Small doubling problems in Baumslag-Solitar groups and sums of dilates

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Papers

Gregory A. Freiman, Marcel Herzog, P. L., Mercede Maj, Yonutz V. Stanchescu

$$Direct\ and\ inverse\ problems$$

$$in\ additive\ number\ theory\ and\ in\ non-abelian\ group\ theory$$

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Additive Number Theory

Direct and Inverse theorems

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Basic definition

Definition

If X, Y are subsets of a group G, then we denote

$$XY := \{xy \mid x \in X, y \in Y\} \quad \text{and} \quad X^2 := \{x_1x_2 \mid x_1, x_2 \in X\} .$$

If $X = \{x\}$, then we denote XY by xY and if $Y = \{y\}$, then we write Xy instead of $X\{y\}$.

If G is an additive group, then we denote

$$X + Y = \{x + y \mid x \in X, y \in Y\} \quad \text{and} \quad 2X = \{x_1 + x_2 \mid x_1, x_2 \in X\} .$$

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Gregory A. Freiman, Structure theory of set addition, *Astérisque*, **258** (1999), 1-33

"Thus a **direct problem** *in additive number theory is a problem which, given summands and some conditions, we discover something about the set of sums. An* **inverse problem** *in additive number theory is a problem in which, using some knowledge of the set of sums, we learn something about the set of summands."*

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Dilates

Subsets of $\mathbb{Z}$ of the form

$$r * A := \{rx : x \in A\},$$

where r is a **positive** integer and A is a **finite** subset of $\mathbb{Z}$, are called *r-dilates*.

Minkowski sums of dilates are defined as follows:

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$$|A + 2 * A| \geq 3|A| - 2.$$

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What about $|A + r * A|$ *, where* $r \geq 3$?

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$$\text{If} \quad |A + 2 * A| < 4|A| - 4, \; |A| \geq 3,$$

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Useful results

Write $[m, n] = \{x \in \mathbb{Z} \mid m \leq x \leq n\}$ and $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$.
Let A and B finite subsets of $\mathbb{Z}$.
It is well known that $|A + B| \geq |A| + |B| - 1$.
Let $A = \{a_0 < a_1 < ... < a_{k-1}\}$ be a finite increasing set of k integers.
By the *length* $\ell(A)$ of A we mean the difference

$$\ell(A) := \max(A) - \min(A) = a_{k-1} - a_0$$

between its maximal and minimal elements and

$$h_A := \ell(A) + 1 - |A|$$

denotes the number of *holes* in A, that is $h_A = |\ [a_0, a_{k-1}] \setminus A\ |$.
Finally, if $k \geq 2$, then we denote

$$d(A) := g.c.d.(a_1 - a_0, a_2 - a_0, ..., a_{k-1} - a_0).$$

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Theorem (V.F. Lev - P.Y. Smelianski and Y.V. Stanchescu)

Let A and B be finite subsets of $\mathbb{N}$ such that $0 \in A \cap B$. Define

$$\delta_{A,B} = \begin{cases} 1, & \text{if } \ell(A) = \ell(B), \\ 0, & \text{if } \ell(A) \neq \ell(B). \end{cases}$$

Then the following statements hold:

(i) *If* $\ell(A) = \max(\ell(A), \ell(B)) \geq |A| + |B| - 1 - \delta_{A,B}$ *and* $d(A) = 1$, *then*

$$|A + B| \geq |A| + 2|B| - 2 - \delta_{A,B}.$$

(ii) *If* $\max(\ell(A), \ell(B)) \leq |A| + |B| - 2 - \delta_{A,B}$, *then*

$$|A+B| \geq (|A|+|B|-1)+\max(h_A, h_B) = \max(\ell(A)+|B|, \ell(B)+|A|).$$

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Let $A = \{a_0 < a_1 < a_2 < \cdots < a_{k-1}\} \subset \mathbb{Z}$ *be a finite set of integers of size* $k = |A| \geq 1$. *Then the following statements hold.*

(a) *If* $1 \leq k \leq 2$, *then* $|A + 2 * A| = 3k - 2$ *and* A **is** *an arithmetic progression of size* k.

