Some inverse problems in Group Theory

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UNIVERSITÀ DEGLI STUDI DI SALERNO

Groups and Topological Groups

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Background

Definition

Let $G(\cdot)$ be a group.
If S is a subset of G, then we denote

$$S^2 = \{xy \mid x, y \in S\}.$$

Problem

Let S be a finite subset of G of size k. Determine the **structure** *of S if*

$$|S^2| \leq f(k),$$

for some function of k.

Problems of this kind are called **inverse** problems.

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Since $xS \subseteq S^2$, for any $x \in S$ we have

$$|S^2| \geq |S|.$$

We shall consider problems of the following type:

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What is the **structure** *of S if $|S^2|$ satisfies*

$$|S^2| \leq \alpha|S| + \beta$$

for some small $\alpha \geq 1$ and small $|\beta|$?

Such problems are called **inverse problems of small doubling** type.

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If S is a finite subgroup of G, then $S^2 = S$, hence $|S^2| = |S|$.

This is a **direct** result.

The corresponding **inverse** problem is:

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This follows from the following theorem of Gregory Freiman that is the first inverse result of "small doubling" type.

Theorem (A)

Let S be a finite non-empty subset of a group G and suppose that

$$|S^2| < \frac{3}{2}|S|.$$

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Proof. Suppose $|S^2| = |S|$, then $S^2 = xS$ for every $x \in S$. By Theorem A, $xS = S^2 = uH$ for some subgroup H of G and some $u \in G$. It follows that $S = x^{-1}uH$, as claimed. Write $a = x^{-1}u$, then $S = aH$. In particular $|S^2| = |S| = |H|$. Furthermore we have:
$|S^2| = |aHaH| = |a^2H^aH| = |H^aH|$, hence $|H^aH| = |H|$, then $H^aH = H$, since $H \subseteq H^aH$. Therefore $H^a \subseteq H$ and $H^a = H$, as required.

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The first version of Theorem A was published in 1951 in a Russian journal. It was the beginning of what is now called the

Freiman′s structural theory of set addition.

The foundations for this theory were laid in the book:

G.A. Freiman,
Foundations of a structural theory of set addition
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By now, Freiman's theory had been extended tremendously.

It was shown by Freiman and others that problems in various fields may be looked at and treated as Structure Theory problems, including Additive and Combinatorial Number Theory, Group Theory, Integer Programming and Coding Theory.

H. Halberstam, **B.J. Green**, **I.Z. Ruzsa**, **T. Sanders**, **Y.V. Stanchescu**, **T.C. Tao**, ...

Background

Consider the group of the integers $\mathbb{Z}(+)$.

If S is subset of the integers consider:

$$S + S = \{x + y \mid x, y \in S\}.$$

It is easy to prove that if S is finite with k elements, then :

$$|S + S| \geq 2k - 1 \ , \ |S + S| \leq k(k-1)/2.$$

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Moreover

$$|S + S| = 2k - 1$$

if and only if S is an arithmetic progression of lenght k.

An arithmetic progression of length k and difference d is a set

$$\{a, a + d, a + 2d, ..., a + (k - 1)d\},$$

where a, d, k are integers, $d, k \geq 1$.

Problem

Let S be a finite subset of the integers of order k. What is the **structure** *of S if $|S + S|$ is not much greater than the minimal value $2k - 1$?*

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G. Freiman proved the following:

Theorem (B)

Let S be a finite set of integers with $k \geq 3$ elements and suppose that

$$|S+S| \leq 2k-1+b,$$

where $0 \leq b \leq k-3$.
Then S is contained in an arithmetic progression of length $k+b$ and difference q,

$$P = \{a, a+q, a+2q, \cdots, a+(k+b-1)q\},$$

where a, q are integers with $q > 0$.
In particular $|P| \leq 2k-3$.

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Corollary

Let S be a finite set of integers with $k \geq 3$ elements and suppose that

$$|S+S| \leq 3k-4,$$

Then S is contained in an arithmetic progression.

Freiman studied also the case $|S+S| \leq 3|S|-3$ and $|S+S| \leq 3|S|-2$.

Theorem (C)

Let S be a finite set of integers with $k > 6$ elements and suppose that

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Then either S is a subset of an arithmetic progression of length at most $2k-1$ or S is a bi-arithmetic progression.

