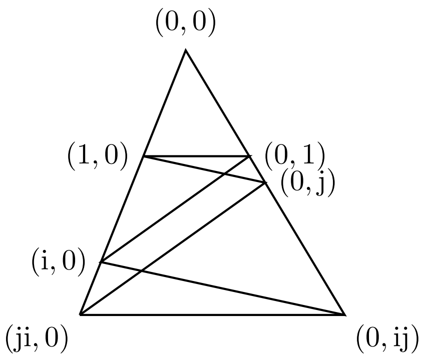


Remark. I learned this short proof from some students' papers. I had previously found a longer argument, which *did* use alternation.

Really, Euclid's VI.2 gives us only (for example) $DF : FB :: CG : GB$; this is equivalent to $DB : FB :: CB : GB$ by V.17 and 18.

As noted, we don't really need to use proportions, just that, in the diagram here, the parallelograms $ABEG$ and $BCKF$ are equal (by cutting and pasting) if and only if FE is parallel to AC. Let's use $BA.BE$ and $BF.BC$ to denote these parallelograms respectively. In the problem then, we have $BF.BC = BG.BD = BA.BE$, so $AE \parallel BE$. This problem is inspired by Descartes, who, in his *Geometry,* observes that, if (in the original diagram) BF is a unit length, and $BG = a$, while $BD = b$, then we can define the product ba as (the length of) BC. Descartes does not show that the multiplication so defined is commutative. But it *is* commutative, by this problem. Indeed, if $BE = BF$, then $BA = ab$, but also $BA = BC$, so $ab = ba$.

However, if you know about the skew-field $\mathbb{H}$ of *quaternions,* then suppose the diagram sits in the vector-space $\mathbb{H}^2$ as shown below. Then the assumptions of parallelism in the problem hold here, since for example $(0, \mathrm{ij}) - (\mathrm{i}, 0)$ is a scalar multiple of $(0, \mathrm{j}) - (1, 0)$. However, $(0, \mathrm{ij}) - (\mathrm{ji}, 0) = \mathrm{ij}(-1, 1)$, which is not a scalar multiple of $(0, 1) - (1, 0)$.



Bonus. *How can this exam and this course be improved? (Responses may be submitted also by email in the next few days:* `dpierce@metu.edu.tr`*. Meanwhile,* iyi çalışmalar; ondan sonra, iyi tatiller!*)*

Solution. [I shall summarize the responses and make my own comments elsewhere.]

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