Countable Borel Equivalence Relations

Burak Kaya

Rutgers University

bkaya@math.rutgers.edu

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Standard Borel Spaces

Definition

- A topological space (X, τ) is called a Polish space if it is separable and completely metrizable.
- A measurable space (X, Ω) is called a standard Borel space if Ω is the Borel σ-algebra $\mathcal{B}(\tau)$ of some Polish topology τ on X.
- E.g. $\mathbb{R}$, $[0,1]$, 2^ω, ω^ω,$[0,1]^\omega$

Definition

Let X, Y be standard Borel spaces.

- A map $\varphi : X \to Y$ is called Borel if it is measurable, i.e. $f^{-1}[B]$ is Borel for all Borel subsets $B \subseteq Y$.
- Equivalently, $\varphi : X \to Y$ is Borel if $graph(\varphi)$ is a Borel subset of $X \times Y$

Polish Spaces vs. Standard Borel Spaces

Theorem

A subspace S of a Polish space X is Polish if and only if it is a G_δ subset of X.

This means that we cannot pass to some arbitrary subspace if we want to keep the induced topology same and Polish. On the other hand:

Theorem

Let (X, τ) be a Polish space and $S \subseteq X$ be any Borel subset. Then there exists a Polish topology $\tau_S \supseteq \tau$ on X such that $\mathcal{B}(\tau_S) = \mathcal{B}(\tau)$ and S is clopen in τ_S.

Corollary

If $(X, \mathcal{B})$ is a standard Borel space and $Y \in \mathcal{B}$, then $(Y, \mathcal{B} \upharpoonright Y)$ is a standard Borel space.

The Borel Isomorphism Theorem

Definition

Two standard Borel spaces (X, Ω_1) and (Y, Ω_2) are called isomorphic if there exists a bimeasurable bijection between X and Y.

A bimeasurable version of Schroder-Bernstein theorem holds for standard Borel spaces. For any uncountable standard Borel space X, by embedding 2^ω into X, X into $[0,1]^\omega$ and $[0,1]^\omega$ into 2^ω in a bimeasurable way, we have:

Theorem (Kuratowski)

Any two uncountable standard Borel spaces are isomorphic.

Coding countably infinite first-order structures into Polish spaces

Let $\mathcal{L} = \{R_i : i \in I\}$ be a countable language where R_i is an n_i-ary relation symbol and let $X_{\mathcal{L}} = \prod_{i \in I} 2^{\omega^{n_i}}$. Then $X_{\mathcal{L}}$ is a Polish space elements of which code $\mathcal{L}$-structures with universe ω as follows. For any $x = (x_i)_{i \in I} \in X_{\mathcal{L}}$, the structure

$$M_x = (\omega, \{R_i^x\}_{i \in I})$$

represented by x is defined by:

$$R_i^x(k_1, ..., k_{n_i}) \Leftrightarrow x_i(k_1, ..., k_{n_i}) = 1$$

Example

If we let $\mathcal{L}$ consist of a single binary relation E, then the Polish space $2^{\omega \times \omega}$ codes the space of countable graphs with underlying set ω. For any such "graph" $x \in 2^{\omega \times \omega}$, there is an edge between the vertices i and j if and only if $x(i,j) = 1$

Coding countably infinite first-order structures into Polish spaces

Remark

If we consider the infinite symmetric group $Sym(\omega)$ as a subspace of the Baire space ω^ω, it becomes a Polish group with a natural Borel action on $X_{\mathcal{L}}$. Then $x, y \in X_{\mathcal{L}}$ are in the same $Sym(\omega)$-orbit if and only if $M_x \cong M_y$.

Given any $\mathcal{L}_{\omega_1,\omega}$ sentence ψ, the class of all structures with underlying set ω that models ψ, $Mod(\psi) = \{x \in X_{\mathcal{L}} : M_x \models \psi\}$ is an isomorphism-invariant Borel subset of $X_{\mathcal{L}}$.

Example

Let $\mathcal{L}$ consist of a single ternary relation. If we associate any countable group $(\omega, \cdot)$ with the characteristic function of $\cdot \subseteq \omega \times \omega \times \omega$, then the class of countable groups, being axiomatized by a $\mathcal{L}_{\omega_1,\omega}$-sentence, is a Borel subset of $X_{\mathcal{L}}$ and thus itself is a standard Borel space.

The Isomorphism Relation on $Mod(\psi)$

The isomorphism relation on $Mod(\psi)$ is given by

$$x \cong y \Leftrightarrow \exists g \in Sym(\omega)\ g \cdot x = y$$

and is an analytic equivalence relation, being the projection of graph of a Borel action, and need not be Borel in general.