A set of the form $I \cup J$ is called a bi-arithmetic progression of length k with difference d if both I and J are arithmetic progressions of difference d, $|I|+|J|=k$, and $I+I, I+J, J+J$ are pairwise disjoint.

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Our aim is to generalize Freiman's results to finite subsets of **ordered groups**.

Definition

Let G be a group and suppose that a total order relation $\leq$ is defined on the set G.
We say that $(G, <)$ is an *ordered group* if for all $a, b, x, y \in G$, the inequality $a \leq b$ implies that $xay \leq xby$.

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A group G is *orderable* if there exists a total order relation $\leq$ on the set G, such that $(G, <)$ is an ordered group.

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Obviously the group of the integers with the usual order is an ordered group.

Theorem (F.W. Levi)

An **abelian group** *G is orderable if and only if it is torsion-free.*

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Moreover, if $|S^2| = 2k - 1$, then S is a geometric progression, i.e. there exists $g \in G$, $x \in S$ such that

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Now new problems arise.

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$$|S^2| \leq 2k - 1 + b,$$

where $0 \leq b \leq k$?

In particular:

Problem

What is the maximal upper bound on $|S^2|$ which implies that the subgroup $\langle S \rangle$ is abelian?

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*Let $(G, \leq)$ be an ordered group, S a finite subset of G of order k. What is the **structure** of $\langle S \rangle$, the subgroup generated by S, if $|S^2|$ satisfies*

$$|S^2| \leq 2k - 1 + b,$$

where $0 \leq b \leq k$?

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What is the maximal upper bound on $|S^2|$ which implies that the subgroup $\langle S \rangle$ is abelian?

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Some results

Theorem (1)

Let $(G, \leq)$ be an ordered group and let $S = \{x_1, x_2, \cdots, x_k\}$ be a finite subset of G of size $k \geq 3$, with $x_1 < x_2 \cdots < x_k$.
Assume that $t = |S^2| \leq 3k - 4$.
Then $\langle S \rangle$ is abelian.
Moreover, there exists $g \in G$, $g > 1$, such that $gx_1 = x_1 g$ and S is a subset of

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Then $< S >$ is abelian and at most 3-generated.

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There exists an ordered group G with a subset S of order k such that $< S >$ is not abelian and $|S^2| \leq 3k - 2$.

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Let $G = A \rtimes \langle b \rangle$ be a semidirect product of an abelian group $A(\cdot)$ isomorphic to the additive rational group $(\mathbb{Q}, +)$ by an infinite cyclic group $\langle b \rangle$, such that $a^b = a^2$ for each $a \in A$.

Then G is torsion-free and it is orderable.

Take $a \in A \setminus 1$ and let

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. Since $ab = ba^2$, it is easy to see that

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Let G be an ordered group and let S be a subset of G of finite size k. Suppose

$$|S^2| \leq 3|S| - 3.$$

If $|S| = 3$, then $|S^2| \leq 6 = 3k - 3$. So assume $k > 3$.

We know that $\langle S \rangle$ is abelian and at most 3-generated, then the structure of S can be described using previous results of G. Freiman.

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Theorem (3)

Let G be an ordered group and let S be a subset of G of finite size $k > 3$. If

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then one of the following holds:

- (1) $|S| \leq 6$;
- (2) S is a subset of a geometric progression;
- (3) S is a bi-geometric progression,

$$S = \{ac^t \mid 0 \leq t \leq t_1 - 1\} \cup \{bc^t \mid 0 \leq t \leq t_2 - 1\}.$$

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Proposition

Let G be an ordered group. Let $S \subseteq G$, $S = \{x_1, x_2, x_3\}$, $x_1 < x_2 < x_3$. If $|S^2| \leq 7$ then one of the following holds:
(i) $S \cap Z(\langle S \rangle) \neq \emptyset$,
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Let G be an ordered group and let S be a subset of G of finite size $k > 3$. Suppose

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If $\langle S \rangle$ is abelian, then the structure of S can be explicitly described using previous results due to G. Freiman.