(b) *If* $k \geq 3$, *assume that* $|A + 2 * A| = (3k - 2) + h < 4k - 4.$
Then $h \geq 0, \quad |A + 2 * A| \geq 3k - 2$
and the set A *is a* **subset** *of an arithmetic progression*
$$P = \{a_0, a_0 + d, a_0 + 2d, \ldots, a_0 + (t-1)d\}$$
of size $|P|$ *bounded by* $|P| \leq k + h = |A + 2 * A| - 2k + 2 \leq 2k - 3.$

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Claim 1:

$$\ell(A) \leq k + \max(m, n) - 2 \leq 2k - 3.$$

For the proof of Claim 1 we shall use (i) of the Theorem of V.F. Lev - P.Y. Smelianski and Y.V. Stanchescu.

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$$|A + 2 * A| \geq (3k - 2) + h_A.$$

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$$\ell(A) \leq k + \max(m, n) - 2 \leq 2k - 3.$$

For the proof of Claim 1 we shall use (i) of the Theorem of V.F. Lev - P.Y. Smelianski and Y.V. Stanchescu.

Claim 2:

$$|A + 2 * A| \geq (3k - 2) + h_A.$$

Recall that $h_A = \ell(A) + 1 - |A|$. For the proof of Claim 2 we shall use Claim 1 and (ii) of the Theorem of V.F. Lev - P.Y. Smelianski and Y.V. Stanchescu.
We distinguish between two cases: $m \leq n$ and $n < m$.

Proof of the Theorem - sketch

In both cases we obtain that h_A, the total number of holes in the normal set A, satisfies

$$0 \leq h_A \leq |A + 2 * A| - (3k - 2) =: h \leq k - 3.$$

Hence

$$h \geq h_A \geq 0 \quad \text{and} \quad |A + 2 * A| \geq (3k - 2).$$

Moreover, the set A is contained in the arithmetic progression

$$P = \{a_0, a_0 + 1, a_0 + 2, ..., a_{k-1}\} = \{0, 1, 2, ..., a_{k-1}\}$$

of size $a_{k-1} + 1 = k + h_A \leq k + h \leq 2k - 3.$

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It follows that Theorem (b) holds for **normal** sets A satisfying the hypothesis.

Let now A be an **arbitrary** finite set of $k = |A| \geq 3$ integers satisfying the hypothesis. We define

$$B = \frac{1}{d(A)}(A - a_0) = \{\frac{1}{d(A)}(x - a_0) : x \in A\}.$$

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Small doubling property

Let G be a group and S a finite subset of G.
Let $S^2 = \{s_1 s_2 \mid s_1, s_2 \in S\}$.

Problem

What if the structure of S if $|S^2|$ satisfies

$$|S^2| \leq \alpha|S| + \beta,$$

for some small $\alpha \geq 1$ and small $|\beta|$?

Definition

The subset S of G is said to satisfy the *small doubling property* if

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The groups $\mathcal{BS}(m, n)$

For integers m and n, the general Baumslag-Solitar group $\mathcal{BS}(m, n)$ is a group with two generators a, b and one defining relation $b^{-1}a^m b = a^n$:

$$\mathcal{BS}(m, n) := \langle a, b \mid \ a^m b = ba^n \rangle.$$

"The Baumslag-Solitar groups are a particular class of two-generator one-relator groups which have played a surprisingly useful role in combinatorial and, more recently (the 1990s), geometric group theory. In a number of situations they have provided examples which mark boundaries between different classes of groups and they often provide a testbed for theories and techniques."

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When $\mathcal{BS}(m,n)$ is a Hopfian group

More generally:

$$\mathcal{BS}(m,n) = \langle a, b \mid\ a^m b = b a^n \rangle$$

is Hopfian if and only if :
(*i*) $|m| = |n|$ or
(*ii*) $|m| = 1$ or
(*iii*) $|n| = 1$ or
(*iv*) $\pi(m) = \pi(n)$ where $\pi(m)$ denotes the set of prime divisors of m.

We shall concentrate on the Baumslag-Solitar groups

$$\mathcal{BS}(1,n) = \langle a, b \mid\ ab = ba^n \rangle.$$

They are extensions of a copy of the additive group of n-adic rational numbers by an infinite cyclic group. They are orderable groups.