Example (Mekler)

The isomorphism relation on the space of countable groups $\cong_{\mathcal{G}}$ is not Borel.

On the other hand, for the structures that are of "finite rank" in a broad sense, the isomorphism relation is a Borel relation. E.g. Finitely generated groups, finite rank torsion-free abelian groups, connected locally finite graphs,...

Standard Borel Space of Torsion-Free Abelian Groups of rank n

Recall that, up to isomorphism, torsion-free abelian groups of rank n are exactly additive subgroups of $\mathbb{Q}^n$ with n linearly independent elements. Then, for any $n \geq 1$, we can regard the set

$$R(\mathbb{Q}^n) = \{x \in 2^{\mathbb{Q}^n} : x \leqslant \mathbb{Q}^n \ \wedge \ \text{"}x \text{ contains } n \text{ linearly independent elements"}\}$$

as the space of torsion-free abelian groups of rank n. Observe that this set is a Borel subset of $2^{\mathbb{Q}^n}$ and is itself a standard Borel space.

Remark

If $A, B \in R(\mathbb{Q}^n)$, then

$$A \cong B \Leftrightarrow \exists \varphi \in GL_n(\mathbb{Q}) \ \varphi[A] = B$$

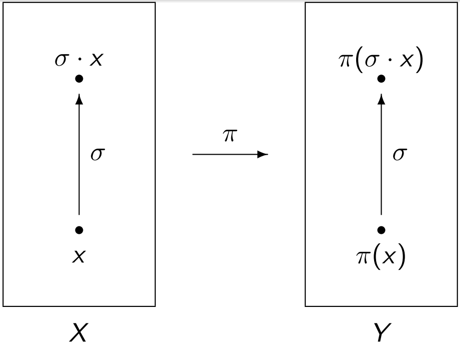
. This shows that $\cong$ on $R(\mathbb{Q}^n)$ is a Borel equivalence relation.

An example from topological dynamics

Fix some $n \geq 2$.

Definition

- A closed infinite subset S of the Cantor space $n^{\mathbb{Z}}$ is called a subshift if it is invariant under the shift operator $(\sigma(x))(k) = x(k+1)$.
- Two subshifts S and T are called topologically conjugate if there exists a homeomorphism $\psi : S \to T$ such that $\psi \circ \sigma = \sigma \circ \psi$



Standard Borel Space of Subshifts

Definition

Let X be a Polish space and $K(X)$ be the set of all non-empty compact subsets of X. Then the Vietoris topology on $K(X)$ generated by the sets $\{K \in K(X) : K \subseteq U\}$ and $\{K \in K(X) : K \cap U \neq \emptyset\}$ for U open in X is a Polish topology. If d is a complete metric on X inducing its Polish topology, then the Hausdorff metric

$$\delta_H(K, L) = max\{max_{x \in K} d(x, L), max_{x \in L} d(x, K)\}$$

is a compatible metric for the Vietoris topology.

Theorem

The collection $\mathcal{S}_n$ of subshifts of $n^{\mathbb{Z}}$ is a Borel subset of $K(n^{\mathbb{Z}})$, and hence is itself a standard Borel space and the topological conjugacy relation on it is a Borel equivalence relation.

Borel Equivalence Relations

Definition

- Let X be a standard Borel space. An equivalence relation $E \subseteq X^2$ is called Borel if it is a Borel subset of $X \times X$. A Borel equivalence relation is called countable if every E-equivalence class is countable.
- Let G be a Polish group. A standard Borel G-space is a standard Borel space X equipped with a Borel G-action. The corresponding orbit equivalence relation is denoted by E_G^X.

Example

Let G be a countable group endowed with discrete topology and X be a standard Borel G-space. Then, E_G^X is a countable Borel equivalence relation.

Borel Reducibility

Definition

Let E, F be Borel equivalence relations on standard Borel spaces X and Y respectively.

- We say E is Borel reducible to F, denoted by $E \leq_B F$, if there exists a Borel map $f : X \to Y$ such that for all $x, y \in X$

$$x \; E \; y \Leftrightarrow f(x) \; F \; f(y)$$

  In this case, f is said to be a reduction from E to F.
- $E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$.
- $E <_B F$ if $E \leq_B F$ but $F \nleq_B E$.

If E is Borel reducible to F, then the classification with respect to E is, intuitively speaking, no harder than the classification with respect to F. The intuition behind the requirement that f is Borel is that Borel maps are thought as "explicit computations".

Climbing up in the hierarchy of $\leq_B$

Remark

If we have a Borel equivalence relation on a countable standard Borel space of cardinality n (for $1 \leq n \leq \omega$), then it is trivially reducible to the identity relation Δ_n since any function that chooses an element from each class is a reduction.