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Torsion-free nilpotent groups of class 2

Theorem (5)

Let G be a torsion-free nilpotent group of class 2 and let $S \subseteq G$ be non-abelian and of order $k \geq 4$. Then

$$|S^2| = 3k - 2$$

if and only if

$$S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\},$$

with $1 + i + 1 + j = k$ *and* $ab = bac$ *or* $ba = abc$, $c > 1$.

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The structure of $\langle S \rangle$ if $|S^2| \leq 3|S| - 1$

We could prove the following generalization of the previous Corollaries if k is big enough.

Theorem (6)

Let G be an ordered group and let S be a subset of G of finite size $k \geq 8$. If

$$|S^2| \leq 3k - 1,$$

then $\langle S \rangle$ is metabelian, and it is nilpotent of class 2 if G is nilpotent.

Torsion-free nilpotent groups of class 2

Theorem (7)

Let G be an ordered nilpotent group of class 2 and let S be a subset of G with $\langle S\rangle$ non-abelian of order $k \geq 5$. Then $|S^2| = 3k-1$ if and only if one of the following holds:

(i)

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A general result on the structure of $\langle S \rangle$

Arguing as in Theorem [3], it is possible to prove that for any positive integer s, if k is big enought and S is a subset of finite size k of an ordered group G and $|S^2| \leq 3k - 2 + s$, then $\langle S \rangle$ is metabelian, and it is nilpotent of class 2 if G is nilpotent. In fact we have:

Theorem (8)

Let G be an ordered group, s be any positive integer, and let k be an integer such that $k \geq 2^{s+2}$. If S is a subset of G of finite size k and such that

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Proof of Theorems 1 and 3

Definizione

Let A be a finite subset of an abelian group $G(+)$ and B a finite subset of an abelian group $H(+)$. A map $\varphi : A \longrightarrow B$ is a Freiman isomorphism if it is bijective and from

$$a_1 + a_2 = b_1 + b_2$$

it follows

$$\varphi(a_1) + \varphi(a_2) = \varphi(b_1) + \varphi(b_2).$$

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Esempio

If $H(+)$ and $G(+)$ are finite groups, an isomorphism $\varphi : G \longrightarrow H$ is a Freiman isomorphism. *Let $r \geq 5$,*

$$A = \{0, 1, 2, r, r+1, 2r\} \subseteq \mathbb{Z},$$

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. *The map φ defined by putting:*

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If A and B are Freiman isomorphic, then

$$|A| = |B|$$

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Remark

If $\varphi : A \longrightarrow B$ is a Freiman isomorphism and

$$A = \{a, a + d, a + 2d, \cdots, a + (k - 1)d\}$$

is an arithmetic progression with difference d, then B is an arithmetic progression with difference $\varphi(a + d) - \varphi(a)$.

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Proof of Theorems 1 and 3

Let G be a torsion free abelian group. If S is a finite subset of G, we write $m(G)$ the rank of the abelian group $\langle S\rangle$, i.e. the number m such that $\langle S\rangle$ is isomorphic to $\mathbb{Z}^m$.

Definizione

The Freiman dimension of S, $d(S)$, is the maximum positive integer d such that there exists a Freiman isomorphism between S and a subset T of $\mathbb{Z}^d$, not situated on an affine hyperplane (where an affine hyperplane of a d-dimensional linear space L is shift of a $(d-1)$-dimensional subspace by a vector of L.

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It is possible to prove that:

$$m(S) \leq d(S) + 1.$$

Moreover Freiman proved that:

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If S is a finite subset of an abelian group, $d = d(S)$ the Freiman dimension of S, then

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Proof of Theorems 1 and 3

If $d = 2$, we obtain the contradiction

$$-1 \geq 0.$$

If $d \geq 3$, using $|S| \geq d + 1$ we obtain

$$d^2 + d - 8 \geq (d - 2)(d + 1),$$

and

$$d^2 - 3d + 4 \leq 0,$$

a contradiction. Therefore $d = 1$, then by Freiman Theorem (B), S is Freiman isomorphic to a set contained in an aritmetic progression, then S is contained in a geometric progression.

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If

$$|S^2| \leq 3k - 3,$$

arguing similarly, we get that the only possibilities are

$$d = 1$$

or

$$d = 2,$$

and Theorems (B) and (C) apply.