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The groups $\mathcal{BS}(1,n) = \langle a, b \mid \ ab = ba^n \rangle$

Let S be a finite subset of $\mathcal{BS}(1,n)$ of size k contained in the coset $b^r < a >$ for some $\mathbf{r} \geq \mathbf{0}$. Then

$$S = \{b^r a^{x_0}, b^r a^{x_1}, \ldots, b^r a^{x_{k-1}}\},$$

where $A = \{x_0, x_1, \ldots, x_{k_1-1}\}$ is a subset of $\mathbb{Z}$. We introduce now the notation

$$S = \{b^r a^x : x \in A\} =: b^r a^A.$$

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From $a^x b = ba^{nx}$ for each $x \in \mathbb{Z}$ it follows

$$(b^r a^x)(b^s a^y) = b^r(a^x b^s)a^y = b^r(b^s a^{n^s x})a^y = b^{r+s} a^{n^s x + y}.$$

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In particular

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Theorem

Suppose that $S = b^r a^A \subseteq \mathcal{BS}(1,n)$, $T = b^s a^B \subseteq \mathcal{BS}(1,n)$, *where* $r, s \in \mathbb{Z}$, $r, s \geq 0$ *and* A, B *are finite subsets of* $\mathbb{Z}$*. Then*

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The group $\mathcal{BS}(1,2) = \langle a, b \mid ab = ba^2 \rangle$

Theorem (J. Cilleruelo, M. Silva, C. Vinuesa)

If A is a finite set of integers, then $|A + 2 * A| \geq 3|A| - 2$ and $|A + 2 * A| = 3|A| - 2$ if and only if A is an arithmetic progression.

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If A is a finite set of integers, then $|A + 2 * A| \geq 3|A| - 2$ and $|A + 2 * A| = 3|A| - 2$ if and only if A is an arithmetic progression.

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

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What is the structure of an arbitrary subset of $\mathcal{BS}(1,2)$, satisfying some small doubling condition?

Very difficult!

Definition

Consider the submonoid

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Example

Let

$$S := a^{A_0} \cup \{b\} \subset \mathcal{BS}^{+}(1,2),$$

where

$$A_0 = \{0, 1, 2, ..., k-2\} \text{ and } k > 2 \text{ is even.}$$

The set S is clearly non-abelian, and it intersects non-trivially the two distinct cosets $1\langle a \rangle$ and $b\langle a \rangle$ of $\langle a \rangle$ in $\mathcal{BS}^{+}(1,2)$.
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Since

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Let $S \subseteq BS^+(1,2)$ be a finite set of size $k = |S|$. Suppose that $t \geq 1$ and there is $0 \leq j \leq t$ such that $k_j = |S_j| \geq 2$. Then S generates a non-abelian group.

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It remains to examine the following two cases:

(i) $j \geq 1$.

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If $j \geq 1$, then $m_j \geq 1$ and $k_j = |S_j| = |b^{m_j} a^{A_j}| \geq 2$ implies that $|A_j| \geq 2$. On the other hand, if $j = 0$ and $m_0 \geq 1$, then $k_0 = |S_0| = |b^{m_0} a^{A_0}| \geq 2$ implies that $|A_0| \geq 2$. In both cases, let $m = m_j$. Then $m \geq 1$ and there are two integers $x \neq y$ such that $\{b^m a^x, b^m a^y\} \subseteq S_j$. We conclude that

$$(b^m a^x)(b^m a^y) = b^{2m} a^{y+2^m x} \neq (b^m a^y)(b^m a^x) = b^{2m} a^{x+2^m y},$$

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$S = S_0 \cup S_1 \cup \ldots \cup S_t,\ t \geq 0\ ,\ S_i = b^{m_i} a^{A_i},\ 0 \leq m_0 < m_1 < \ldots < m_t$

Let $S \subseteq BS^+(1,2)$ be a finite set of size $k = |S|$.

Lemma (2)

Suppose that $t = 1$. *Then* $|S^2| \geq \frac{7}{2}|S| - 4$.

Lemma (3)

Suppose that $t \geq 2$. *If* $k_0 = |S_0| \geq 2$ *and* $k_i = |S_i| = 1$ *for every* $1 \leq i \leq t$, *then* $|S^2| \geq 4k - 5 > \frac{7}{2}|S| - 4$ *and the inequality is tight.*

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Thank you for the attention !

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