Theorem (Silver)

Let E be a Borel equivalence relation on a standard Borel space. Then either $E \leq_B \Delta_\omega$ or $\Delta_{2^\omega} \leq_B E$.

Definition

A Borel equivalence relation E is called smooth if $E \leq_B \Delta_X$ for some (equivalently every) uncountable standard Borel space X.

Examples of smooth Borel equivalence relations

Example

Let E be a finite Borel equivalence relation on X, that is, a Borel equivalence relation with finite classes. Fix a Borel linear ordering $\leq$ on X. Then, E is smooth via the Borel map $f(x) =$ the $\leq$ -least element of $[x]_E$.

A more interesting example:

Example

The class of countable divisible abelian groups in the space of countable groups can be axiomatized by a $\mathcal{L}_{\omega_1,\omega}$-sentence ψ and hence, forms a standard Borel space on its own. Let $\cong_\psi$ denote the isomorphism relation on it. Any countable divisible abelian group G can be written as $(\oplus_{i \in r_0(G)} \mathbb{Q}) \oplus (\oplus_{p \in \mathbb{P}} \oplus_{j \in r_p(G)} \mathbb{Z}[p^\infty])$ where $0 \leq r_0(G), r_p(G) \leq \omega$ and these ranks determine G up to isomorphism. Then, the Borel map $f(G) = (r_0(G), r_2(G), r_3(G), ...)$ witnesses the fact that $\cong_\psi$ is smooth.

Examples of non-smooth Borel equivalence relation

Example

Let $\mathbb{Z}$ act on S^1 by $n \cdot e^{i\theta} \mapsto e^{i(\theta+n)}$. The orbit equivalence relation $E_{\mathbb{Z}}^{S^1}$ is non-smooth for if it were smooth, then there would be a Borel set of S^1 intersecting each equivalence class at exactly one point. But all such sets are necessarily non-measurable since the action is measure preserving.

Example

Let E_0 be the countable Borel equivalence relation on 2^ω defined by:

$$x \ E_0 \ y \Leftrightarrow \exists n \ \forall m \geq n \ x(m) = y(m)$$

Assume that there is a Borel reduction $f : 2^\omega \to [0,1]$ from E_0 to $\Delta_{[0,1]}$. If we endow 2^ω with its usual product probability measure, then both $f^{-1}[0,1/2]$ and $f^{-1}[1/2,1]$ are Borel tail events, and one of them has to have measure 1 by Kolmogorov 0-1 law. Continuing in this manner, we see that f is constant almost everywhere, which is a contradiction.

Back to climbing up in the hierarchy

It turns out that E_0 is the immediate successor of Δ_{2^ω} with respect to $\leq_B$

Theorem (Harrington-Kechris-Louvea)

Let E be a Borel equivalence relation on a standard Borel space. Then either $E \leq_B \Delta_{2^\omega}$ or $E_0 \leq_B E$.

Example ($\cong_1 \sim_B E_0$)

Let $\cong_1$ be the isomorphism relation for torsion-free abelian groups of rank 1. For any $G \in R(\mathbb{Q})$, $0 \neq x \in G$ and prime $p \in \mathbb{P}$, set the p-height of x to be $h_p(x) = sup\{n \in \omega : \exists y \in G \ p^n y = x\} \in \omega \cup \{\infty\}$ and let the characteristic of x be the sequence $\chi(x) = (h_p(x))_{p \in \mathbb{P}}$. In 1937, Baer proved that $G, H \in R(\mathbb{Q})$ are isomorphic if and only if for any non-zero $x \in G$, $y \in H$, $\chi(x)$ and $\chi(y)$ take the same values on almost all primes and they take the value ∞ on exactly the same indices. This condition defines an equivalence relation on $(\mathbb{N} \cup \{\infty\})^{\mathbb{P}}$ which is bireducible with E_0.

The Feldman-Moore Theorem

Recall that any countable discrete group G acting a standard Borel G-space induces a countable Borel equivalence relation as its orbit equivalence relation. Remarkably, the converse of this is also true:

Theorem (Feldman-Moore)

Let E be a countable Borel equivalence relation on a standard Borel space X. Then, there exists a countable discrete group G and a Borel G-action on X such that $E = E_G^X$. Moreover, G can be chosen such that

$$x \; E \; y \Leftrightarrow \exists g \in G \; g^2 = 1 \; \wedge \; g \cdot x = y$$

In order to prove this, we will need

Theorem (Lusin-Novikov Uniformization Theorem)

Let X, Y be standard Borel spaces and $E \subseteq X \times Y$ be a Borel relation such that each section E_x is countable. Then $proj_X(E)$ is Borel and $E = \bigcup_n f_n$ where f_n are partial Borel functions of X.