Proofs of Theorems concerning the structure of $\langle S \rangle$

Let $(G, \leq)$ be an ordered group, $S = \{x_1, x_2, \cdots, x_{k-1}, x_k\}$ a subset of G, $|S| = k$, $|S^2| \leq 3k - v, v \in \{1, 2, 3, 4\}$.
Suppose $x_1 < x_2 < \cdots < x_{k-1} < x_k$.
Write

$$T = \{x_1, \cdots, x_{k-1}\}.$$

We show that either

$$|T^2| \leq 3(k-1) - v, \textit{ or } \langle T \rangle \textit{ is abelian.}$$

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Proof of Theorem 2

Theorem (2)

Let $(G, \leq)$ be an ordered group and let $S = \{x_1, x_2, \cdots, x_k\}$ be a finite subset of G of size $k \geq 2$, with $x_1 < x_2 \cdots < x_k$.
Assume that

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Then $< S >$ is abelian.

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Suppose that $S = \{x_1, x_2, \cdots, x_k\}$ is a subset of an ordered group, $x_1 < x_2 < \cdots < x_k$.
Assume $|S^2| \leq 3|S| - 3$.
We want to show that $\langle S \rangle$ is abelian.
If $k = 2$ or $k = 3$, we prove directly the result.
Suppose $k > 3$ and argue by induction on k. Write $T = \{x_1, \cdots, x_{k-1}\}$.
Then either $\langle T \rangle$ is abelian or $|T^2| = 3|T| - 3$, by the previous remarks.
By induction we can assume that $\langle T \rangle$ is abelian.
If $x_i x_k \in T^2$, for some $i < k$, then $x_k \in \langle T \rangle$ and $\langle S \rangle \subseteq \langle T \rangle$ is abelian , as required. Hence we can assume that $x_1 x_k, \cdots, x_{k-1} x_k, x_k^2 \notin T^2$, then $|T^2| \leq |S^2| - k = 3k - 3 - k = 2(k-1) - 1$. Then

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Proof of Theorem 2

Write

$$V = \{x_2, \cdots, x_k\}.$$

Considering the order opposite to $<$ and arguing on V as we did on T we get that V is abelian.
Moreover $|V| \leq 3$, since $k > 3$. Then there exist $i \neq j$ such that $[x_k, ac^i] = [x_k, ac^j] = 1$. Then $[x_k, c^{i-j}] = 1$ and

$$[x_k, c] = 1.$$

since we are in an ordered group.
From $[x_k, ac^j] = 1$, we get that also

$$[x_k, a] = 1.$$

Thus $x_k \in C_G(T)$ and $\langle S \rangle$ is abelian, as required.

Problems

Problem

Let $(G, \leq)$ *be an ordered group,* S *a finite subset of* G *of order* $k > 3$*.*
What is the **structure** *of* S *, if*

$$|S^2| \leq 3k - 2?$$

Theorem (4)

Let G be an ordered group and let S be a subset of G of finite size $k > 3$. If

$$|S^2| \leq 3k - 2,$$

then one of the following holds:

- (1) $\langle S \rangle$ *is abelian;*
- (2) $\langle S \rangle = \langle a, b \,|[a, b] = c, [c, a] = [c, b] = 1\rangle$;
- (3) $\langle S \rangle = \langle a, b \,|[a, b] = c, [c, a] = 1, c^b = c^2\rangle$;
- (4) $\langle S \rangle = \langle a, b \,|[a, b] = c, [c, a] = 1, (c^2)^b = c\rangle$;
- (5) $\langle S \rangle = \langle a, b \mid a^b = a^2\rangle$;
- (6) $\langle S \rangle = \langle a, b \mid ba^2 = ab^2, a^2ba^{-2} = bab^{-1}\rangle$.

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Let S be a finite subset of an ordered group. What is the maximal upper bound on $|S^2|$ which implies that the subgroup $\langle S\rangle$ is soluble of fixed length s?

We have solved the problem if $s = 1$. If $|S^2| \leq 3|S| - 3$ the group $\langle S\rangle$ is abelian and there exists an ordered group with a subset S of order k (for any k) such that $|S^2| = 3k - 2$ and $\langle S\rangle$ non-abelian.

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Thank you for the attention !

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