Sketch Proof of the Feldman-Moore Theorem

Let $E \subseteq X^2$ be a countable Borel equivalence relation with countable sections E_x for all x.

- By Lusin-Novikov uniformization theorem, there exists partial Borel functions on X such that $E = \bigcup_n f_n$. Without loss of generality, assume that $f_m \cap f_n = \emptyset$.
- Using the isomorphism between X and $[0,1]$, find disjoint Borel subsets A_p, B_p for each $p \in \omega$ such that $X^2 - \Delta_X = \bigcup_{p \in \omega} A_p \times B_p$.
- If we set $f_{nmp} = f_n \cap {f_m}^{-1} \cap (A_p \times B_p)$, then each f_{nmp} is a partial Borel bijection whose domain and range are disjoint.
- Extend each f_{nmp} to some Borel automorphism g_{nmp} of X so that $E = \bigcup g_{nmp}$ (this can be done in such a way that each g_{nmp} is an involution).
- Then $E = E_G^X$ for $G = \langle g_{nmp} \rangle$.

An important consequence of Feldman-Moore

Theorem (Dougherty-Jackson-Kechris)

There exists a universal countable Borel equivalence relation E_ω, i.e. for all countable Borel equivalence relations E we have $E \leq_B E_\omega$.

Definition

- Let $\mathbb{F}_\omega$ be the free group on ω-many generators.
- Define the Borel action of $\mathbb{F}_\omega$ on

$$(2^\omega)^{\mathbb{F}_\omega} = \{f | f : \mathbb{F}_\omega \to 2^\omega\}$$

by setting

$$(g.p)(h) = p(g^{-1}h)$$

for all $p : \mathbb{F}_\omega \to 2^\omega$ and let E_ω be the orbit equivalence relation of this action.

Proof that E_ω is universal

- Let E be a countable Borel equivalence relation on X. Then, by Feldman-Moore, there exists G such that $E = E_G^X$.
- G is a homomorphic image of $\mathbb{F}_\omega$, so we can find some Borel action of $\mathbb{F}_\omega$ inducing E as its orbit equivalence relation.
- Let $\{U_i\}_{i\in\omega}$ be a sequence of Borel subsets of X separating points and define the Borel map $f : X \to (2^\omega)^{\mathbb{F}_\omega}$ by $x \mapsto f_x$ where

$$(f_x(h))(i) = 1 \Leftrightarrow x \in h(U_i)$$

- Since U_i separates points, f is injective and

$$(g \cdot f_x(h))(i) = 1 \Leftrightarrow x \in (f_x(g^{-1}h))(i) = 1$$

$$\Leftrightarrow x \in g^{-1}h(U_i) \Leftrightarrow g \cdot x \in h(U_i)$$

$$\Leftrightarrow (f_{g\cdot x}(h))(i) = 1$$

Other examples of universal countable Borel equivalence relations

Remark

More generally, for any Borel action $G \curvearrowright X$ of some countable G, the corresponding orbit equivalence relation E_G^X is Borel reducible to the orbit equivalence relation of the shift action $G \curvearrowright (2^\omega)^G$ by the same proof.

Theorem (Clemens, 2009)

Topological conjugacy on the space of subshifts $\mathcal{S}_n$ is a universal countable Borel equivalence relation.

Theorem (Thomas-Velickovic, 1998)

The isomorphism relation on the space of finitely generated groups $\mathcal{G}_{fg}$ is a universal countable Borel equivalence relation.

Do we have anything between E_0 and E_ω?

Theorem (Hjorth, 1998 (for $n = 1$), Thomas, 2001 (for $n \geq 2$))

Let $\cong_n$ denote the isomorphism relation of torsion-free abelian groups of rank n. Then, $\cong_n <_B \cong_{n+1}$ for all $n \geq 1$.

Theorem (Adams-Kechris, 2000)

There exists 2^ω-many incomparable countable Borel equivalence relations.

Back to the space of subshifts

Definition

A subshift $S \subseteq n^{\mathbb{Z}}$ is called minimal if S has no proper σ-invariant closed subsets.

The subshifts constructed by Clemens to show universality of topological conjugacy on $\mathcal{S}_n$ are not minimal. If we restrict topological conjugacy to the standard Borel space of minimal subshifts $\mathcal{M}_n$, where does it fit in the picture?

Theorem (Gao-Jackson-Seward, 2011)

The topological conjugacy relation for minimal subshifts is not smooth.

Conjecture (Thomas)

The topological conjugacy relation for minimal subshifts is a universal countable Borel equivalence relation.